DIFFERENTIAL-GEOMETRIC POISSON BRACKETS ON A LATTICE

B. A. Dubrovin

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The concept of differential-geometric Poisson brackets (DGPB) was introduced in [1] in connection with an investigation of the properties of Poisson brackets of hydrodynamic type [2] and their generalizations. Recall that homogeneous DGPBs of m-th order on a phase space of fields $u^i(x)$, $i = 1, \ldots, N$, $x \in \mathbf{R}$ (in this note we confine attention to the spatially one-dimensional case), taking values in a manifold $\mathcal{M}^N$, are defined by

$$\{u^i(x), u^j(y)\} = \sum_{k=0}^{m} G_k^{ij}(u(x), u'(x), \ldots, u^{(k)}(x))\, \delta^{(m-k)}(x-y), \tag{1}$$

where the coefficients G_k^{ij} are graded-homogeneous polynomials of u, u', ..., $u^{(k)}$ of degree k, where by definition $\deg u^{(\ell)} = \ell$, $\ell = 0, 1, \ldots$. (The standard properties of brackets — bilinearity, skew symmetry, Leibniz and Jacobi identities — are implied.) Under local transformations of the field variables

$$u^i(x) \to v^i(u(x)),\ i = 1, \ldots, N, \tag{2}$$

determined by changes of local coordinates $u^i \to v^i(u)$ on $\mathcal{M}^N$, the class of brackets of type (1) remains invariant, with the coefficients G_k^{ij} understood as "differential-geometric objects of order k," so that, for example, $G_0^{ij} = G_0^{ij}(u)$ is a metric on $\mathcal{M}^N$, $G_1^{ij} = \Gamma_s^{ij}(u)u_x^s$ defines a connection on the same manifold, and so on. The conditions for the coefficients G_k^{ij} that imply skew symmetry and the Jacobi identity may also be phrased in differential-geometric language. A general DGPB is a sum of homogeneous brackets of different orders. For a survey of results in the theory of DGPSs and its applications see [3].

In this note we describe a discrete variant of DGPBs, where the continuous variables x, y are replaced by discrete variables $m, n \in \mathbf{Z}$. The phase space is the set of sequences $u_n = \{u_n^i\} \in \mathcal{M}^N$, $n \in \mathbf{Z}$. A DGPB on a (homogeneous) integer lattice is defined by

$$\{u_m^i, u_n^j\} = g_{m-n}^{ij}(u_m, u_n), \tag{3}$$

where $g_k^{ij} \equiv 0$ for $|k| > M$. The brackets (3) are invariant under local transformations

$$u_n^i \to v_n^i = v^i(u_n),\quad n \in \mathbf{Z}, \tag{4}$$

analogous to (2), under which the coefficients transform as

$$g_k^{ij}(u', u'') \to \frac{\partial v^i(u')}{\partial u^p}\frac{\partial v^j(u'')}{\partial u^q} g_k^{pq}(u', u''). \tag{5}$$

In the "continuous limit" exemplified by $u_n^i = u^i(n\varepsilon)$, where $\varepsilon \to 0$ is the lattice spacing, the bracket (3) defines a DGPB on the space of (nonlocal) fields, which depends on ε:

$$\{u^i(x), u^j(y)\}_\varepsilon \equiv \sum_{q=0}^{\infty} \varepsilon^q \{u^i(x), u^j(y)\}_q = \sum_{k=-M}^{M} g_k^{ij}(u(x), u(y))\, \delta(x-y-k\varepsilon). \tag{6}$$

The lowest-order term $\{u^i(x), u^j(y)\}_m$ in the series (6), where $\{u^i(x), u^j(y)\}_k \equiv 0$ for $k < m$, defines a homogeneous DGPB of order m.

We confine attention to the case M = 1 (the case M = 0 is trivial; the general case M > 0 is reduced to M = 1 by consolidation of the lattice). In this case the DGPB is defined by a pair of matrices

$$g_1^{ij}(u, v) \equiv g^{ij}(u, v),\quad g_0^{ij}(u, u) \equiv h^{ij}(u). \tag{7}$$

We shall assume that the following nondegeneracy condition holds:

$$\det g^{ij}(u, u) \neq 0. \tag{8}$$

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It turns out that the manifold $\mathcal{M}^N$ is then (locally) endowed with a Hamilton–Lie group structure (see [4]), $\mathcal{M}^N = G$. Recall [4] that a Hamilton–Lie group with Lie algebra L = L(G) is locally defined by a Lie algebra structure on the dual space L*, provided that the structure is compatible with L, i.e., the structure constant tensor $f^{\alpha\beta}_{\gamma} \in \mathrm{Hom}\,(L, L \otimes L)$ is a 1-cocycle on L. We shall say that a Hamilton–Lie group C is admissible if:

1) there are two Lie algebra structures on L*, say L_1^* and L_2^*, both compatible with L;

2) there are mtually dual Lie algebra homomorphisms r_i: $L_1^* \to L$, i = 1, 2, $r_2 = r_1^+$, i.e., if $r_i = (r_i^{\alpha\beta})$, i = 1, 2, then $r_2^{\alpha\beta} = r_1^{\alpha\beta}$;

3) the structure constants $f^{\alpha\beta}_{1\gamma}$ and $f^{\alpha\beta}_{2\gamma}$ of the Lie algebras L_1^* and L_2^* define cohomologous cocycles

$$f^{\alpha\beta}_{1\gamma} - f^{\alpha\beta}_{2\gamma} = c^{\alpha}_{\varepsilon\gamma}h^{\varepsilon\beta} + h^{\alpha\varepsilon}c^{\beta}_{\varepsilon\gamma}, \tag{9}$$

where $h^{\alpha\beta} = -h^{\beta\alpha}$ is a matrix and $c^{\alpha}_{\beta\gamma}$ the structure constants of L;

4) the matrix $h^{\alpha\beta}$ satisfies the equation

$$c^{\alpha}_{\mu\nu}h^{\mu\beta}h^{\nu\gamma} + h^{\alpha\mu}c^{\beta}_{\mu\nu}h^{\nu\gamma} + h^{\alpha\mu}h^{\beta\nu}c^{\gamma}_{\mu\nu} = f^{\alpha\beta}_{1\mu}h^{\mu\gamma} + f^{\gamma\alpha}_{1\mu}h^{\mu\beta} + f^{\beta\gamma}_{1\mu}h^{\mu\alpha}. \tag{10}$$

A Hamilton–Lie group will be called strongly admissible if $r \equiv r_1 = r_2^+$ is an isomorphism. Such an object is locally defined by a Lie algebra L and a nonsingular matrix $r = (r^{\alpha\beta})$ satisfying the above conditions.

THEOREM. Any DGPB on a lattice of type (3) with M = 1, satisfying (8), is locally defined by a strongly admissible Hamilton-Lie group G, r through the formulas:

$$\begin{gathered}\{\varphi\,(u_n), \psi\,(u_{n+1})\} = -r^{\alpha\beta}\partial_\alpha\varphi\,(u_n)\partial'_\beta\psi\,(u_{n+1}),\\ \{\varphi\,(u_n), \psi\,(u_n)\} = \{\varphi\,(u_n), \psi\,(u_n)\}_0 + h^{\alpha\beta}\partial'_\alpha\varphi\,(u_n)\partial'_\beta\psi\,(u_n),\\ \{\varphi\,(u_n), \psi\,(u_m)\} = 0 \text{ for } |m-n| > 1.\end{gathered} \tag{11}$$

Here φ, ψ are any smooth functions on G, ∂_α and ∂'_α are left- and right-invariant vector fields on G, respectively, which coincide at the identity, $\{,\}_0$ is a group Poisson bracket on G, defined by the Lie algebra L_1^* as in [4]. For any admissible Hamilton-Lie group, formula (11) also defines a DGPB on the lattice, but the latter does not necessarily satisfy the nondegeneracy condition.

Note that a DGPB on a lattice with the nondegeneracy condition uniquely determines the group G (locally), but corresponds to an entire family of matrices $r^{\alpha\beta}(u) = g^{\alpha\beta}(u, u)$.

Example 0. If G is an abelian group, the bracket (11) is constant.

Example 1. For a simple 2-dimensional nonabelian group G, r may be any nonsingular matrix. We obtained the following family of brackets:

$$\begin{gathered}g^{ij}(u, v) = \begin{pmatrix}\sigma y^2 & 0\\ 0 & \frac{1}{2}y\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ x' & \frac{1}{2}y'\end{pmatrix}, \quad u = (x, y), \quad v = (x', y'),\\ h^{ij}(u) = g^{ij}(u, u) - g^{ji}(u, u), \quad \sigma = \pm 1.\end{gathered} \tag{12}$$

If $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\sigma & 0\\ 0 & 2\end{pmatrix}$, formula (12) yields the second Hamilton structure for a Toda system [5]. (This example is due to V. P. Cherkashin.)

Example 2. Let $r^{\alpha\beta}$ be a skew-symmetric matrix satisfying the classical Yang-Baxter equation on the Lie algebra L = L(G) (see, e.g., [4]). It defines a DGPB on a lattice,

$$\begin{gathered}\{\varphi\,(u_n), \psi\,(u_{n+1})\} = -r^{\alpha\beta}\partial_\alpha\varphi\,(u_n)\,\partial'_\beta\psi\,(u_{n+1}),\\ \{\varphi\,(u_n), \psi\,(u_n)\} = r^{\alpha\beta}\,(\partial_\alpha\varphi\,(u_n)\,\partial_\beta\psi\,(u_n) + \partial'_\alpha\varphi\,(u_n)\,\partial'_\beta\psi\,(u_n)).\end{gathered} \tag{13}$$

This bracket satisfies the nondegeneracy condition (8) if det $r^{\alpha\beta} \neq 0$ (such Lie algebras are known as quasi-Frobenius algebras [6]). The bracket (13) (more precisely, its transformed version (6)) is in fact the same as that occurring in [7] in the context of solution of a problem in quantization of a current algebra.

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SUM REGIONS OF WEAKLY CONVERGENT SERIES

V. M. Kadets

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Let X be a Banach space. The set $SR\left(\sum_1^\infty x_k\right)$ of all $x \in X$ to which for some permutation π the rearrangement $\sum_{k=1}^\infty x_{\pi(k)}$ converges is called the sum region of the series $\sum_1^\infty x_k$. Similarly, the set of all those elements to which the series $\sum_{k=1}^\infty x_{\pi(k)}$ can converge weakly is called the weak sum region, $WSR\left(\sum_1^\infty x_k\right)$. It is well known (Steinitz's theorem [1]) that, for a series in a finite-dimensional space, the sum region, and hence also the weak sum region, is a linear set, i.e., with any two distinct points it also contains the segment joining them. It has been shown [2] that in every infinite-dimensional Banach space there are series with nonlinear sum regions. At the same time, no one has succeeded in constructing an example of a series in a Hilbert space, let us say, for which the weak sum region is nonlinear. The difficulty revolves around the fact that the weak topology is much closer in character to the topology of a finite-dimensional space than the strong topology is. In this note we construct an example of a series with a nonlinear weak sum region, and, using the techniques of [2] and [3], we establish the existence of similar series in every infinite-dimensional Banach space.

<u>THEOREM</u>. In a Hilbert space there exist a series $\sum_{k=1}^\infty z_k$ and two points a and b in $SR(\Sigma z_k)$ such that

$$\frac{a+b}{2} \notin COC(\Sigma z_k).$$

<u>Proof</u>. We index an orthonormal basis in the Hilbert space ℓ_2 in the following way:

$$e_0, e_{1,1}, e_{2,1}, e_{2,2}, e_{3,1}, \ldots, e_{3,4}, e_{4,1}, \ldots, e_{4,8}, e_{5,1}, \ldots$$

We choose the constant $1/2 < \varepsilon < \sqrt{2}/2$. We construct the vectors $x_{i,k}$, $k \le 2^{i-1}$, $i \in N$, with the following properties:

a) $$x_{1,1} = e_0,$$

b) $$x_{k+1,2n-1} = \frac{1}{2} x_{k,n} + a_k e_{k,n},$$
$$x_{k+1,2n} = \frac{1}{2} x_{k,n} - a_k e_{k,n},$$

where a_k is such that

c) $$\|x_{k+1,2n-1}\| = \|x_{k+1,2n}\| = \frac{1}{2\varepsilon}\|x_{k,n}\|.$$

Then for any indices k, j, n, $j \neq n$, we have $\|x_{k,j}\| = \left(\frac{1}{2\varepsilon}\right)^{k-1}$; $x_{k\,n} = x_{k+1,2n-1} + x_{k+1,2n}$; $\langle x_{k,j}, x_{k,n}\rangle < 0$; and the vectors $\{x_{k,j}\}_{j=1}^{2^{k-1}}$ are linearly independent. Because $1/(2\varepsilon) < 1$, we have that $\lim_{k\to\infty} \|x_{k,j}\| = 0$.

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