ON POISSON BRACKETS OF HYDRODYNAMIC TYPE

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I. **Riemannian geometry of multidimensional Poisson brackets of hydrodynamic type.** In [1] we developed the Hamiltonian formalism of general one-dimensional systems of hydrodynamic type. Now suppose that a system of the type of an ideal fluid (possibly with internal degrees of freedom) is described by a set of field variables $u = (u^i(x))$, $i = 1, \dots, N$, in a space with coordinates $x = (x^\alpha)$, $\alpha = 1, \dots, n$. The equations of motion of this system have the form of first order equations, linear in the derivatives:

$$u^i_t = v^{i\alpha}_j(u)u^j_\alpha, \quad i = 1, \dots, N, \quad u^j_\alpha \equiv \partial u^j/\partial x^\alpha \tag{1}$$

(here and later we assume summation over repeated indices). The multidimensional analog of the Poisson brackets of hydrodynamic type considered in [1] has the form

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}(u(x))\delta_\alpha(x-y) = u^k_\alpha b^{ij\alpha}_k(u(x))\delta(x-y). \tag{2}$$

We define Hamiltonians of hydrodynamic type as functionals of the form

$$H = \int h(u)\, d^n x, \tag{3}$$

that do not depend on the derivatives $u_\alpha, u_{\alpha\beta}, \dots$. As in the one-dimensional case, we have that the class of equations (1) and Hamiltonians (3), and the form of the Poisson brackets (2) are invariant under local transformations of the fields $u = u(w)$ (not containing the derivatives). For each α the coefficient $g^{ij\alpha}$ transforms like the set of components of a tensor of the second rank (metric with superscripts i, j); if these tensors do not degenerate, then the coefficients $b^{ij\alpha}_k = g^{is\alpha}\Gamma^{j\alpha}_{sk}$ transform like $\Gamma^{i\alpha}_{sk}$ the Christoffel symbols of a differential-geometric connection. Under linear unimodular changes of the spatial variables $x^\alpha = c^\alpha_\beta \tilde{x}^\beta$, $c^\alpha_\beta = \text{const}$, $\det(c^\alpha_\beta) = 1$, the quantities $g^{ij\alpha}$ and $b^{ij\alpha}_k$ for fixed i, j, k transform like vectors:

$$g^{ij\alpha} = c^\alpha_\beta \tilde{g}^{ij\beta}, \quad b^{ij\alpha}_k = c^\alpha_\beta \tilde{b}^{ij\beta}_k. \tag{4}$$

Here we consider *nondegenerate* brackets of the form (2); that is, $\det g^{ij\alpha} \neq 0$, $\alpha = 1, \dots, n$. (Because of the invariance of (4) it is sufficient to assume that the metric $g^{ij\alpha}$ is nondegenerate for one value of α.) From the results of [1] it follows immediately that for a Poisson bracket of the form (2) and for each α the metric $g^{ij\alpha}$ is symmetric, the connection $\Gamma^{i\alpha}_{jk}$ is compatible with the metric $g^{ij\alpha}$ and has zero curvature and torsion. However, for $n \geqslant 2$ it is in general impossible to reduce simultaneously, by a transformation of the form $u = u(w)$, all the metrics $g^{ij\alpha}(u)$ to

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constants, and the coefficients of connection to zero. An obstruction is the tensor (in the field variables)

$$T^{i\alpha\beta}_{jk} = \Gamma^{i\beta}_{jk} - \Gamma^{i\alpha}_{jk}. \tag{5}$$

It is more convenient to consider a tensor with superscripts

$$T^{ijk\alpha\beta} = -g^{ip\alpha} g^{kq\beta} T^{j\alpha\beta}_{pq} = b^{ij\alpha}_l g^{lkb} - b^{kj\beta}_l g^{li\alpha}, \tag{6}$$

defined also in the case of degenerate metrics.

Example. For Poisson brackets that correspond to the Lie algebra of vector fields in R^n (see [2])

$$\{p^i(x), p^j(y)\} = p^i(x)\delta_j(x-y) - p^j(y)\delta_i(y-x), \quad N = n, \tag{7}$$

the metrics $g^{ij\alpha}$ and the coefficients $b^{ij\alpha}_k$ have the form

$$g^{ij\alpha} = p^i \delta^{j\alpha} + p^j \delta^{i\alpha}, \quad b^{ij\alpha}_k = \delta^i_k \delta^{j\alpha}. \tag{8}$$

Here the simultaneous reduction of metrics to constant form is impossible for $n \geqslant 2$, since $T^{ijk\alpha\beta} \neq 0$. Metrics $g^{ij\alpha}$ of the form (8) are degenerate for $N > 2$. The same is true for all the examples, considered in [2], of multi-dimensional brackets of hydrodynamic type with more than two fields.

Theorem 1. *Nondegenerate Poisson brackets of the form* (2) *with* $N \geqslant 3$ *fields can be reduced by a transformation* $u = u(w)$ *to linear form,* $g^{ij\alpha} = c^{ij\alpha}_k u^k + g^{ij\alpha}_0$, $\alpha = 1, \dots, n$, $n \geqslant 2$, *where* $c^{ij\alpha}_k, g^{ij\alpha}_0$ *and* $b^{ij\alpha}_k$ *are constants.*

We examine separately the cases $N = 1$ and $N = 2$. For $N = 1$ a Poisson bracket of the form (2) is specified by the set $g^\alpha(u)$, $\alpha = 1, \dots, n$, which can be arbitrary functions of the field variable u. Now suppose that $N = 2$. The conditions on the metric $g^{ij\alpha}$ and the symmetric connections compatible with them for zeros of the curvature $b^{ij\alpha}_k = g^{is\alpha}\Gamma^{i\alpha}_{sk}$, which follow from the Jacobi identity for the bracket (2), have the form

$$T^{ijk\alpha\beta} = -T^{ijk\beta\alpha} = T^{kij\alpha\beta}, \tag{9}$$

where the tensor $T^{ijk\alpha\beta}$ is defined by (6), and also

$$T^{ik\alpha\beta}_j T^{jl\alpha\beta}_m = T^{il\alpha\beta}_j T^{jk\alpha\beta}_m, \tag{10}$$

where

$$T^{ik\alpha\beta}_j = g^{il\alpha} T^{k\alpha\beta}_{lj}. \tag{11}$$

Let us consider a classification of *admissible* (in the sense of (9) and (10)) nondegenerate metrics $g^{ij\alpha}$ in the first nontrivial case $n = N = 2$. Let λ_1 and λ_2 be the of the pair of quadratic forms g^{ij1}, g^{ij2}; that is, the roots of the equation

$$\det |g^{ij1} - \lambda g^{ij2}| = 0. \tag{12}$$

Theorem 2. *Admissible pairs of metrics* g^{ij1} *and* g^{ij2} *reduce to one of the following canonical forms*:

a) *For nonconstant and noncoincident* λ_1, λ_2, *the metrics can be written in the coordinates* $u^1 = \lambda_1$, $u^2 = \lambda_2$ *in the form*

$$g^{ij1} = h_i^{-2}\delta_{ij}, \quad g^{ij2} = (u^i)^{-1} h_i^{-2} \delta_{ij}, \tag{13}$$

where the coefficients h_1, h_2 satisfy the system of equations

$$\begin{gathered}\frac{\partial h_1}{\partial u^2} = -h_2 \frac{\partial}{\partial u^1}\left(\frac{\psi}{\sqrt{u^1 u^2}}\right), \quad \frac{\partial h_2}{\partial u^1} = h_1 \frac{\partial}{\partial u^2}\left(\frac{\psi}{\sqrt{u^1 u^2}}\right), \\ 2(u^1 - u^2)\frac{\partial^2 \psi}{\partial u^1 \partial u^2} + \frac{\partial \psi}{\partial u^1} - \frac{\partial \psi}{\partial u^2} = 0.\end{gathered} \tag{14}$$

b) $\lambda_1 \neq \mathrm{const}$, $\lambda_2 = c$; *in certain coordinates the metrics reduce to the form*

$$g^{ij1} = h_i^{-2}\delta_{ij}, \quad g^{ij2} = \lambda_i^{-1} h_i^{-2}\delta_{ij}, \qquad \lambda_1 = u^1,\ \lambda_2 = c, \tag{15}$$

where

$$\frac{\partial h_1}{\partial u^2} = -\left[\frac{c(au^2 + b)}{2\sqrt{u^1(c-u^1)^3}} + \psi(u^1)\right] h_2, \quad \frac{\partial h_2}{\partial u^1} = a\sqrt{\frac{u^1}{c-u^1}}\, h_1, \tag{16}$$

a, b are any constants, and $\psi(u^1)$ is an arbitrary function.

c) $\lambda_1 \neq \lambda_2$ *are constant numbers*; *in certain coordinates the metrics reduce to the form*

$$\begin{gathered} g^{ij1} = h_i^{-2}\delta_{ij}, \quad g^{ij2} = \lambda_i^{-1} h_i^{-2}\delta_{ij}, \quad h_1 = f(u^1, u^2), \\ h_2 = \partial f(u^1, u^2)/\partial u^2, \quad \partial^2 f/\partial u^1\, \partial u^2 = f. \end{gathered} \tag{17}$$

d) *Coincident eigenvalues $\lambda_1 \equiv \lambda_2$, and the metrics are indefinite*; *in certain coordinates they reduce to the form*

$$(g^{ij1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (g^{ij2}) = \begin{pmatrix} 0 & \mu(u^1) \\ \mu(u^1) & u^2\varphi(u^1) + \psi(u^1) \end{pmatrix}, \tag{18}$$

where μ, φ and ψ are arbitrary functions.

It is not difficult to determinee the degree of arbitrariness in the choice of "canonical" coordinates that reduce admissible pairs of metrics to the form (13), (15), (17) or (18). Metrics in general position (type a)) depend on four "essential" arbitrary functions of one variable (plus an arbitrary change of coordinates), those of type b) and d) on three functions of one variable, and those of type c) on two. The brackets (7) reduce to the form (18).

II. **Poisson brackets for nonhomogeneous one-dimensional systems of hydrodynamic type.** Along with homogeneous systems of hydrodynamic type it is natural to consider the class of nonhomogeneous systems

$$u_t^i = v_j^i(u)u_x^i + f^i(u), \quad i = 1, \dots, N \tag{19}$$

(here we restrict ourselves to the one-dimensional case). Correspondingly we introduce the class of nonhomogeneous Poisson brackets of the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + [b_k^{ij}(u)u_x^k + h^{ij}(u)]\delta(x-y), \tag{20}$$

with respect to which the systems (19) can be written in Hamiltonian form with Hamiltonians of the form (3).

Theorem 3. 1) *For nonhomogeneous brackets of the form* (20) *the tensor $h^{ij}(u)$ determines the usual* (*finite-dimensional*) *Poisson bracket on functions of u.*

2) *If the metric $g^{ij}(u)$ is not degenerate, then by a transformation of coordinates $u = u(w)$ the bracket* (20) *reduces to the form*

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij}\delta'(x-y) + [c_k^{ij} w^k + d^{ij}]\delta(x-y), \tag{21}$$

where the coefficients $\tilde{g}^{ij}, c_k^{ij}$ *and* d^{ij} *are constants, the* c_k^{ij} *are the structural constants of a semisimple Lie algebra with Killing metric* $\tilde{g}^{ij}$, *and* $d^{ij} = -d^{ji}$ *is an arbitrary cocycle on this Lie algebra.*

Example. The equations of the problem of n-waves [3]

$$M_t - \varphi(M_x) = [M, \varphi(M)], \tag{22}$$

where $M = (M_{ij})$ is an $n \times n$ matrix with zero trace (possibly with additional symmetries), and $\varphi(M) = (\lambda_{ij} M_{ij})$, are Hamiltonian with respect to the brackets (21), $d^{ij} = 0$, with quadratic Hamiltonian $2H = \int \operatorname{Tr}(M^* \cdot \varphi(M))\, dx$. As we know (see [3]), these equations are integrable by the method of the inverse problems for $\lambda_{ij} = (a_i - a_j)/(b_i - b_j)$.

III. **Differential-geometric brackets of higher orders.**

Definitions. 1. A *general homogeneous differential-geometric Poisson bracket of order* m *on a space of fields* $u^j(x)$ is a bracket of the form

$$\begin{aligned}\{u^i(x), u^j(x)\} = g^{ij}(u(x))\delta^{(m)}(x-y) + b_k^{ij}(u(x))u_x^k \delta^{(m-1)}(x-y) \\ + [c_k^{ij}(u(x))u_{xx}^k + d_{kl}^{ij}(u(x))u_x^k u_x^l]\delta^{(m-2)}(x-y) + \cdots,\end{aligned} \tag{23}$$

so that any Hamiltonian of hydrodynamic type (3) generates equations of the form

$$u_t^i = v_j^i(u)\partial^m u^j / \partial x^m + \cdots, \tag{24}$$

where the right-hand side is a graded-homogeneous polynomial of degree m in the derivatives, where the derivative $\partial^k u/\partial x^k$ has index k.

2. A *nonhomogeneous bracket* is the sum of homogeneous Poisson brackets of different orders.

According to a conjecture of the authors, the properties of these brackets are as follows: a) if the "metric" g^{ij} is nondegenerate, then the homogeneous bracket of a local variable $u(w)$ reduces to the form $g_0^{ij}\delta^{(m)}(x-y)$, $g_0^{ij} = \text{const}$; and b) for nonhomogeneous brackets with nondegenerate highest term of order m all the homogeneous components of order $m-k$ depend on no more than the kth powers of the original fields u^j and their derivatives (in those coordinates u^j where the highest term of order m reduces to a canonical constant of the form $g_0^{ij}\delta^{(m)}$). In particular, the term corresponding to $k = 1$ generates a Lie algebra (cf. Theorem 3 above).

Remark. Since the group of translations $u_t^i = u_x^i$ can be a Hamiltonian system only if there is no homogeneous term of the first order in the bracket, it is of particular interest to classify nonhomogeneous brackets with this property.

Just as in the case of one special variable $n = 1$ (see above), this class of brackets admits a natural generalization to the case $n > 1$.

References

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