Algebro-geometric Poisson brackets for real, finite-zone solutions of the Korteweg–de Vries (KdV) equation were studied in [1]. The transfer of this theory to the Toda lattice and the sinh-Gordon equation is more or less obvious. The complex part of the finite-zone theory for the nonlinear Schrödinger equation (NS) and the sine-Gordon equation (SG) is analogous to KdV, but conditions that solutions be real require serious investigation.

I. Complex, “finite-zone” solutions of SG and NS. Poisson brackets. The SG equation \( (u_{tt} - u_{xx} + \sin u = 0) \) and the NS equation \( (ir_{tt} = -r_{xx} + 2r^2q, \ iq_t = q_{xx} - 2q^2r) \) can be represented as commutation conditions for \( \lambda \)-pencils (see [2]):

\[
[L, \partial_t + B] = 0,
\]

(SG) \[
L = -\partial_x + \sqrt{\lambda} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{i}{4}(u_t + u_x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{16\sqrt{\lambda}} \begin{pmatrix} 0 & e^{-iu} \\ -e^{iu} & 0 \end{pmatrix},
\]

(NS) \[
L = \partial_x + \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} i(r - q) & r + q \\ r + q & i(q - r) \end{pmatrix}.
\]

In the periodic or quasiperiodic case (\( \exp(iu) \) is quasiperiodic for SG) the operator \( L \) has a Bloch eigenfunction \( \psi \) which with suitable normalization is meromorphic on a Riemann surface \( \Gamma \) over the \( \lambda \) plane:

(SG) \[
y^2 = \prod_{j=0}^{2g}(\lambda - \lambda_j), \quad \lambda_0\lambda_1 \ldots \lambda_{2g} = 0;
\]

(NS) \[
y^2 = \prod_{j=0}^{2g+1}(\lambda - \lambda_j).
\]

The function \( \psi \) possesses poles \( \gamma_j \) (or zeros \( \gamma_j(x) \) of the first component of \( \psi \), \( j = 0, \ldots, g \) for NS, \( j = 1, \ldots, g \) for SG. These equations are Hamiltonian with standard Hamiltonians and Poisson brackets \( \{\cdot, \cdot\} \), where the nonzero brackets are the following:

(SG) \[
\{u(x), \pi(x')\}_1 = \delta(x - x'), \quad \pi = u_t,
\]

(NS) \[
\{r(x), q(x')\}_1 = \delta(x - x').
\]

\textit{Date:} Received 17/JUN/82.
\textit{1980 Mathematics Subject Classification.} Primary 35Q20, 35J10. UDC 513.835.
\textit{Translated by} J. R. SCHULENBERGER.
Formulas for solutions in terms of $\theta$ functions, the derivations of which differ little from the KdV case, can be found in [3] for NS and also in [4] and [5] (Theorem 1 and the example of §4) for SG. We shall not discuss them here. It is important only that these formulas have the form $F(Ux + Wt + K_0)$, where $F$ is a complex function of $g$ variables and $U$, $W$ and $K_0$ are constant vectors. In the case of NS the formula of this type characterizes only the quantity $rq$; $r$ and $q$ themselves contain $g + 1$ periods including the “phase rotation”. The vector $U$ has the components

$$U_j = \oint_{b_j} dp, \quad \oint_{a_j} dp = 0, \quad j = 1, 2, \ldots, g,$$

where $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is a canonical basis of cycles in $H_1(Y)$, $z = \lambda^{-1}$,

(NS) \hspace{1cm} dp = dz \left( \pm \frac{1}{z^2} + O(1) \right), \quad \sigma^* dp = -dp

near both infinitely distant points $\infty_{\pm} \in \Gamma$, and $\sigma$ is the holomorphic involution $\sigma(\lambda, +) = (\lambda, -)$, $\sigma^2 = 1$;

SG \hspace{1cm} dp_+ = dz(-1/z^2 + O(1)), \quad z = \lambda^{-1/2} \to 0;
\hspace{1cm} dp_- = dw(1/16w^2 + O(1)), \quad w = \lambda^{1/2} \to 0,

(SG) \hspace{1cm} dp = dp_+ + dp_-, \quad \oint_{a_j} dp_\pm = 0, \quad j = 1, 2, \ldots, g.

For SG there is the “mean density of topological charge”

$$2\pi \bar{c} = \lim_{L \to \infty} \frac{1}{L} \int_0^L u_\lambda \, dx.$$

The function $p(\lambda)$ represents the “quasimomentum” in the periodic case, i.e.,

$$\psi_\pm(x + T, \lambda) = \exp(\pm ip(\lambda)/T)\psi_\pm(x, \lambda).$$

The coefficients of the expansion of $p(\lambda)$ are called the Hamiltonians of the “higher SG” or “higher NS”:

$$p(\lambda) = \lambda + c_0 + c_1/2\lambda + \cdots, \quad \lambda \to \infty, \quad \text{NS};$$

(SG) \hspace{1cm} p(\lambda) = \begin{cases} \sqrt{\lambda} + 2\pi \bar{c} + c_+ (16\sqrt{\lambda})^{-1} + \cdots, \quad \lambda \to \infty, \\ - (15\sqrt{\lambda})^{-1} + \pi \bar{c} - c_- \sqrt{\lambda} + \cdots \quad \lambda \to 0; \end{cases}

here $2c_0 = \int (\ln q)_x \, dx$, $c_1 = - \int rq \, dx$ is the generator of the phase rotation and $c_+ + c_- = H$ is the SG Hamiltonian, and $p(\lambda)$ is single-valued on $\hat{\Gamma}$ (see below).

The algebra-geometric Poisson brackets [1] are

$$\{\lambda_{j_1}, \lambda_{j_2}\} = \{\gamma_{q_1}, \gamma_{q_2}\} = 0.$$

Since for NS the number of indices $j$ is equal to $g + 1$, the Abel transformation linearizes the dynamics of only $g$ complex quantities on the torus $J(\Gamma)$. There still remains the “phase variable” in the kernel of the Abel transformation. This is a typical situation for matrix systems where the number of poles $\gamma_j$ is greater than the genus. The SG case is essentially scalar.

The analytic brackets are given by a meromorphic 1-form $Q(\lambda) \, d\lambda$ on $\Gamma$ or on the covering $\hat{\Gamma} \to \Gamma$ which preserves the closedness of all cycles $(a_j)$; here

$$\{Q(\gamma_j), \gamma_k\} = \delta_{jk}, \quad \{Q(\gamma_j), Q(\gamma_k)\} = 0.$$
The bracket \( \{ \cdot, \cdot \} \) is said to be consistent with the SG (or NS) dynamics if all its higher analogues are Hamiltonian.

**Example 1.** The standard bracket \( \{ \cdot, \cdot \}_1 \) is analogous to [6] for NS and to Example 4 of [1] for SG:

\[
\text{(SG)} \quad Q_1(\lambda) \, d\lambda = 4i p(\lambda) \lambda^{-1} d\lambda \quad \text{(on } \hat{\Gamma} \text{)},
\]

\[
\text{(NS)} \quad Q_1(\lambda) \, d\lambda = -2i p(\lambda) \lambda^{-2} d\lambda.
\]

The annihilators of these brackets consist of the periods \( T_1, \ldots, T_g \) together with the condition \( \prod \lambda_j = 0 \) for SG, and of \( T_1, \ldots, T_{g+1} \) for NS.

**Example 2.** The Poisson bracket \( \{ \cdot, \cdot \}_2 \) of the stationary problem

\[
\sum_j c_j \delta H_j = 0,
\]

where the \( H_j \) are Hamiltonians of the higher analogues of SG or NS. According to [7], these Poisson brackets are consistent with the SG and NS dynamics; the bracket \( \{ \cdot, \cdot \}_2 \) is algebro-geometric and analytic in analogy to [8]:

\[
\text{(SG)} \quad Q_2(\lambda) \, d\lambda = 2i \left( 1 + 16 \prod_{\lambda_j \neq 0} \lambda_j \right) \sqrt{\prod (\lambda - \lambda_j)} \lambda^{-2} \, d\lambda,
\]

\[
\text{(NS)} \quad Q_2(\lambda) \, d\lambda = -2i \sqrt{\prod (\lambda - \lambda_j)} \, d\lambda.
\]

The annihilator of the bracket \( \{ \cdot, \cdot \}_2 \) consists of the quantities \( (c_j) \) which can be expressed in one-to-one fashion in terms of the following symmetric functions of the end points of the zones:

\[
\text{(SG)} \quad \sigma_1, \sigma_2, \ldots, \sigma_{g-1}, \pm \sqrt{2g}, \quad \sigma_1, \sigma_2, \ldots, \sigma_{g+1},
\]

where \( \sigma_k = \prod_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \)

**Remark.** According to an analogue of Lemma 3 of [1], for brackets consistent with the SG or NS dynamics the Hamiltonians of the higher SG or NS equations are generated by the same coefficients of the expansion of \( Q(\lambda) \) near the points \( \lambda = \infty \) for SG or \( \lambda = 0, \infty \) for NS as for the standard bracket \( \{ \cdot, \cdot \}_1 \). All the remaining coefficients of the expansion belong to the annihilator.

II. **Conditions for real SG and NS solutions in the \( \gamma \) representation.** The action variables. Suppose that a solution \( u(x,t) \) is real for SG or \( r = \pm \bar{q} \) for NS (notation: \( NS_{\pm} \)). The most difficult question is the precise description of the location of the quantities \( \gamma_j \) on \( \Gamma \). The case \( NS_+ \) where \( L \) is selfadjoint is an exception. In this case all \( \lambda_j \in R, \ j = 0, 1, \ldots, 2g + 1 \). Cycles \( a_j \) on \( \Gamma \) are selected which lie over the lacunae \( [\lambda_{2j}, \lambda_{2j+1}] \), \( j = 0, 1, \ldots, g \). In analogy to KdV, \( \gamma_j \in a_j \). We obtain the torus \( T^{g+1} = (a_0 \times a_1 \times \cdots \times a_g) \). The action variables \( I_q \) conjugate to the angles \( \phi_q \) (mod 2\( \pi \)) on \( T^{g+1} \) have the form

\[
I_j = \frac{1}{2\pi} \oint_{a_j} Q(\lambda, \Gamma) \, d\lambda, \quad j = 0, 1, \ldots, g.
\]

Since the collection of cycles \( a_j \) cuts \( \Gamma \) into two parts \( \Gamma = \Gamma_+ \cup \Gamma_- \), we have

\[
\sum_q I_q = \sum_{P_k} \text{res}_{\lambda = \lambda_k} [Q(\lambda) d\lambda], \quad P_k \in \Gamma_+.
\]
We shall henceforth assume that the form $Q \, d\lambda$ is “real” for real $\lambda$ and has a unique pole on $\Gamma_+$ at the point $\lambda = \infty_+$. Under these conditions the following result holds.

**Theorem 1.** Suppose that the Poisson bracket is consistent with the dynamics of all NS. Then the following assertions are true:

a) The action variables conjugate to the angles on the torus $T^{q+1}$ have the form (1).

b) The sum $\sum^\infty_0 T_q = \text{res}_\infty [Q \, d\lambda]$ coincides with the generator of the phase transformation $r \to re^{i\phi}$.

c) The Hamiltonians of the “higher NS” are obtained from the expansion of $Q(\lambda) \, d\lambda$ at $\lambda = \infty_+$ in terms of $z = \lambda^{-1}$ at the same sites as in the expansion of $Q(\lambda) \, d\lambda = 2ip \, d\lambda$ (and with the same coefficients). The remaining terms of the expansion belong to the annihilator.

We now proceed to the involved cases of SG and NS... Using the results of [9], [10] and [3], we can easily describe an admissible class of surfaces $\Gamma$:

1) The branch points come in complex conjugate pairs $(\lambda_{2j+1}, \lambda_{2j+2} = \lambda_{2j+1})$; among them there is no real pair (NS$_\lambda$).

2) Part of the branch points $\lambda_0 < \lambda_1 < \cdots < \lambda_2k-2 < \lambda_{2k-1} < \lambda_{2k} = 0$ is real and negative; the other part of the branch points $(\lambda_{2j+1}, \lambda_{2j+2} = \lambda_{2j+1})$ comes in complex conjugate pairs, $j > k$ (SG).

As $x \in R$ varies the zeros $\gamma_j(x)$ cover sets $M_j \in \Gamma$ containing cycles $[M_j]$ with the natural orientation; the projections of these on the $\lambda$ plane are invariant under the mapping $\lambda \to \bar{\lambda}$. Let $x_\alpha \in R$, $|x_\alpha| \to \infty$ if $\alpha \to \infty$, where $\gamma_j(x_\alpha) \approx \gamma_j(x_0)$, and $\gamma_j$; $[x_0, x_\alpha] \to M_j$.

**Definition.** The average “number of oscillations” is

$$m_j = \lim_{\alpha \to \infty} \frac{\deg \gamma_j}{x_\alpha - x_0} \geq 0,$$

where $\deg \gamma_j$ is the torsion number in the homology group $H_1(\Gamma)$.

**Lemma 1.** Let $a_j$ be the homology class of the $\gamma$-cycle $[M_j]$; then $a_{j_1} \circ a_{j_2} = 0$, and $\tau \cdot a_j = a_j$, where $\tau(y, \lambda) = (-\bar{y}, \bar{\lambda})$.

Using the collection $(a_j)$, we choose a canonical basis of cycles and normalize $dp(\lambda)$ with respect to this basis. There arises the vector $U_j = \int_{a_j} dp$.

**Lemma 2.** $2\pi m_j = U_j > 0$.

We introduce the “natural” numeration of the cycles $a'_q = a_q$, $q = q(j)$, where $\cdots < m_{q-1} < m < \cdots$. The following results can be proved.

**Theorem 2.** The homology classes $a_q$ possess representations which are curves $M'_q$ without self-intersections having the properties that their projections $N'_q$ onto the $\lambda$ plane are without self-intersections and do not intersect pairwise, and that they are invariant under the mapping $\lambda \to \bar{\lambda}$. In the case of SG the curves $N'_q$ are closed for $1 \leq j \leq k$, and they intersect the semiaxis $(0, \infty)$ once at points $\mu_q$ and the segment $[\lambda_{2j-2}, \lambda_{2j-1}]$: they intersect the real axis nowhere else; the curves $N'_q$ for $j > k$ and all $N'_q$ for NS$_\lambda$ terminate at the branch points $\lambda_{2j-1}, \lambda_{2j}$, and intersect the real axis once at points $\mu_q$ of the semiaxis $(0, \infty)$. Here $0 < \cdots < \mu_{q-1} < \mu_q < \cdots$ under the natural ordering of $q(j)$. The subgroup of the group $H_1(\Gamma, Z)$ generated by the cycles $(a_q)$ does not depend on the ordering. A basis of
where \( \Gamma \) is uniquely determined by these properties with the condition \( U_q \geq 0 \). There is the formula for the average density of topological charge

\[
\bar{e} = \sum_{j \leq k} \sigma_j m_j = (2\pi)^{-1} \sum_{j \leq k} \sigma_j U_j; \quad \sigma_j = \pm 1, \quad j = 1, 2, \ldots, k,
\]

where the signs depend on the “index” \( \sigma = (\sigma_1, \ldots, \sigma_k) \), \( \sigma_j = \pm 1 \), \( j = 1, \ldots, k \), of the connected components of real solutions for given \( \Gamma \) (see Theorem 3). For \( \text{NS}_- \) there are no real branch points and \( \sum a_j = 0 \).

Using [11] and [12], we can prove the following assertion.

**Theorem 3.**

1) For the SG equation with \( k = 0 \) and \( \text{NS}_- \) there is only one real torus for given branch points—"the spectrum" of the operator \( L \).

2) For SG with \( k \neq 0 \) there are \( 2^k \) connected components numbered by collections \( \sigma = (\sigma_1, \ldots, \sigma_k) \), \( \sigma_j = \pm 1 \), \( j = 1, \ldots, k \), with the collections of \( \gamma \)-cycles

\[
(M_1^\sigma, M_2^\sigma, \ldots, M_j^\sigma, \ldots, M_k^\sigma, M_{k+1}, \ldots, M_q) = M(\sigma),
\]

where \( M_j^- = \tau(M_j^+) \) and \( \tau \) is the anti-involution \( \tau(y, \lambda) = (-y, \lambda) \). For \( j \leq k \) the anti-involution \( \tau \) reverses the direction of the projection \( N_j^- \) and changes the sign of \( m_j \) in (3).

3) To each component with index \( \sigma \) there corresponds a collection of covering \( \gamma \)-cycles \( \hat{M}(\sigma) = (\hat{M}_1^\sigma, \ldots, \hat{M}_j^\sigma, \ldots, \hat{M}_k^\sigma, \hat{M}_{k+1}, \ldots, \hat{M}_q) \) on \( \hat{\Gamma} \) which jointly form part of the boundary of one of the copies of \( \Gamma \) in \( \Gamma \) (we recall that the surface \( \hat{\Gamma} \) is glued together from an infinite number of copies of \( \Gamma \) cut along the cycles \( a_j \)). Suppose that \( \sigma' \) is obtained from \( \sigma \) by changing only one sign with index \( j \) (\( \sigma_j = +1 \rightarrow \sigma_j' = -1 \)). Then the collection \( \hat{M}(\sigma') \) is obtained from \( \hat{M}(\sigma) \) by superposition of the operation \( \tau \) on the cycle \( \hat{M}_j^+ \) (the curve \( \hat{M}_j^- \) is replaced by the curve \( \hat{M}_j^- \) homologous to it on \( \hat{\Gamma} \) which covers the curve \( \hat{M}_j^- = \tau(\hat{M}_j^+) \)) and the shift of all \( \gamma \)-cycles by the monodromy transformation \( \kappa_j : \hat{\Gamma} \to \hat{\Gamma} \) corresponding to the cycle \( b_{q(j)} \):

\[
\hat{M}(\sigma') = \kappa_j(\hat{M}_1^\sigma, \hat{M}_2^\sigma, \ldots, \tau \hat{M}_j^\sigma, \ldots, \hat{M}_k^\sigma, \ldots, \hat{M}_q).
\]

**Corollary 1.** If the form \( Q \, d\lambda \) is meromorphic on \( \Gamma \) with poles only at \( \lambda = 0, \infty \), then the action variables of distinct components \( \sigma = (\sigma_1, \ldots, \sigma_k) \), \( \sigma' = (\sigma'_1, \ldots, \sigma'_k) \) differ for those \( j \) where \( \sigma_j \neq \sigma_j' \):

\[
I_{q(j)}(\sigma) - I_{q(j)}(\sigma') = \frac{1}{2}(\sigma_j - \sigma_j') \text{ res}_{\lambda=0} [Q \, d\lambda].
\]

**Corollary 2.** For the standard bracket \( \{\cdot, \cdot\} \), the form \( Q_1 \, d\lambda = 4ip \, d\lambda/\lambda \) is meromorphic on \( \Gamma \) (\( \kappa, p(\lambda) = p(\lambda) + U_j \)); passage from the component \( \sigma = (\sigma_1, \ldots, \sigma_k) \) to the component \( \sigma' = (\sigma'_1, \ldots, \sigma'_k) \) implies the change of action variables

\[
I_{q(s)}(\sigma) = \frac{1}{2\pi} \int_{M_s(\sigma)} Q_1 \, d\lambda \rightarrow I_{q(s)}(\sigma') = \frac{1}{2\pi} \int_{M_s(\sigma')} Q_1 \, d\lambda,
\]

where

\[
I_{q(s)}(\sigma') = I_{q(s)}; \quad s > k,
\]

\[
I_{q(s)}(\sigma') = I_{q(s)} + 8\pi \sum_{j=1}^{k} m_s \frac{\sigma_j \sigma_j - \sigma'_j \sigma'_j}{2}, \quad j \leq k.
\]
Remark 1. In a recent preprint [13] the action variables for SG, $k = 0$, $g = 2$, are actually indicated in a certain integral basis of the group of $a$-cycles which is defined without using the natural numeration.

Remark 2. In the recent paper [14], where effective conditions for real SG solutions are obtained expressed in terms of $\theta$ functions, a random basis of $a$-cycles was used. For applications it is natural to use the canonical basis of $a$-cycles found here in which the structure of the formulas is considerably simplified.

References


Landau Institute of Theoretical Physics, Cernogolovka, Moscow District