ALGEBRO-GEOMETRIC POISSON BRACKETS FOR REAL FINITE-ZONE SOLUTIONS OF THE SINE-GORDON EQUATION AND THE NONLINEAR SCHRÖDINGER EQUATION

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Algebro-geometric Poisson brackets for real, finite-zone solutions of the Korteweg–de Vries (KdV) equation were studied in [1]. The transfer of this theory to the Toda lattice and the sinh-Gordon equation is more or less obvious. The complex part of the finite-zone theory for the nonlinear Schrödinger equation (NS) and the sine-Gordon equation (SG) is analogous to KdV, but conditions that solutions be real require serious investigation.

I. **Complex, "finite-zone" solutions of SG and NS. Poisson brackets.** The SG equation ($u_{tt} - u_{xx} + \sin u = 0$) and the NS equation ($ir_t = -r_{xx} + 2r^2 q$, $iq_t = q_{xx} - 2q^2 r$) can be represented as commutation conditions for λ-pencils (see [2]):

$$[L, \partial_t + B] = 0,$$

$$\text{(SG)} \quad L = -\partial_x + \sqrt{\lambda}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{i}{4}(u_t + u_x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{16\sqrt{\lambda}}\begin{pmatrix} 0 & e^{-iu} \\ -e^{iu} & 0 \end{pmatrix},$$

$$\text{(NS)} \quad L = \partial_x + \lambda\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} i(r-q) & r+q \\ r+q & i(q-r) \end{pmatrix}.$$

In the periodic or quasiperiodic case ($\exp(iu)$ is quasiperiodic for SG) the operator L has a Bloch eigenfunction ψ which with suitable normalization is meromorphic on a Riemann surface Γ over the λ plane:

$$\text{(SG)} \quad y^2 = \prod_{j=0}^{2g}(\lambda - \lambda_j), \quad \lambda_0\lambda_1 \dots \lambda_{2g} = 0;$$

$$\text{(NS)} \quad y^2 = \prod_{j=0}^{2g+1}(\lambda - \lambda_j).$$

The function ψ possesses poles γ_j (or zeros $\gamma_j(x)$ of the first component of ψ), $j = 0, \dots, g$ for NS, $j = 1, \dots, g$ for SG. These equations are Hamiltonian with standard Hamiltonians and Poisson brackets $\{\cdot,\cdot\}$, where the nonzero brackets are the following:

$$\text{(SG)} \quad \{u(x), \pi(x')\}_1 = \delta(x - x'), \quad \pi = u_t,$$

$$\text{(NS)} \quad \{r(x), q(x')\}_1 = \delta(x - x').$$

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Formulas for solutions in terms of θ functions, the derivations of which differ little from the KdV case, can be found in [3] for NS and also in [4] and [5] (Theorem 1 and the example of §4) for SG. We shall not discuss them here. It is important only that these formulas have the form $F(Ux + Wt + K_0)$, where F is a complex function of g variables and U, W and K_0 are constant vectors. In the case of NS the formula of this type characterizes only the quantity rq; r and q themselves contain $g+1$ periods including the "phase rotation". The vector U has the components

$$U_j = \oint_{b_j} dp, \quad \oint_{a_j} dp = 0, \quad j = 1, 2, \dots, g,$$

where $(a_1, \dots, a_g, b_1, \dots, b_g)$ is a canonical basis of cycles in $H_1(Y)$, $z = \lambda^{-1}$,

$$\text{(NS)} \qquad dp = dz\left(\pm\frac{1}{z^2} + O(1)\right), \quad \sigma^*\, dp = -dp$$

near both infinitely distant points $\infty_\pm \in \Gamma$, and σ is the holomorphic involution $\sigma(\lambda, +) = (\lambda, -)$, $\sigma^2 = 1$;

$$\text{(SG)}^- \qquad \begin{gathered} dp_+ = dz(-1/z^2 + O(1)), \quad z = \lambda^{-1/2} \to 0, \\ dp_- = dw(1/16w^2 + O(1)), \quad w = \lambda^{1/2} \to 0, \\ dp = dp_+ + dp_-, \quad \oint_{a_j} dp_\pm = 0, \quad j = 1, 2, \dots, g. \end{gathered}$$

For SG there is the "mean density of topological charge"

$$2\pi\bar{e} = \lim_{L\to\infty} \frac{1}{L}\int_0^L u_x\, dx.$$

The function $p(\lambda)$ represents the "quasimomentum" in the periodic case, i.e.,

$$\psi_\pm(x + T, \lambda) = \exp\{\pm ip(\lambda)T\}\psi_\pm(x, \lambda).$$

The coefficients of the expansion of $p(\lambda)$ are called the *Hamiltonians of the "higher SG" or "higher NS"*:

$$p(\lambda) = \lambda + c_0 + c_1/2\lambda + \dots, \quad \lambda \to \infty_+, \ \text{NS};$$

$$\text{(SG)} \qquad p(\lambda) = \begin{cases} \sqrt{\lambda} + 2\pi\bar{e} + c_+(16\sqrt{\lambda})^{-1} + \cdots, & \lambda \to \infty, \\ -(15\sqrt{\lambda})^{-1} + \pi\bar{e} - c_-\sqrt{\lambda} + \cdots & \lambda \to 0; \end{cases}$$

here $2c_0 = \int (\ln q)_x\, dx$, $c_1 = -\int rq\, dx$ is the generator of the phase rotation and $c_+ + c_- = H$ is the SG Hamiltonian, and $p(\lambda)$ is single-valued on $\hat{\Gamma}$ (see below).

The *algebro-geometric Poisson brackets* [1] are

$$\{\lambda_{j_1}, \lambda_{j_2}\} = \{\gamma_{q_1}, \gamma_{q_2}\} = 0.$$

Since for NS the number of indices j is equal to $g+1$, the Abel transformation linearizes the dynamics of only g complex quantities on the torus $J(\Gamma)$. There still remains the "phase variable" in the kernel of the Abel transformation. This is a typical situation for matrix systems where the number of poles γ_j is greater than the genus. The SG case is essentially scalar.

The *analytic brackets* are given by a meromorphic 1-form $Q(\lambda)\, d\lambda$ on Γ or on the covering $\hat{\Gamma} \to \Gamma$ which preserves the closedness of all cycles (a_j); here

$$\{Q(\gamma_j), \gamma_k\} = \delta_{jk}, \quad \{Q(\gamma_j), Q(\gamma_k)\} = 0.$$

The bracket $\{\cdot,\cdot\}$ is said to be *consistent with the SG* (*or NS*) *dynamics* if all its higher analogues are Hamiltonian.

Example 1. The standard bracket $\{\cdot,\cdot\}_1$ is analogous to [6] for NS and to Example 4 of [1] for SG:

$$Q_1(\lambda)\,d\lambda = 4ip(\lambda)\lambda^{-1}d\lambda \quad (\text{on } \hat{\Gamma}), \tag{SG}$$

$$Q_1(\lambda)\,d\lambda = -2ip(\lambda)d\lambda \sim 2i\lambda\,dp(\lambda). \tag{NS}$$

The annihilators of these brackets consist of the periods $T_1,\dots,T_g$ together with the condition $\prod \lambda_j = 0$ for SG, and of $T_1,\dots,T_{g+1}$ for NS.

Example 2. The Poisson bracket $\{\cdot,\cdot\}_2$ of the stationary problem

$$\sum c_j \delta H_j = 0,$$

where the H_j are Hamiltonians of the higher analogues of SG or NS. According to [7], these Poisson brackets are consistent with the SG and NS dynamics; the bracket $\{\cdot,\cdot\}_2$ is algebro-geometric and analytic in analogy to [8]:

$$Q_2(\lambda)\,d\lambda = 2i\left(1+16\sqrt{\prod_{\lambda_j\neq 0}\lambda_j}\right)\sqrt{\prod(\lambda-\lambda_j)}\,\lambda^{-2}\,d\lambda, \tag{SG}$$

$$Q_2(\lambda)\,d\lambda = -2i\sqrt{\prod(\lambda-\lambda_j)}\,d\lambda. \tag{NS}$$

The annihilator of the bracket $\{\cdot,\cdot\}_2$ consists of the quantities (c_j) which can be expressed in one-to-one fashion in terms of the following symmetric functions of the end points of the zones:

$$\sigma_1,\sigma_2,\dots,\sigma_{g-1},\pm\sqrt{\sigma_{2g}}, \tag{SG}$$

$$\sigma_1,\sigma_2,\dots,\sigma_{g+1}, \tag{NS}$$

where $\sigma_k = \prod_{i_1<\dots<i_k}\lambda_{i_1}\cdots\lambda_{i_k}$.

Remark. According to an analogue of Lemma 3 of [1], for brackets consistent with the SG or NS dynamics the Hamiltonians of the higher SG or NS equations are generated by the same coefficients of the expansion of $Q(\lambda)$ near the points $\lambda=\infty_+$ (NS) or $\lambda = 0,\infty$ (SG) as for the standard bracket $\{\cdot,\cdot\}_1$. All the remaining coefficients of the expansion belong to the annihilator.

II. **Conditions for real SG and NS solutions in the γ representation. The action variables.** Suppose that a solution $u(x,t)$ is real for SG or $r=\pm\bar{q}$ for NS (notation: $\mathrm{NS}_\pm$). The most difficult question is the precise description of the location of the quantities γ_j on Γ. The case NS_+ where L is selfadjoint is an exception. In this case all $\lambda_j\in R$, $j=0,1,\dots,2g+1$. Cycles a_j on Γ are selected which lie over the lacunae $[\lambda_{2j},\lambda_{2j+1}]$, $j=0,1,\dots,g$. In analogy to KdV, $\gamma_j\in a_j$. We obtain the torus $T^{g+1}=(a_0\times a_1\times\cdots\times a_g)$. The action variables I_q conjugate to the angles ϕ_q (mod 2π) on T^{g+1} have the form

$$I_j = \frac{1}{2\pi}\oint_{a_j} Q(\lambda,\Gamma)\,d\lambda, \quad j=0,1,\dots,g. \tag{1}$$

Since the collection of cycles a_j cuts Γ into two parts $\Gamma=\Gamma_+\cup\Gamma_-$, we have

$$\sum_q I_q = \sum_{P_k}\operatorname*{res}_{\lambda=P_k}[Q(\lambda)d\lambda], \quad P_k\in\Gamma_+. \tag{2}$$

We shall henceforth assume that the form $Q\,d\lambda$ is "real" for real Γ and has a unique pole on Γ_+ at the point $\lambda = \infty_+$. Under these conditions the following result holds.

Theorem 1. *Suppose that the Poisson bracket is consistent with the dynamics of all NS. Then the following assertions are true*:

a) *The action variables conjugate to the angles on the torus T^{g+1} have the form* (1).

b) *The sum $\sum_0^g I_q = \mathrm{res}_{\infty_+}[Q\,d\lambda]$ coincides with the generator of the phase transformation $r \to re^{i\phi}$.*

c) *The Hamiltonians of the "higher NS" are obtained from the expansion of $Q(\lambda)\,d\lambda$ at $\lambda = \infty_+$ in terms of $z = \lambda^{-1}$ at the same sites as in the expansion of $Q_1(\lambda)\,d\lambda = 2ip\,d\lambda$ (and with the same coefficients). The remaining terms of the expansion belong to the annihilator.*

We now proceed to the involved cases of SG and NS$_-$. Using the results of [9], [10] and [3], we can easily describe an admissible class of surfaces Γ:

1) The branch points come in complex conjugate pairs $(\lambda_{2j+1}, \lambda_{2j+2} = \bar\lambda_{2j+1})$; among them there is no real pair (NS$_-$).

2) Part of the branch points $\lambda_0 < \lambda_1 < \cdots < \lambda_{2k-2} < \lambda_{2k-1} < \lambda_{2k} = 0$ is real and negative; the other part of the branch points $(\lambda_{2j+1}, \lambda_{2j+2} = \bar\lambda_{2j+1})$ comes in complex conjugate pairs, $j > k$ (SG).

As $x \in R$ varies the zeros $\gamma_j(x)$ cover sets $M_j \in \Gamma$ containing cycles $[M_j]$ with the natural orientation; the projections of these on the λ plane are invariant under the mapping $\lambda \to \bar\lambda$. Let $x_\alpha \in R$, $|x_\alpha| \to \infty$ if $\alpha \to \infty$, where $\gamma_j(x_\alpha) \approx \gamma_j(x_0)$, and $\gamma_{j_\alpha} \colon [x_0, x_\alpha] \to M_j$.

Definition. The *average "number of oscillations"* is

$$m_j = \lim_{\alpha\to\infty} \frac{\deg \gamma_{j_\alpha}}{x_\alpha - x_0} \geq 0.$$

where $\deg \gamma_{j_\alpha}$ is the torsion number in the homology group $H_1(\Gamma)$.

Lemma 1. *Let a_j be the homology class of the γ-cycle $[M_j]$; then $a_{j_1} \circ a_{j_2} = 0$, and $\tau_* a_j = a_j$, where $\tau(y, \lambda) = (-\bar y, \bar\lambda)$.*

Using the collection (a_j), we choose a canonical basis of cycles and normalize $dp(\lambda)$ with respect to this basis. There arises the vector $U_j = \oint_{b_j} dp$.

Lemma 2. $2\pi m_j = U_j > 0$.

We introduce the "natural" numeration of the cycles $a'_q = a_j$, $q = q(j)$, where $\cdots < m_{q-1} < m < \cdots$. The following results can be proved.

Theorem 2. *The homology classes a_q possess representations which are curves M'_q without self-intersections having the properties that their projections N'_q onto the λ plane are without self-intersections and do not intersect pairwise, and that they are invariant under the mapping $\lambda \to \bar\lambda$. In the case of SG the curves $N'_{q(j)}$ are closed for $1 \leq j \leq k$, and they intersect the semiaxis $(0, \infty)$ once at points μ_q and the segment $[\lambda_{2j-2}, \lambda_{2j-1}]$; they intersect the real axis nowhere else; the curves $N'_{q(j)}$ for $j > k$ and all N'_q for NS$_-$ terminate at the branch points $\lambda_{2j-1}, \lambda_{2j}$, and intersect the real axis once at points μ_q of the semiaxis $(0, \infty)$. Here $0 < \cdots < \mu_{q-1} < \mu_q < \cdots$ under the natural ordering of $q(j)$. The subgroup of the group $H_1(\Gamma, Z)$ generated by the cycles (a_q) does not depend on the ordering. A basis of*

γ-cycles $[M'_q] \in H_1(\Gamma)$ is uniquely determined by these properties with the condition $U_q \geq 0$. There is the formula for the average density of topological charge

$$\bar{e} = \sum_{j \leq k} \sigma_j m_j = (2\pi)^{-1} \sum_{j \leq k} \sigma_j U_j; \quad \sigma_j = \pm 1, \quad j = 1, 2, \ldots, k, \tag{3}$$

where the signs depend on the "index" $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\sigma_j = \pm 1$, $j = 1, \ldots, k$, of the connected components of real solutions for given Γ (see Theorem 3). For NS_- there are no real branch points and $\sum_0^g a_j = 0$.

Using [11] and [12], we can prove the following assertion.

Theorem 3. 1) *For the SG equation with $k = 0$ and NS_- there is only one real torus for given branch points—"the spectrum" of the operator L.*

2) *For SG with $k \neq 0$ there are 2^k connected components numbered by collections $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\sigma_j = \pm 1$, $j = 1, \ldots, k$, with the collections of γ-cycles*

$$(M_1^{\sigma_1}, M_2^{\sigma_2}, \ldots, M_j^{\sigma_j}, \ldots, M_k^{\sigma_k}, M_{k+1}, \ldots, M_g) = M(\sigma),$$

where $M_j^- = \tau(M_j^+)$ and τ is the anti-involution $\tau(y, \lambda) = (-\bar{y}, \bar{\lambda})$. For $j \leq k$ the anti-involution τ reverses the direction of the projection N'_j and changes the sign of m_j in (3).

3) *To each component with index σ there corresponds a collection of covering γ-cycles $\hat{M}(\sigma) = (\hat{M}_1^{\sigma_1}, \ldots, \hat{M}_k^{\sigma_k}, \hat{M}_{k+1}, \ldots, \hat{M}_g)$ on $\hat{\Gamma}$ which jointly form part of the boundary of one of the copies of Γ in $\hat{\Gamma}$ (we recall that the surface $\hat{\Gamma}$ is glued together from an infinite number of copies of Γ cut along the cycles a_j). Suppose that σ' is obtained from σ by changing only one sign with index j ($\sigma_j = +1 \to \sigma'_j = -1$). Then the collection $\hat{M}(\sigma')$ is obtained from $\hat{M}(\sigma)$ by superposition of the operation τ on the cycle $\hat{M}_j^+$ (the curve $\hat{M}_j^+$ is replaced by the curve $\hat{M}_j^-$ homologous to it on $\hat{\Gamma}$ which covers the curve $M_j^- = \tau(M_j^+)$) and the shift of all γ-cycles by the monodromy transformation $\kappa_j \colon \hat{\Gamma} \to \hat{\Gamma}$ corresponding to the cycle $b_{q(j)}$:*

$$\hat{M}(\sigma') = \kappa_j(\hat{M}_1^{\sigma_1}, \hat{M}_2^{\sigma_2}, \ldots, \tau \hat{M}_j^{\sigma_j}, \ldots, \hat{M}_k^{\sigma_k}, \ldots, \hat{M}_g).$$

Corollary 1. *If the form $Q\, d\lambda$ is meromorphic on Γ with poles only at $\lambda = 0, \infty$, then the action variables of distinct components $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\sigma' = (\sigma'_1, \ldots, \sigma'_k)$ differ for those j where $\sigma_j \neq \sigma'_j$:*

$$I_{q(j)}(\sigma) - I_{q(j)}(\sigma') = \frac{1}{2}(\sigma_j - \sigma'_j) \operatorname*{res}_{\lambda=0} [Q\, d\lambda].$$

Corollary 2. *For the standard bracket $\{\cdot, \cdot\}$, the form $Q_1\, d\lambda = 4ip\, d\lambda/\lambda$ is meromorphic on $\hat{\Gamma}$ ($\kappa_j p(\lambda) = p(\lambda) + U_j$); passage from the component $\sigma = (\sigma_1, \ldots, \sigma_k)$ to the component $\sigma' = (\sigma'_1, \ldots, \sigma'_k)$ implies the change of action variables*

$$I_{q(s)}(\sigma) = \frac{1}{2\pi} \oint_{\hat{M}_s(\sigma)} Q_1\, d\lambda \to I_{q(s)}(\sigma') = \frac{1}{2\pi} \oint_{\hat{M}_s(\sigma')} Q_1\, d\lambda,$$

where

$$I_{q(s)}(\sigma') = I_{q(s)}, \quad s > k,$$

$$I_{q(s)}(\sigma') = I_{q(s)} + 8\pi \left[\sum_{s=1}^{k} m_s \frac{\sigma_s \sigma_j - \sigma'_s \sigma'_j}{2} \right], \quad j \leq k.$$

Remark 1. In a recent preprint [13] the action variables for SG, $k = 0$, $g = 2$, are actually indicated in a certain integral basis of the group of a-cycles which is defined without using the natural numeration.

Remark 2. In the recent paper [14], where effective conditions for real SG solutions are obtained expressed in terms of θ functions, a random basis of a-cycles was used. For applications it is natural to use the canonical basis of a-cycles found here in which the structure of the formulas is considerably simplified.

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