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Algebraic-Geometrical Darboux Coordinates In R-Matrix Formalism.

P. DIENER[1] and B. A. DUBROVIN[2]

SISSA International School for Advanced Studies,
Via Beirut 4, 34014 Trieste, Italy.

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Abstract

We propose a simple algorithm to compute the poles of properly normalized eigenvector of an $n \times n$ matrix $\mathrm{T}(\lambda)$ on the spectral curve of this matrix. The main advantage of the algorithm is in the separation of the equation for the λ-coordinates of the poles. The algorithm does not depend neither on size n nor on a particular structure of $\mathrm{T}(\lambda)$. We apply it to separation of variables in algebraic-geometrical integrable systems. The construction of the algebraic-geometrical Darboux coordinates is obtained using the r-matrix formalism.

(1) email: diener@tsmi19.sissa.it
(2) email: dubrovin@tsmi19.sissa.it

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1. Introduction

The concept of algebraic-geometrical integrable systems was discovered on the basis of the theory of periodic and almost periodic solutions of the KdV equation (although some of their features were known for classical integrable problems of analytic mechanics). The main property of these integrable systems is that, their complexified Liouville tori can be compactified to give Abelian varieties [6]. On this basis some classification results on algebraic-geometrically integrable Hamiltonian systems were developed in the cycle of papers [1, 2].

The theory of action-angle variables for algebraic-geometrical integrable systems was started in the pioneer works of Flaschka and McLaughlin and of S. Alber [11, 3]. The calculation of action- angle variables was based on the discovery of remarkable Darboux coordinates in the theory of KdV equation [11] and of certain finite dimensional Hamiltonian systems related to KdV [3]. These are constructed as the coordinates of the poles of Baker-Akhiezer function meromorphic on the spectral curve. The observations of these papers were generalized by Novikov and Veselov [16, 17] for essentially all known algebraically integrable systems (some of these ideas were developed later also in [14], see also references therein).

Quantum theory applications of these algebraic-geometrical Darboux coordinates were initiated in the paper [19] of Sklyanin. Using these coordinates for the periodic Toda lattice he elaborated the general scheme of functional Bethe ansatz (see also [19, 20, 18]).
The main technical difficulty in application of the functional Bethe ansatz to higher rank systems (where the number of sheets of the spectral curve is greater than 2) is to separate the equation for the algebraic-geometrical canonical coordinates from the equation for the canonical momenta. This difficulty was overcome by a sophisticated algebraic technique in particular cases [20, 18].
We propose a simple solution of the problem essentially independent on the concrete structure of the algebraic-geometrically integrable system.. Let the integrable system be represented by an evolution of a $n \times n$ matrix $\mathrm{T}(\lambda)$ depending on the spectral parameter λ. Let $P_a = (\lambda_a, \mu_a)$ be the poles of

properly normalized eigenvector of $\mathbf{T}(\lambda)$ on the spectral curve

$$\det(\mathbf{T}(\lambda) - \mu I) = 0.$$

We find an explicit equation

$$D(\lambda) = 0$$

for the λ-projections $\lambda = \lambda_a$ of the poles, where $D(\lambda)$ is an explicit simple polynomial of the entries of $\mathbf{T}(\lambda)$ (see (9) below).
Algebraic-geometrical Darboux coordinate, according to the general scheme of [16, 17] are obtained from the coordinates of the poles as

$$q_a := g(\lambda_a) \quad p_a := f(\mu_a, \lambda_a) \tag{1}$$

where f and g are some functions (the function f can be depend also on the spectral curve). The action angles variables then are given by the periods of the differential

$$f(\mu, \lambda) dg(\lambda)$$

and the correspondent angle variables are given via Abel-Jacobi map [9, 16, 17]

We address then the problem of relation between the concrete structure (1) of the Darboux coordinates and the R-matrix Hamiltonian structure of the integrable system. For the simplest R-matrix

$$r(\lambda) = \frac{P}{\lambda}$$

(P is the permutation operator) and for linear/quadratic Poisson brackets for the matrix $\mathbf{T}(\lambda)$ we show that the canonical Darboux coordinates are represented, respectively, as

$$q_a = \lambda_a \quad p_a = \mu_a$$

and

$$q_a = \lambda_a \quad p_a = \log \mu_a.$$

This generalizes (and gives a proof on the basis of R-matrix machinery) the results of [11, 3].

An interesting problem is to analyze further the correspondence

$$R - matrix \longrightarrow alg. - geom.\ Darboux\ coordinates$$

for other R-matrices.

2. Algorithm to find the poles.

Let $\mathbf{T}(\lambda)$ be an entire $n \times n$ matrix-valued function of the complex parameter λ. We assign to the matrix a collection of *spectral data* (Γ, D) where Γ is a n-sheet Riemann surface over the λ-plane and D is a divisor on Γ. To construct Γ we consider first the *spectral curve* Γ_0 of the matrix $\mathbf{T}(\lambda)$

$$R(\lambda, \mu) \equiv \det(\mathbf{T}(\lambda) - \mu I) = 0 \tag{2}$$

where

$$\det(\mathbf{T}(\lambda) - \mu I) \equiv (-\mu)^n - \alpha_1(\lambda)\mu^{n-1} - \ldots - \alpha_n(\lambda), \tag{3}$$

I is the unity matrix. Let us assume that for generic complex λ the equation (2) has precisely n roots $\mu_1(\lambda), \ldots, \mu_n(\lambda)$. These are the eigenvalues of the matrix $\mathbf{T}(\lambda)$. So they become the branches of the function $\mu = \mu(P)$, $P = (\lambda, \mu) \in \Gamma_0$ being single-valued on Γ_0.

The ramification points of Γ_0 over the λ-plane are those λ for which the matrix $\mathbf{T}(\lambda)$ has less than n linearly independent eigenvectors. They are among the zeroes of the discriminant:

$$\mathcal{D}(\lambda) = \begin{array}{c} greatest\ common\ divisor \\ of\ R(\lambda, \mu)\ and\ \frac{\partial R(\lambda,\mu)}{\partial \mu}\ as\ polynomials\ in\ \mu. \end{array}$$

This is also an entire function of λ. However, some zeroes of $\mathcal{D}(\lambda)$ correspond to singular points of the spectral curve. For example, a double singularity on Γ_0 occurs when for the given λ such that $\mathcal{D}(\lambda) = 0$ two of the eigenvalues coincide but the matrix $\mathbf{T}(\lambda)$ has still n linearly independent eigenvectors. These are double zeroes of the discriminant. In the double point $(\lambda, \mu) \in \Gamma_0$ we have thus

$$rank(\mathbf{T}(\lambda) - \mu I) = n - 2$$

In some part of our considerations we will need the following *genericity assumption* about the matrix $\mathbf{T}(\lambda)$: we assume that the discriminant $\mathcal{D}(\lambda)$

has at most double zeroes. Moreover, we assume that any of the double zeroes is the λ-projection of a double point of the spectral curve. In other words, the assumption is that all the branch points of $\lambda : \Gamma_0 \to \mathbb{C}$ are simple and their λ-projections cannot coincide, and that all the singularities of Γ_0 are double points with pairwise distinct λ-projections. By Γ we denote the desingularization of the spectral curve. This will be the first part of the spectral data.

Under the genericity assumption the eigenvector $\psi = (\psi_1, \ldots, \psi_n)^T$ of the matrix $\mathbf{T}(\lambda)$

$$\mathbf{T}(\lambda)\psi = \mu\psi, \quad \psi = \psi(P), \; P = (\lambda, \mu) \in \Gamma_0 \tag{4}$$

determines a holomorphic map of the desingularization Γ to the projective space [13]

$$\psi : \Gamma \to \mathbf{CP}^{n-1}, \quad P \mapsto (\psi_1(P) : \psi_2(P) : \ldots : \psi_n(P))^T. \tag{5}$$

The second part of the spectral data is the divisor $D = P_1 + P_2 + \ldots$ obtained in the intersection

$$\psi(\Gamma) \cap H$$

with the generic hyperplane $H \subset \mathbf{CP}^{n-1}$. If the hyperplane is specified by the equation

$$K_1\psi_1 + \ldots + K_n\psi_n = 0 \tag{6}$$

then the points P_i of the divisor are the solutions (with their multiplicities) of the equation

$$K_1\psi_1(P) + \ldots + K_n\psi_n(P) = 0, \quad P \in \Gamma. \tag{7}$$

If we normalize the eigenvector ψ of $\mathbf{T}(\lambda)$ by

$$K_1\psi_1 + K_2\psi_2 + \ldots + K_n\psi_n = 1 \tag{8}$$

then the components $\psi_i = \psi_i(\lambda, \mu)$ will be meromorphic functions on Γ having their poles just at the points of the divisor D.

If some infinite points $\infty_1, \ldots, \infty_k$, at $\lambda = \infty$, of the multiplicities $n_1, \ldots, n_k$ (with $n_1 + \ldots + n_k = n$) can be added to Γ to obtain a compact Riemann surface of a genus g in such a way that the map (5) can be

extended onto the compactification then the matrix $\mathbf{T}(\lambda)$ can be uniquely reconstructed from the spectral data (Γ, D) (the multiplicity n_i of the point ∞_i is, by definition, the order of the pole at ∞_i).

Our first result is a simple algorithm to find the points of the divisor D. The principal advantage of the algorithm is a sort of "separation of variables" λ and μ: the λ-projections $\lambda(P_i)$ of the points of the divisor D can be found, independently on the μ-projections as zeroes of an entire function$D(\lambda)$ polynomial on $\mathbf{T}(\lambda)$.

Let $\mathbb{C}^n$ be the linear space where the matrix $\mathbf{T}(\lambda)$ acts. By $\mathbb{C}^{n*}$ we denote the dual space. $\mathbf{T}(\lambda)$ acts on the right in the dual space. By $K = (K_1, \ldots, K_n) \in \mathbb{C}^{n*}$ we denote the covector specifying the hyperplane (6). The function $D(\lambda)$ is defined by the formula

$$D(\lambda) = K \wedge K\mathbf{T} \wedge K\mathbf{T}^2 \wedge \ldots \wedge K\mathbf{T}^{n-1} \in \wedge^n \mathbb{C}^n \tag{9}$$

In the coordinate form for any basis in $\mathbb{C}^n$ $D(\lambda)$ is given by the determinant

$$D(\lambda) := \det \begin{pmatrix} K_1 & K_2 & . & K_i & . & K_n \\ (K\mathbf{T})_1 & (K\mathbf{T})_2 & . & (K\mathbf{T})_i & . & (K\mathbf{T})_n \\ . & . & . & . & . & \\ . & . & . & . & . & \\ (K\mathbf{T}^{n-1})_1 & . & . & . & . & (K\mathbf{T}^{n-1})_n \end{pmatrix} \tag{10}$$

We consider the equation

$$D(\lambda) = 0 \tag{11}$$

Let us assume that all the zeroes $\lambda = \gamma_i$ of the function $D(\lambda)$ are simple.

We also define the analytic functions $q_j^i(\lambda)$ putting

$$\begin{aligned} (-1)^{n-1} Q^i(\lambda, \mu) := \quad & K_i \mu^{n-1} + q_1^i(\lambda)\mu^{n-2} + \ldots + \\ & + \ldots + q_{n-1}^i(\lambda) \qquad i = 1, \ldots, n \end{aligned} \tag{12}$$

where the functions $Q^i(\lambda, \mu)$ are defined as

$$Q^i(\lambda, \mu) := \sum_{j=1}^{n} K_j \Delta_{ij}(\lambda, \mu), \qquad i = 1, \ldots, n$$

and $\Delta_{ij}(\lambda,\mu)$ is the cofactor of the (i,j) entry in the determinant of $\mathbf{T}(\lambda)-\mu I$.

THEOREM 1.

1. *The poles $P = P_a$ of $\psi(P)$, normalized as in (8), on the Riemann surface Γ could be only in the points $(\lambda = \gamma_a, \mu = \mu_a)$ where γ_a is a root of the equation $D(\lambda) = 0$ and μ_a is a solution of the system*

$$Q^i(\gamma_a,\mu) = 0 \qquad i = 1,....,n \tag{13}$$

2. *For any double point $Q \in \Gamma$ the λ-projection $\lambda(Q)$ is a zero of the function $D(\lambda)$.*

3. *If $\mathbf{T}(\lambda)$ satisfies the above genericity assumptions then the poles are in the points $(\lambda = \gamma_a, \mu = \mu_a = \mu(\gamma_a))$ where γ_a is any of the roots of the (11) not coinciding with the λ-projection of a double point of the spectral curve, then the matrix*

$$\begin{pmatrix} K_1 & q_1^1 & \cdot & \cdot & q_{n-2}^1 & q_{n-1}^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_n & q_1^n & \cdot & \cdot & q_{n-2}^n & q_{n-1}^n \end{pmatrix}$$

has rank $n-1$. Assuming the minor

$$\begin{pmatrix} K_1 & q_1^1 & \cdot & \cdot & q_{n-3}^1 & q_{n-2}^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n-1} & q_1^{n-1} & \cdot & \cdot & q_{n-3}^{n-1} & q_{n-2}^{n-1} \end{pmatrix}$$

to be non zero, we obtain

$$\mu_a = \mu(\gamma_a) = -\frac{\begin{vmatrix} K_1 & q_1^1 & \cdot & \cdot & q_{n-3}^1 & q_{n-1}^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n-1} & q_1^{n-1} & \cdot & \cdot & q_{n-3}^{n-1} & q_{n-1}^{n-1} \end{vmatrix}}{\begin{vmatrix} K_1 & q_1^1 & \cdot & \cdot & q_{n-3}^1 & q_{n-2}^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n-1} & q_1^{n-1} & \cdot & \cdot & q_{n-3}^{n-1} & q_{n-2}^{n-1} \end{vmatrix}}, \tag{14}$$

and $q_i^j = q_i^j(\gamma_a)$.

Proof:

The eigenvectors of the matrix $\mathbf{T}(\lambda)$ with an eigenvalue μ can be written as

$$\psi = c_i \begin{pmatrix} \Delta_{i1} \\ \Delta_{i2} \\ \cdot \\ \cdot \\ \Delta_{in} \end{pmatrix}$$

for any i, where c_i is a normalization constant and Δ_{ij} 's are the cofactors of the matrix $(\mathbf{T}(\lambda) - \mu I)$. The normalization $\sum_i K_i \phi = 1$ gives for c_i the following expression

$$c_i = (\sum_j K_j \Delta_{ij})^{-1}$$

So, ψ can have poles only in the points (λ, μ) of Γ that satisfy

$$\sum_j K_j \Delta_{ij}(\lambda, \mu) = 0 \qquad \forall i$$

The Δ_{ij} 's are polynomial in μ at most of order $(n-1)$, and they are analytic in λ; so we have a system, i.e. the system (13), of n equations in λ and μ and we want to check if there are solutions belonging Γ.
We first obtain equation for λ as compatibility conditions of the linear system (13).

Let us denote by K the row-vector

$$K := (K_1, ..., K_n)$$

LEMMA 1.

1. *If the vectors*

$$K, \quad K\mathbf{T}(\lambda_0), \quad K\mathbf{T}^2(\lambda_0), \quad . \quad . \quad , K\mathbf{T}^{n-1}(\lambda_0) \tag{15}$$

for a given λ_0 are linearly independent then any point on Γ of the form (λ, μ) is not a pole of ψ.

2. *If the vectors (15) are linearly dependent then there exists a point $(\lambda_0, \mu) \in \Gamma$ and an eigenvector of $\mathbf{T}(\lambda_0)\psi = \mu\psi$ such that*

$$K_1\psi_1 + K_2\psi_2 + ... + K_n\psi_n = 0$$

Proof:

Let $(\lambda_0, \mu) \in \Gamma$ and the vectors (15) are linearly independent. We consider them as vectors in the dual space $\mathbb{C}^{n*}$. Let $e_1, ..., e_n$ be the basis of $\mathbb{C}^n$ dual to the basis (15). We have

$$\begin{gathered} \mathbf{T}e_n = e_{n-1}, \ \dots \ , \mathbf{T}e_2 = e_1 \\ \mathbf{T}e_1 = \alpha_1(\lambda_0)e_1 + ... + \alpha_n(\lambda_0)e_n \end{gathered} \tag{16}$$

We put

$$\psi := e_1 + \mu e_2 + ... + \mu^{n-1}e_n$$

for any root $\mu = \mu(\lambda_0)$ of the characteristic equation (2). This is an eigenvector of $\mathbf{T}(\lambda_0)$ with the eigenvalue μ due to (16).
From the duality

$$\langle K\mathbf{T}^{i-1}, e_j \rangle = \delta^i_j \quad i, j = (1, ..., n)$$

it follows that

$$K\psi = 1$$

So $(\lambda_0, \mu) \in \Gamma$ is not a pole of ψ.
The first part is proved.

Let us assume now that the vectors (15) are linearly dependent. Let $V \subset \mathbb{C}^{n*}$ (the dual space) be the span of the vectors (15). This is an invariant subspace for the right action of T in the dual space. The annihilator $V^0 \subset \mathbb{C}^n$ of V

$$V^0 = \{\psi : \mathcal{V}\psi = 0 \quad \forall \mathcal{V} \in V\}$$

is an invariant subspace for T. We can find an eigenvector in the subspace

$$\mathrm{T}\psi = \mu\psi, \quad \psi \in V^0$$

This will satisfy

$$K\psi \equiv K_1\psi_1 + ... + K_n\psi_n = 0$$

The lemma is proved.

From this Lemma we obtain that the λ-coordinate of the poles of ψ could be only in a root of the equation (11). Conversely, if λ_0 is a root of (11) than there exists such μ that $(\lambda_0, \mu) \in \Gamma$ and

$$\sum_j K_j \Delta_{ij}(\lambda_0, \mu) = 0 \quad \forall i = (1, ..., n) \tag{17}$$

We show first that, if λ_0 is not a ramification point then μ is specified uniquely in the form (14) by the system (17) and (λ_0, μ) is a pole of ψ.
In this case one can choose n analytic branches $\mu_1 = \mu_1(\lambda), ..., \mu_n = \mu_n(\lambda)$ of the algebraic function $\mu(\lambda)$ for λ close to λ_0, $\mu_i(\lambda_0) \neq \mu_j(\lambda_0)$ for $i \neq j$. There exists a basis of the correspondent eigenvectors of $\mathrm{T}(\lambda)$. Since μ_i's are simple then the eigenvectors are analytic functions of λ and the transformation matrix to the basis of the eigenvectors is an invertible ones and analytic in λ. So D is proportional to the

$$\begin{vmatrix} K_1' & . & . & K_n' \\ K_1'\mu_1 & . & . & K_n'\mu_n \\ . & . & . & . \\ K_1'\mu_1^{n-1} & . & . & K_n'\mu_n^{n-1} \end{vmatrix}$$

with an analytic in λ non vanishing coefficient of proportionality. Here $(K_1', ..., K_n')$ are the components of K in the correspondent basis.
So λ_0 is solution of

$$D(\lambda) = 0 \longleftrightarrow K_1'(\lambda)...K_n'(\lambda) \prod_{i<j} (\mu_i(\lambda) - \mu_j(\lambda)) = 0 \tag{18}$$

The right hand side of (18) has locally a simple zero $\lambda = \lambda_0$ since $D(\lambda)$ has only simple zeroes by assumption. So λ_0 is a zero of, say, $K_j'(\lambda)$, and there are no common solutions of

$$\begin{cases} K_i'(\lambda) = 0 \\ \\ K_j'(\lambda) = 0 \quad i \neq j \end{cases}$$

(otherwise this solution does not give simple zeroes of the (18)).
So, there is a $(n-1)$-minor of the matrix

$$\begin{pmatrix} K' \\ K'\mathbf{T} \\ K'\mathbf{T}^2 \\ \cdot \\ \cdot \\ K'\mathbf{T}^{n-1} \end{pmatrix}$$

that is not zero, more precisely, if λ_0 is solution of $K_i' = 0$ then a non zero minor is obtained by dropping out the i-th column and the last row

$$\prod_{j \neq i}^{n} K_j' \prod_{k=1, k \neq i}^{n} \prod_{s=1, s \neq i}^{k} (\mu_k - \mu_s) \neq 0.$$

The conclusion is that the dimension of the span of the vectors (15) is $n-1$. So the annihilator of this system (see the proof of the lemma 1) is one-dimensional. This is nothing but the eigenvector of $\mathbf{T}(\lambda_0)$ satisfying $K\psi = 0$. So the linear system (13) in the variables μ, $\mu^2, ..., \mu^{n-1}$ for $\lambda = \lambda_0$ has one and only one solution $\mu = \mu_0$. From Lemma 1 we have $(\lambda_0, \mu_0) \in \Gamma$. Applying the Cramer rule to this system system we obtain the (14).

Let us prove now that, the λ-projections of the double points of Γ satisfy (11). However, there are no poles of ψ over these λ.
Let be (λ, μ) a double point of Γ, and ψ', ψ'' two linearly independent eigenvectors

$$\mathbf{T}(\lambda)\psi' = \mu\psi'$$
$$\mathbf{T}(\lambda)\psi'' = \mu\psi''.$$

In the linear span of ψ' and ψ'' we can find a vector ψ such that

$$\sum_i K_i \psi_i = 0. \tag{19}$$

For this (λ_0, μ) the rank of $(\mathbf{T}(\lambda_0) - \mu I)$ is $n-2$. This implies that this point is a common zero of the system

$$\begin{cases} \Delta_{i1} = 0 \\ \Delta_{i2} = 0 \\ \quad \dots \qquad\qquad \forall i \\ \Delta_{in} = 0 \\ \sum_j K_j \Delta_{ij} = 0 \end{cases}$$

because the first n equations are minors of order $n-1$ of $(T - \mu I)$ and the last equation is the same of the (19).
So we can conclude that such λ_0 is not a pole for ψ.

3. Algebraic-Geometrical Darboux Coordinates

The next step of our investigation is to prove that the variables γ_a, $f(\mu_b)$ are canonically conjugated variables w.r.t. some important class of Poisson brackets on the space of λ-matrices $\mathbf{T}(\lambda)$. Here $f(\mu)$ is a function. The following two types of Poisson brackets will be under consideration.

Quadratic Poisson brackets. The Poisson bracket has the form

$$\Big\{\mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)\Big\}_{(q)} = [r(\lambda - \mu), \mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)]\,. \tag{20}$$

Here $r = r(\lambda)$ is a *classical r-matrix*, i.e. a solution of the linearized Yang – Baxter equation (see [10] regarding the definitions and notations).

The main source of the quadratic Poisson brackets comes from consideration of the Poisson brackets of the monodromy matrix of a linear differential operator with, say, periodic coefficients. We recall briefly this construction.

Let us consider the family of operators L depending on the parameter λ.

$$L := \partial_x + U(x, \lambda) \qquad Tr(U) = 0 \tag{21}$$

with periodic coefficients

$$U(x+T,\lambda) = U(x,\lambda).$$

We assume that the matrix-valued function $U(x,\lambda)$ is analytic in x near a point $x = x_0$. We assume also that $U(x,\lambda)$ is an entire function in λ with a possible singularity at infinity (e.g., a polynomial in λ).

The matrix $\mathrm{T} = \mathrm{T}(\lambda, x_0)$ is the monodromy matrix of the operator (it depends on x_0 as on parameter) . This is the matrix of the monodromy operator

$$\hat{T}\phi(x) = \phi(x+T)$$

acting in the space of solutions of the equation

$$L(x,\lambda)\phi(x,\lambda) = 0. \tag{22}$$

A basis in the space of solutions can be constructed from the columns of the fundamental matrix $Y(x_0,x,\lambda) := (Y_i^j(x_0,x,\lambda))$ i.e. the matrix solution of the equation $LY = 0$ with the initial conditions $Y(x_0,x_0,\lambda) = I$.
This is an entire function of the complex variable λ due to the formula

$$\mathrm{T}(\lambda, x_0) = Y(x_0+T, x_0, \lambda). \tag{23}$$

The eigenvectors ψ of the monodromy matrix are in one-to-one correspondence with the Bloch - Floquet eigenfunctions, i.e. with the solutions ϕ of (22) satisfying

$$\phi(x+T) = e^{pT}\phi(x). \tag{24}$$

The correspondence is given by

$$\begin{cases} \phi = & Y\psi \\ e^{pT} = & \mu \end{cases}$$

for $\phi = \phi(P)$, $P = (\lambda,\mu) \in \Gamma$. The multivalued function $p = p(P)$ is an Abelian integral on Γ, i.e. dp is an Abelian differential on Γ with poles only at the infinite points. All the periods of the differential are integer multiples of $2\pi/T$.

I recall [10] that if the coefficients of the operator $L(\lambda)$ satisfy *linear* r-matrix Poisson bracket

$$\Big\{U(\lambda,x)\otimes U(\mu,y)\Big\} = [r(\lambda-\mu), U(\lambda,x)\otimes I + I\otimes U(\mu,y)]\,\delta(x-y) \tag{25}$$

then the monodromy matrix $\mathbf{T}(\lambda, x_0)$ satisfies the quadratic Poisson bracket (20).

The second type will be r-matrix Poisson brackets

$$\left\{\mathbf{T}(\lambda)\otimes\mathbf{T}(\mu)\right\}_{(l)} = [r(\lambda-\mu), \mathbf{T}(\lambda)\otimes I + I\otimes\mathbf{T}(\mu)]\,. \tag{26}$$

These appear in the description of Hamiltonian structure of algebraic-geometrically integrable finite dimensional systems, i.e. of the equations of commutativity of an operator of the form (21) with the operator of multiplication by $\mathbf{T}(\lambda)$

$$[\partial_x + U(x,\lambda), \mathbf{T}(\lambda, x)] = 0. \tag{27}$$

The equation (27) can be rewritten as a finite dimensional Hamiltonian system on the space of matrices $\mathbf{T}(\lambda)$ w.r.t. the Poisson bracket (26). The variable x plays the role of the time.
We will consider the particular case of the r-matrix:

$$r(\lambda) := \frac{P}{\lambda} \tag{28}$$

where:

$$P : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$$

is the permutation operator.
In these cases the important property of both the Poisson brackets is the commutativity of the eigenvalues of $\mathbf{T}(\lambda)$ [10]

$$\{\mu_a(\lambda_1), \mu_b(\lambda_2)\} = 0 \tag{29}$$

for any fixed λ_1 , λ_2 and for any branches a , b.

THEOREM 2.

Let the r-matrix have the form (28), and let $P_i = (\gamma_i, \mu_i) \in \Gamma$ be the points of the divisor of the poles of ψ. Then γ_i, $p_i := \log \mu_i$ have the canonical Poisson brackets w.r.t. the P.B .(20), or γ_i, $p_i := \mu_i$ have the canonical Poisson brackets w.r.t. the P.B .(26).

$$\{\gamma_i, \gamma_j\} = 0 \quad \forall i, j$$

$$\{\gamma_i, p_j\} = \delta_i^j \quad \forall i, j$$

$$\{p_i, p_j\} = 0 \quad \forall i, j$$

Remark. To obtain the complete set of Darboux coordinates for the Poisson brackets (26) or (20) we are to add the Casimirs of the brackets. To do this we need to impose some restrictions on the behaviour of the spectral curve and of $\mu = \mu(\lambda)$ at infinity (see examples below).
In order to proof this theorem we need some lemmas.

LEMMA 2.

The P.B. between $\log D(\lambda_1)$ *and* $\mathbf{T}(\lambda_2)$ *given by*

$$\{\log D(\lambda_1), \mathbf{T}_c^d(\lambda_2)\} = \sum_{a,r,s} \sum_{b \neq a} \frac{\psi_r^d(\lambda_2)(\psi^{-1})_c^s(\lambda_2)}{(\mu_a(\lambda_1) - \mu_b(\lambda_1))} \times$$

$$\times (K')_a^{-1}(\lambda_1) K_b'(\lambda_1) \left(\{\mathbf{T}(\lambda_1) \underset{,}{\otimes} \mathbf{T}(\lambda_2)\}'\right)_{as}^{br} +$$

$$+ \sum_{a,r,s} \sum_{c \neq a} \frac{(\{\mathbf{T}(\lambda_1) \otimes_, \mathbf{T}(\lambda_2)\}')_{as}^{ar} \psi_r^d(\lambda_2)(\psi^{-1})_c^s(\lambda_2)}{(\mu_a(\lambda_1) - \mu_c(\lambda_1))}$$

where $\psi_j^i(\lambda)$ *is the i-th component of the eigenvector of* $\mathbf{T}$ *with eigenvalue* μ_j *and the* $'$ *means the components of the tensors in the eigenvector basis, and*

$$\{\mathbf{T}(\lambda_1) \underset{,}{\otimes} \mathbf{T}(\lambda_2)\}' := \Psi^{-1}(\lambda_1) \otimes \Psi^{-1}(\lambda_2) \{\mathbf{T}(\lambda_1) \underset{,}{\otimes} \mathbf{T}(\lambda_2)\} \Psi(\lambda_1) \otimes \Psi(\lambda_2)$$

$\Psi(\lambda) = (\psi_i^j(\lambda))$ *is a* $n \times n$ *matrix.*

Proof:

What we need is an expression for the variation of the $D(\lambda)$ as a function of $\mathbf{T}(\lambda)$.

Let us consider

$$D := e^1 \wedge e^2 \wedge .... \wedge e^n$$

and $e^1, ..., e^n$ are vectors of $\mathbb{C}^n$. Suppose that

$$\delta e^i = \sum_j A^i_j e^j$$

So

$$\delta \log D = \sum_i A^i_i$$

In the our case let

$$e^i := K\mathbf{T}^{i-1}$$

$$\delta e^i = \sum_{p+q=i-2} K\mathbf{T}^p \, \delta\mathbf{T} \, \mathbf{T}^q$$

Let $F_1, .., F_n$ be the dual basis to $e^1, ..., e^n$, then

$$A^i_j = \langle \delta e^i, F_j \rangle.$$

If we chose the eigenvector basis of $\mathbf{T}$, F_j's are

$$(F')^b_i := (K'_b)^{-1} f^b_i$$

and the f^s_i are the coefficients of the family of the Lagrangean polynomials in μ:

$$P^b(\mu) := \sum_{i=1}^n f^b_i \mu^{i-1} \quad P^b(\mu_a) = \delta^b_a$$

So

$$\begin{aligned} \textstyle\sum_i A^i_i = \quad & \textstyle\sum_{a,b} (K'_a)^{-1} K'_b (\delta T')^b_a \times \\ & \times \textstyle\sum_{i=1}^n \sum_{p+q=i-2} \mu^p_b \mu^q_a f^a_i \end{aligned}$$

LEMMA 3.

$$\sum_{i=1}^n \sum_{p+q=i-2} \mu^p_b \mu^q_a f^a_i = \begin{cases} \frac{1}{\mu_a - \mu_b} & a \neq b \\ \sum_{c \neq a} \frac{1}{\mu_a - \mu_c} & b = a \end{cases}.$$

Proof:

If $a \neq b$ then

$$\sum_{p+q=i-2} \mu_b^p \mu_a^q = \frac{\mu_a^{i-1} - \mu_b^{i-1}}{\mu_a - \mu_b}$$

so

$$\sum_{i=1}^{n} \sum_{p+q=i-2} \mu_b^p \mu_a^q f_i^a =$$

$$= \sum_{i=1}^{n} \frac{\mu_a^{i-1} - \mu_b^{i-1}}{\mu_a - \mu_b} f_i^a =$$

$$= \frac{1}{\mu_a - \mu_b}$$

the last equality follows by the definition of f_i^a.
If $a = b$ then

$$\sum_{i=1}^{n} \sum_{p+q=i-2} \mu_a^{i-2} f_i^a = \sum_i (i-1) \mu_a^{i-2} f_i^a.$$

Let us observe that this is just the derivative of the polynomial $P^a(\mu)$ w.r.t. μ evaluated in the point μ_a.
Using the Lagrange interpolation formula we obtain

$$P^a(\mu) = \frac{\prod_{b \neq a} (\mu - \mu_b)}{\prod_{c \neq a} \mu_a - \mu_c}$$

$$\left. \frac{dP^a(\mu)}{d\mu} \right|_{\mu=\mu_a} = \sum_{c \neq a} \frac{1}{\mu_a - \mu_c}$$

this complete the proof of the Lemma.

We go back to the Lemma 2, using the Lemma 3 we obtain an expression for

$\delta \log D(\lambda)$

$$\delta \log D(\lambda) = \sum_a \sum_{b \neq a} (K'_a)^{-1} K_b (\delta T')^b_a \frac{1}{\mu_a - \mu_b} + \\ + \sum_a \sum_{c \neq a} \frac{(\delta T')^a_a}{\mu_a - \mu_c} \tag{30}$$

Now, the substitution of this expression

$$\{\log D(\lambda_1), \mathbf{T}(\lambda_2)\} = \frac{\partial \log D(\lambda_1)}{\partial \mathbf{T}(\lambda_1)} \{\mathbf{T}(\lambda_1) \otimes \mathbf{T}(\lambda_2)\}$$

gives the Lemma 2.

Let us represent $D(\lambda)$ in the form

$$D(\lambda) = V(\gamma_1, .., \gamma_n, ..) \prod_i (\lambda - \gamma_i)$$

where the coefficient $V(\gamma_1, \ldots, \gamma_n \ldots)$ does not depend on λ.
Then

$$\{\log D(\lambda_1), \log D(\lambda_2)\} = \sum_{i,j} \frac{\{\gamma_i, \gamma_j\}}{(\lambda_1 - \gamma_i)(\lambda_2 - \gamma_j)} + \\ + \sum_i \frac{\{\gamma_i, V(\gamma)\}}{\lambda_1 - \gamma_i} - \sum_i \frac{\{\gamma_i, V(\gamma)\}}{\lambda_2 - \gamma_i}$$

In this way the P.B. between two different γ's is given by

$$\{\gamma_i, \gamma_j\} = \lim_{\lambda_1 \to \gamma_i} \lim_{\lambda_2 \to \gamma_j} [\{\log D(\lambda_1), \log D(\lambda_2)\}(\lambda_1 - \gamma_i)(\lambda_2 - \gamma_j)] \tag{31}$$

Using the Lemma 2 we can write explicitly the P.B. between $\log D$ for two

different values of λ

$$\{\log D(\lambda_1), \log D(\lambda_2)\} =$$

$$\sum_{a,s}\sum_{b\neq a}\sum_{r\neq s}\frac{(K_a')^{-1}(\lambda_1)K_b'(\lambda_1)(K_s')^{-1}(\lambda_2)K_r'(\lambda_2)\,(\{\mathbf{T}(\lambda_1)\otimes,\mathbf{T}(\lambda_2)\}')^{br}_{as}}{(\mu_a-\mu_b)(\lambda_1)(\mu_s-\mu_r)(\lambda_2)}+$$

$$+\sum_{a,s}\sum_{b\neq a}\sum_{r\neq s}\frac{(K_a')^{-1}(\lambda_1)K_b'(\lambda_1)\,(\{\mathbf{T}(\lambda_1)\otimes,\mathbf{T}(\lambda_2)\}')^{bs}_{as}}{(\mu_a-\mu_b)(\lambda_1)(\mu_s-\mu_r)(\lambda_2)}+$$

$$+\sum_{a,s}\sum_{b\neq a}\sum_{r\neq s}\frac{(K_s')^{-1}(\lambda_2)K_r'(\lambda_2)\,(\{\mathbf{T}(\lambda_1)\otimes,\mathbf{T}(\lambda_2)\}')^{ar}_{as}}{(\mu_a-\mu_b)(\lambda_1)(\mu_s-\mu_r)(\lambda_2)}$$

Let us suppose that there exist some u,v s.t. γ_i and γ_j are simple zeroes of $K_u'(\lambda)=0$ and $K_v'(\lambda)=0$ respectively (observe that u and v label the sheets of the spectral curve).
So, the only contribution in the limit (31) is given by

$$\{\gamma_i,\gamma_j\} = \lim_{\lambda_1\to\gamma_i}\lim_{\lambda_2\to\gamma_j}\frac{(\lambda_1-\gamma_i)(\lambda_2-\gamma_j)}{K_u'(\lambda_1)K_v'(\lambda_2)}\times$$

$$\times\sum_{b\neq u}\sum_{r\neq v}\frac{(K_b'(\gamma_i)K_r'(\gamma_j)(\{\mathbf{T}(\gamma_i)\otimes,\mathbf{T}(\gamma_j)\}')^{br}_{uv}}{(\mu_u-\mu_b)(\gamma_i)(\mu_v-\mu_r)(\gamma_j)}.$$

By now all the equations hold for arbitrary r-matrix.
Now we use the form (28) of the r matrix and alternatively the linear P.B or the quadratic P.B. for the T matrix. We obtain

$$(\{\mathbf{T}(\gamma_i)\otimes,\mathbf{T}(\gamma_j\}')^{br}_{uv} = -(\mu_b(\gamma_i)\mu_r(\gamma_j)-\mu_u(\gamma_i)\mu_v(\gamma_j))\times$$

$$\times\sum_{nd}\frac{(\psi^{-1})^b_n(\gamma_i)\psi^n_v(\gamma_j)(\psi^{-1})^r_d(\gamma_j)\psi^d_u(\gamma_i)}{\gamma_i-\gamma_j} \tag{32}$$

for the quadratic case and

$$(\{\mathbf{T}(\gamma_i)\otimes,\mathbf{T}(\gamma_j\}')^{br}_{uv} = (\mu_v(\gamma_j) - \mu_r(\gamma_j) + \mu_u(\gamma_i) - \mu_b(\gamma_i))\times$$

$$\sum_{nd} \times \frac{(\psi^{-1})^b_n(\gamma_i)\psi^n_v(\gamma_j)(\psi^{-1})^r_d(\gamma_j)\psi^d_u(\gamma_i)}{\gamma_i-\gamma_j} \tag{33}$$

for the linear case.
The substitution of these formulas in the previous one gives

$$\{\gamma_i,\gamma_j\} = F(\gamma_i,\gamma_j)K'_v(\gamma_j) + K'_u(\gamma_i)G(\gamma_i,\gamma_j)$$

where

$$F(\gamma_i,\gamma_j) := \lim_{\lambda_1\to\gamma_i}\lim_{\lambda_2\to\gamma_j}\frac{(\lambda_1-\gamma_i)(\lambda_2-\gamma_j)}{K'_u(\lambda_1)K'_v(\lambda_2)}\times$$

$$\times\sum_{r,d}\frac{K'_r(\gamma_j)\mu_r(\gamma_j)(\psi^{-1})^r_d(\gamma_j)\psi^d_u(\gamma_i)}{(\mu_v-\mu_r)(\gamma_j)(\gamma_i-\gamma_j)}$$

$$G(\gamma_i,\gamma_j) := \lim_{\lambda_1\to\gamma_i}\lim_{\lambda_2\to\gamma_j}\frac{(\lambda_1-\gamma_i)(\lambda_2-\gamma_j)}{K'_u(\lambda_1)K'_v(\lambda_2)}\times$$

$$\times\sum_{b,d}\frac{K'_b(\gamma_i)\mu_u(\gamma_i)(\psi^{-1})^b_d(\gamma_i)\psi^d_v(\gamma_j)}{(\mu_u-\mu_b)(\gamma_i)(\gamma_i-\gamma_j)}$$

for quadratic case and

$$F(\gamma_i,\gamma_j) := \lim_{\lambda_1\to\gamma_i}\lim_{\lambda_2\to\gamma_j}\frac{(\lambda_1-\gamma_i)(\lambda_2-\gamma_j)}{K'_u(\lambda_1)K'_v(\lambda_2)}\times$$

$$\times\sum_{r,d}\frac{K'_r(\gamma_j)(\gamma_j)(\psi^{-1})^r_d(\gamma_j)\psi^d_u(\gamma_i)}{(\mu_v-\mu_r)(\gamma_j)(\gamma_i-\gamma_j)}$$

$$G(\gamma_i,\gamma_j) := \lim_{\lambda_1\to\gamma_i}\lim_{\lambda_2\to\gamma_j}\frac{(\lambda_1-\gamma_i)(\lambda_2-\gamma_j)}{K'_u(\lambda_1)K'_v(\lambda_2)}\times$$

$$\times \sum_{b,d} \frac{K_b'(\gamma_i)(\psi^{-1})_d^b(\gamma_i)\psi_v^d(\gamma_j)}{(\mu_u - \mu_b)(\gamma_i)(\gamma_i - \gamma_j)}$$

for the linear case. But by hypothesis $K_u'(\gamma_i) = 0$ and $K_v'(\gamma_j) = 0$ so

$$\{\gamma_i, \gamma_j\} = 0$$

LEMMA 4.

Let P_0 be a pole of ψ, and Q an arbitrary point of the spectral curve with a fixed (i.e., independent on $\mathbf{T}$) value of $\lambda(Q)$. Then for the matrix (28)

1. *if $\mathbf{T}$ satisfies the Poisson bracket (20) then*

$$\{\lambda(P_0), \log \mu(Q))\} = \operatorname*{res}_{P=P_0} \frac{\sum_{i,s} K_i g_s^i(Q) g_j^s(P) d\lambda(P)}{(\lambda(P) - \lambda(Q)) \sum_i K_i g_j^i(P)} \quad \forall j$$

2. *if $\mathbf{T}$ satisfies the Poisson bracket (26) then*

$$\{\lambda(P_0), \mu(Q))\} = \operatorname*{res}_{P=P_0} \frac{\sum_{i,s} K_i g_s^i(Q) g_j^s(P) d\lambda(P)}{(\lambda(P) - \lambda(Q)) \sum_i K_i g_j^i(P)} \quad \forall j$$

Here $g_i^j(P)$ is defined as

$$g_i^j(P) := \psi_a^i(\lambda_1)(\psi^{-1})_j^a(\lambda_1)$$

if P is represented in local coordinates as $P = (\lambda_1, \mu_a)$.

Proof:

Firstly, we compare the P.B.

$$\{\mathbf{T}_j^i(\lambda_1), \mu_c(\lambda_2)\}$$

for any λ_1 , λ_2; at this end let us consider the $\log\det(\mathbf{T}(\lambda) - \mu I)$ for any complex μ being not an eigenvalue of $\mathbf{T}$, and let us consider the P.B.

$$\{\mathbf{T}_i^j(\lambda_1), \log\det(\mathbf{T}(\lambda_2) - \mu I)\} = Tr[\{T_i^j(\lambda_1), \mathbf{T}(\lambda_2) - \mu I\}(\mathbf{T}(\lambda_2) - \mu I)^{-1}].$$

Using the explicit form of the r-matrix we obtain for the quadratic case

$$\{\mathbf{T}^i_j(\lambda_1), \log\det(\mathbf{T}(\lambda_2) - \mu I)\} =$$

$$= \frac{[\mathbf{T}(\lambda_1)(\mathbf{T}(\lambda_2)-\mu I)^{-1}\mathbf{T}(\lambda_2)]^i_j - [\mathbf{T}(\lambda_2)(\mathbf{T}(\lambda_2)-\mu I)^{-1}\mathbf{T}(\lambda_1)]^i_j}{\lambda_1-\lambda_2}$$

and, for the linear case

$$\{\mathbf{T}^i_j(\lambda_1), \log\det(\mathbf{T}(\lambda_2) - \mu I)\} =$$

$$= \frac{[(\mathbf{T}(\lambda_2)-\mu I)^{-1}(\mathbf{T}(\lambda_2)-\mathbf{T}(\lambda_1))]^i_j + [(\mathbf{T}(\lambda_1)-\mathbf{T}(\lambda_2))(\mathbf{T}(\lambda_2)-\mu I)^{-1}]^i_j}{\lambda_1-\lambda_2}$$

Let us observe that $\det(\mathbf{T}(\lambda) - \mu I)$ is a polynomial in μ of the degree n, so it can be written as

$$\det(\mathbf{T}(\lambda) - \mu I) := W(\mu_1, \ldots, \mu_n) \prod_k (\mu - \mu_k(\lambda)).$$

It follows immediately that

$$\{\mathbf{T}^i_j(\lambda_1), \mu_c(\lambda_2)\} = -\lim_{\mu\to\mu_c(\lambda_2)} (\mu - \mu_c(\lambda_2))\{\mathbf{T}^j_i(\lambda_1), \log\det(\mathbf{T}(\lambda_2) - \mu I)\}. \tag{34}$$

Now

$$\{\log D(\lambda_1), \mu_c(\lambda_2)\} = \frac{\partial \log D(\lambda_1)}{\partial(\mathbf{T}')^b_a} (\psi^{-1})^b_i(\lambda_1)\psi^j_a(\lambda_1)\{\mathbf{T}^i_j(\lambda_1), \mu_c(\lambda_2)\}$$

the sum on the repeated indices is assumed here. So, for the quadratic case we obtain

$$\Sigma_{ij}(\psi^{-1})^b_i(\lambda_1)\psi^j_a(\lambda_1)\{T^i_j(\lambda_1), \mu_c(\lambda_2)\}_{(q)} = \lim_{\mu\to\mu_c}(\mu - \mu_c)\times$$

$$\times \Sigma_r \left[\mu_b(\lambda_1)\left(\psi^{-1}(\lambda_1)\psi(\lambda_2)\right)^b_r \frac{\mu_r(\lambda_2)}{\mu_r(\lambda_2)-\mu}\left(\psi^{-1}(\lambda_1)\psi(\lambda_2)\right)^r_a - \right.$$

$$\left. -\mu_a(\lambda_1)\left(\psi^{-1}(\lambda_2)(\psi(\lambda_1)\right)^r_a \frac{\mu_r(\lambda_2)}{\mu_r(\lambda_2)-\mu}\left(\psi^{-1}(\lambda_1)\psi(\lambda_2)\right)^b_r\right] \frac{1}{\lambda_1-\lambda_2} =$$

$$= \frac{\mu_c(\lambda_2)}{\lambda_2 - \lambda_1}(\mu_a - \mu_b)(\lambda_1)\left(\psi^{-1}(\lambda_2)(\psi(\lambda_1)\right)_a^c \left(\psi^{-1}(\lambda_1)(\psi(\lambda_2)\right)_c^b$$

and, for the linear case

$$\sum_{ij}(\psi^{-1})_i^b(\lambda_1)\psi_a^j(\lambda_1)\{\mathbf{T}_j^i(\lambda_1), \mu_c(\lambda_2)\}_{(l)} =$$

$$= \frac{1}{\lambda_2 - \lambda_1}(\mu_a - \mu_b)(\lambda_1)\left(\psi^{-1}(\lambda_2)(\psi(\lambda_1)\right)_a^c \left(\psi^{-1}(\lambda_1)(\psi(\lambda_2)\right)_c^b$$

The substitution of the explicit form of the derivative of $\log D(\lambda)$ gives for the quadratic case

$$\{\log D(\lambda_1), \mu_c(\lambda_2)\}_{(q)} = \frac{\mu_c(\lambda_2)}{\lambda_2 - \lambda_1}\sum_a \sum_{b\neq a}(K_a')^{-1}(\lambda_1)K_b'(\lambda_1) \times$$

$$\times \left(\psi^{-1}(\lambda_2)\psi(\lambda_1)\right)_a^c \left(\psi^{-1}(\lambda_1)\psi(\lambda_2)\right)_c^b$$

and for the linear case

$$\{\log D(\lambda_1), \mu_c(\lambda_2)\}_{(l)} = \frac{1}{\lambda_2 - \lambda_1}\sum_a \sum_{b\neq a}(K_a')^{-1}(\lambda_1)K_b'(\lambda_1) \times$$

$$\times \left(\psi^{-1}(\lambda_2)\psi(\lambda_1)\right)_a^c \left(\psi^{-1}(\lambda_1)\psi(\lambda_2)\right)_c^b$$

Let us observe that the previous two expressions differ by a factor μ_c, i.e.

$$\{\log D(\lambda_1), \mu_c(\lambda_2)\}_{(q)} = \mu_c(\lambda_2)\{\log D(\lambda_1), \mu_c(\lambda_2)\}_{(l)}$$

So

$$\{\log D(\lambda_1), \log(\mu_c(\lambda_2))\}_{(q)} = \{\log D(\lambda_1), \mu_c(\lambda_2)\}_{(l)}$$

and we continue the proof only for one of these.
If the pole of ψ is $P_0 = (\gamma_i, \mu_u)$ then

$$\{\gamma_i, \log \mu_c(\lambda_2)\}_{(q)} = \lim_{\lambda_1 \to \gamma_i} \frac{\lambda_1 - \gamma_i}{K_u'(\lambda_1)} \frac{K_c'(\lambda_2)}{\lambda_2 - \lambda_1}\left(\psi^{-1}(\lambda_2)\psi(\lambda_1)\right)_u^c.$$

If we multiply by $(\psi^{-1})^u_j(\lambda_1)$ the numerator and the denominator of this expression for any j and remember the definition of K' and g^j_i we obtain for all j

$$\{\gamma_i, \log \mu_c(\lambda_2)\}_{(q)} = \lim_{\lambda_1 \to \gamma_i} \frac{\lambda_1 - \gamma_i}{\sum_r K_r g^r_j(\lambda_1, u)} \frac{\sum_{r,p} K_r g^r_p(\lambda_2, c) g^p_j(\lambda_1, u)}{\lambda_2 - \lambda_1}.$$

This completes the proof of the lemma.

Let us consider a point Q_0 on the spectral curve s.t. that is a pole of ψ, (possibly, coinciding with the pole P_0) and compute the P.B.

$$\{\lambda(P_0), f(\mu(Q_0))\} = \{\lambda(P_0), f(\mu(Q))\}|_{\lambda(Q)=\lambda(Q_0)} + \\ + \frac{\partial f(\mu(Q))}{\partial \lambda} \{\lambda(P_0), \lambda(Q_0)\}$$

where f is either the identity or the logarithm.
The second term of this expression is zero by the equation $\{\gamma_i, \gamma_j\} = 0$ proved before. So

$$\{\lambda(P_0), f(\mu(Q_0))\} = \operatorname*{res}_{P=P_0} \frac{\sum_{i,s} K_i g^i_s(Q_0) g^s_j(P) d\lambda(P)}{(\lambda(P) - \lambda(Q_0)) \sum_i K_i g^i_j(P)} \quad \forall j$$

so

$$\{\lambda(P_0), f(\mu(Q_0))\} = \begin{cases} 0 & P_0 \neq Q_0 \\ 1 & P_0 = Q_0 \end{cases}$$

this is because in the first case $\sum_i K_i g^i_s(Q_0) = 0$, but in the second case the P.B. became

$$\{\lambda(P_0), f(\mu(P_0))\} = \operatorname*{res}_{P=P_0} \frac{d\lambda(P)}{\lambda(P) - \lambda(P_0)} = 1$$

The P.B. between two different p_i's is given, using (29) and the commutativity of $\lambda(P_0)$ and $\lambda(Q_0)$, and of $\lambda(P_0)$ and $f(\mu(Q_0))$ for distinct poles

P_0, Q_0

$$\{p_i, p_j\} = \{f(\mu(P)), f(\mu(Q))\}|_{P=P_0,Q=Q_0} +$$

$$+ \left(\frac{\partial f(\mu)}{\partial \mu}\frac{d\mu}{d\lambda}\right)(P_0)\left(\frac{\partial f(\mu)}{\partial \mu}\frac{d\mu}{d\lambda}\right)(Q_0)\{\lambda(P_0), \lambda(Q_0)\}+$$

$$+ \left(\frac{\partial f(\mu)}{\partial \mu}\frac{d\mu}{d\lambda}\right)(P_0)\left(\frac{\partial f(\mu)}{\partial \mu}\right)(Q_0)\{\lambda(P_0), \mu(Q_0)\}+$$

$$+ \left(\frac{\partial f(\mu)}{\partial \mu}\right)(P_0)\left(\frac{\partial f(\mu)}{\partial \mu}\frac{d\mu}{d\lambda}\right)(Q_0)\{\mu(P_0), \lambda(Q_0)\} = 0$$

this complete the proof of the theorem.

4. Examples

Example 1. For the Sturm-Liouville operator with periodic coefficients

$$L := -\partial_x^2 + u(x), \quad u(x+T) = u(x)$$

the monodromy matrix

$$\mathrm{T}(x_0, \lambda) = \begin{pmatrix} \mathrm{T}_{11}(\lambda, x_0) & \mathrm{T}_{12}(\lambda, x_0) \\ \mathrm{T}_{21}(\lambda, x_0) & \mathrm{T}_{22}(\lambda, x_0) \end{pmatrix}$$

with $\det\mathrm{T} = 1$, is defined in the standard way (23). the eigenvectors of T correspond to the Bloch-Flouquet eigenfunctions of L

$$L\phi = \lambda\phi, \quad \phi(x+T) = \mu\phi(x)$$

$$\mathrm{T}(\lambda, x_0)\begin{pmatrix} \phi(x_0) \\ \phi'(x_0) \end{pmatrix} = \mu\begin{pmatrix} \phi(x_0) \\ \phi'(x_0) \end{pmatrix}.$$

Using the normalization $\phi(x_0) = 1$ (i.e. $K = (1,0)$) we obtain for the λ-projections of the poles the well known [6] equation

$$D(\lambda) = \det \begin{pmatrix} 1 & 0 \\ \mathbf{T}_{11} & \mathbf{T}_{12} \end{pmatrix} = \mathbf{T}_{12}(\lambda, x_0) = 0$$

Example 2. Consider a first order matrix operator with potential $U(x,\lambda)$, linear in variable λ, i.e.:

$$U(x,\lambda) := V(x) - \lambda A \qquad Tr(V) = 0 \tag{35}$$

Here A is a diagonal matrix with pairwise distinct entries.
We assume that the matrix-valued function $V(x)$ is analytic in x near a point $x = x_0$.

Following the scheme, say of [5] we construct a Poisson structure on appropriate space of functionals of $V(x)$.
To a matrix X with entries X_{ij} belongings to a suitable space of functions $\mathcal{A}$ of the variables V_{ij} a vector field corresponds

$$\partial_X = \sum X_{ij} \frac{\partial}{\partial V_{ij}}.$$

The space of vector fields $\{\partial_X\}$ is a Lie algebra $\mathfrak{g}$, w.r.t. the standard commutator. Moreover let Ω^0 be the space of functionals

$$\mathbf{f}[V] := \int f dx \quad f \in \mathcal{A}$$

Let set a pairing between $\mathfrak{g}$ and the space of matrix-valued functions of $V(x)$ by

$$\langle \partial_X, Y \rangle := \int Tr(XY) dx.$$

So the dual space Ω^1 of the Lie algebra $\mathfrak{g}$ is defined as the space of matrix-valued functions $\{X \ s.t. \ X_{lm} \in \mathcal{A}\}$.
The family of Poisson structures depending on the parameter λ is defined by the map

$$H_\lambda : \Omega^1 \to \mathfrak{g}$$

(i.e. 1-forms to vector fields) of the form:

$$X \in \Omega^1 \to H_\lambda(X) = \ [X' + V(x) - \lambda A, X]$$

$$X \in \Omega^1 \to \partial_{[L,X]}$$

The Poisson bracket can be written as:

$$\{\mathbf{f}, \mathbf{g}\} = \int TrH\left(\frac{\delta f}{\delta V}\right)\frac{\delta g}{\delta V}dx.$$

where the variational derivative of the functionals are defined by

$$\delta \mathbf{f} = \int Tr\left(\frac{\delta f}{\delta V}\delta V\right)dx$$

To each functional $\mathbf{f}[V]$ a Hamiltonian vector field corresponds

$$\mathbf{f}[V] \to \partial_{H(\delta f/\delta V)} = \partial_{[\partial + V - \lambda A, \frac{\delta f}{\delta V}]}.$$

The limiting cases are: $\lambda = 0$ and $\lambda = \infty$,

$$\{\mathbf{f}, \mathbf{g}\}^{(\infty)} = \frac{1}{T}\int_{x_0}^{x_0+T} Tr\left\{\frac{\delta g}{\delta V}\left[A, \frac{\delta f}{\delta V}\right]\right\}dx \tag{36}$$

$$\{\mathbf{f}, \mathbf{g}\}^{(0)} := \frac{1}{T}\int_{x_0}^{x_0+T} Tr\left\{\frac{\delta g}{\delta V}\left[\partial + V(x), \frac{\delta f}{\delta V}\right]\right\}dx \tag{37}$$

The matrix entries of $V(x)$ and so of $U(x, \lambda)$ are local functionals of $V(x)$. for their P.B. we obtain from (36)

$$\begin{aligned}\{V_{ij}(x), V_{kl}(y)\}^{(\infty)} &= \{U_{ij}(x, \lambda), U_{kl}(y, \mu)\}^{(\infty)} = \\ &= \delta_{li}\delta_{jk}(a_j - a_i)\delta(x - y)\end{aligned} \tag{38}$$

where a_i are the entries of the matrix A; and from (37)

$$\begin{aligned}\{V_{ij}(x), V_{lm}(y)\}^{(0)} &= \delta_{lj}\delta_{mi}\partial_y\delta(x - y) + \\ &+ \delta(x - y)\left(V_{lj}(x)\delta_{mi} - V_{im}(x)\delta_{lj}\right).\end{aligned} \tag{39}$$

The P.B. (38) is called ultralocal Poisson bracket because on the right-hand side of the (38) there are no terms with derivative of the δ function. This P.B. can be put in a r-matrix form:

$$\{U(x,\lambda) \underset{,}{\otimes} U(y,\mu)\}^{(\infty)} = [r(\lambda-\mu), U(x,\lambda)\otimes 1 + 1 \otimes U(y,\lambda)]\delta(x-y)$$

where the r-matrix is given by:

$$r(\lambda) := \frac{P}{\lambda} \tag{40}$$

P being the permutation matrix in $\mathbb{C}^n \otimes \mathbb{C}^n$.
Suppose that $V(x)$ is a periodic matrix, with period T. We recall here the derivation of the P.B. (38) between the entries of the monodromy matrix $\mathrm{T}(\lambda)$, $\mathrm{T}(\mu)$. In order to do this we use the representation (23) of $\mathrm{T}(\lambda)$ via the solution of the equation (22). We consider the variation equation, for fixed λ,

$$\delta\partial_x Y(x) + \delta V(x) Y + V(x)\delta Y(x) - \lambda A \delta Y(x) = 0$$

This is a first order matrix differential equation in $\delta Y(x)$ with initial condition $\delta Y(x_0) = 0$ .The solution has the form:

$$\delta Y(x) = Y(x) C(x)$$

and the matrix $C(x)$ satisfies the following equation:

$$Y(x)\partial_x C(x) + \delta V Y(x) = 0$$

so

$$\partial_x C(x) = -Y^{-1}(x)\delta V(x) Y(x)$$

and

$$\delta Y(x) = -Y(x)\int_{x_0}^{x} Y^{-1}(x')\delta V(x') Y(x') dx'.$$

So for $x = x_0 + T$ we obtain:

$$\frac{\delta \mathrm{T}_{ij}(\lambda)}{\delta V_{ks}(x)} = -\sum_l \mathrm{T}_{il}(\lambda) Y^{-1}_{lk}(x) Y_{sj}(x) \tag{41}$$

So:

$$\{\mathrm{T}(\lambda) \underset{,}{\otimes} \mathrm{T}(\mu)\}^{ab}_{cd} =$$

$$= \sum_{i,j,k,l} \int_{x_0}^{x_0+T} \int_{x_0}^{x_0+T} dx\, dy \frac{\delta \mathbf{T}_c^a(\lambda)}{\delta V_j^i(x)} \frac{\delta \mathbf{T}_d^b(\mu)}{\delta V_l^k(y)} \{V(x) \underset{,}{\otimes} V(y)\}_{jl}^{ik}. \qquad (42)$$

Using the (41) and (38) we obtain:

$$\{\mathbf{T}_c^a(\lambda), \mathbf{T}_d^b(\mu)\}^{(\infty)} =$$

$$= \sum_{m,s} \mathbf{T}_m^a(\lambda)\mathbf{T}_s^b(\mu) \int dx \left(Y^{-1}(x,\lambda)Y(x,\mu)\right)_d^m \left(Y^{-1}(x,\mu)AY(x,\lambda)\right)_c^s -$$

$$-\mathbf{T}_m^a(\lambda)\mathbf{T}_s^b(\mu) \int dx \left(Y^{-1}(x,\lambda)AY(x,\mu)\right)_d^m \left(Y^{-1}(x,\mu)Y(x,\lambda)\right)_c^s.$$

Because $Y(x,\lambda)$ is a matrix solution of the (22) the following identity:

$$Y^{-1}(x,\lambda)AY(x,\mu) = \frac{(Y^{-1}(x,\lambda)Y(x,\mu))_x}{\mu - \lambda}$$

holds.

The substitution of this identity in the previous formula gives :

$$\{\mathbf{T}_c^a(\lambda), \mathbf{T}_d^b(\mu)\}^{(\infty)} =$$

$$= \sum_{m,s} \frac{\mathbf{T}_m^a(\lambda)\mathbf{T}_s^b(\mu)}{\lambda - \mu} \left[\left(Y^{-1}(x,\lambda)Y(x,\mu)\right)_d^m \left(Y^{-1}(x,\mu)Y(x,\lambda)\right)_c^s\right]_{x_0}^{x_0+T}$$

$$= \frac{\mathbf{T}_d^a(\mu)\mathbf{T}_c^b(\lambda) - \mathbf{T}_d^a(\lambda)\mathbf{T}_c^b(\mu)}{\lambda - \mu}.$$

The last expression can be rewritten as:

$$\{\mathbf{T}(\lambda) \underset{,}{\otimes} \mathbf{T}(\mu)\}^{(\infty)} = [r(\lambda - \mu), \mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)]. \qquad (43)$$

with the same r-matrix (40)

The hierarchy of commuting Hamiltonian systems related with the operator L can be constructed as follows [7]. We consider a solution $R := \sum_{i=0}^{\infty} R_i \lambda^{-i}$ with an arbitrary constant diagonal matrix R_0 (this is a formal series) of the equation:

$$[L, R] = 0 \qquad (44)$$

For the coefficients one obtains the following recursion equations

$$R_i' + [V, R_i] - [A, R_{i+1}] = 0 \quad i = -1, \ldots \qquad (45)$$

here $R_{-1} = 0$. Then [12] all the entries of the coefficients R_i are polynomial on $V, V', \ldots,$.
The equation of the hierarchies are linear combinations of the equations of the form

$$\dot{V} = -[A, R_{k+1}] \tag{46}$$

for the matrix-valued function $V = V(x,t)$. This is a Hamiltonian system with the Hamiltonian

$$h_{k+1} = -\frac{1}{k+1}\int Tr(AR_{k+2})dx$$

in the $H^{(\infty)}$ structure;
and:

$$h_k = \frac{1}{k}\int Tr(AR_{k+1})dx$$

in the $H^{(0)}$ structure.

Taking $k = 2$ for an arbitrary diagonal matrix $B = R_0$ we obtain the Hamiltonian system with a quadratic non linearity

$$\dot{V} = ad_B\ ad_A^{-1}V' + [V, ad_B\ ad_A^{-1}V]. \tag{47}$$

here the operator ad_A have the form:

$$ad_A(X) = [A, X]$$

All these systems commute pairwise. Imposing the reality constrains of the form (52) (see below) we obtain from (47) a system describing various type of non linear n-wave interaction [21].

The spectral curve (2) is an n-sheet covering of the Riemann λ sphere. It has n distinct infinite points $\infty_1, \ldots, \infty_n$ such that

$$\log \mu(\lambda) \sim a_i\lambda + O(1)$$

near ∞_i. For the $H^{(\infty)}$ structure we have that the Darboux coordinates are, in the previous notations, $\log \mu(\gamma_i)$, and γ_i, as a direct consequence of the

Theorem 2.
Periods of the Abelian integral

$$\oint d\log\mu$$

over the cycles on the spectral curve are the Casimirs of the first P.B. (38). It easy to see from the recursion relation (45) that the Darboux coordinates for the second Poisson structure (39) can be obtained from the poles (γ_i, μ_i) in the form

$$q_i = \log\gamma_i, \quad p_i = \log\mu_i.$$

Example 3. Suppressing the x-dependence in (47) we obtain a system of ODE

$$\dot{V} = [V, ad_B\ ad_A^{-1}\ V]. \tag{48}$$

This is a Hamiltonian system on the Lie algebra $sl(n)$ with the quadratic Hamiltonian

$$H = -\frac{1}{4}Tr\left(V\ ad_B\ ad_A^{-1}\ V\right) \tag{49}$$

depending on the parameters A and B.
This coincides with the multidimensional analog [4] of the Euler equations describing free rotations of a solid.
We introduce the matrix

$$\mathbf{T}(\lambda) = \lambda A - V.$$

Then the equations (48) coincide with the commutativity conditions [15]

$$\left[\partial_t + \lambda B - ad_B\ ad_A^{-1}\ V, \mathbf{T}(\lambda)\right] = 0$$

The Hamiltonian structure of (49) can be described by linear r- matrix Poisson brackets (26).

The spectral curve

$$\det(\lambda A - V - \mu I) = 0$$

generically is a plane algebraic curve of degree n (the genus equals $(n-1)(n-2)/2$).
Our Theorem 2 gives an algorithm of construction of canonical Darboux coordinates γ_i, $\mu(\gamma_i)$ for the Hamiltonian system (48). The function $D(\lambda)$ in

this case will be a polynomial of degree $n(n-1)/2$. Generically all the roots of this polynomial corresponds to the poles of the eigenvector.

The Casimirs $c_1, \ldots, c_n$ in this case are the coefficients of the expansions

$$\mu = a_i\lambda + c_i + O(\frac{1}{\lambda})$$

near the infinite points

$$\infty_i = \{\lambda \to \infty, \quad \frac{\mu}{\lambda} \to a_i\}$$

$A = diag(a_1, \ldots, a_n)$. These are nothing but the diagonal entries of the matrix V being constant due to the equations (48).

Remark.

1. *Changes of the normalization (8) of the eigenvectors ψ give canonical transformations of the variables $(p(\gamma), \gamma)$.*

2. *All of these canonical transformations can be covered by a similarity transformations:*

$$\mathbf{T} \to M^{-1}\mathbf{T}M \tag{50}$$

where $M = diag(m_1, ..., m_n)$.

The part 1) follows immediately if we consider another normalization for the eigenvector

$$\sum_i m_i\psi_i = 1 \quad m_i \in \mathbb{C}.$$

This condition changes the γ_i and $p(\gamma_i)$ but the Poisson brackets between the new γ_i's and $p(\gamma_i)$' s are the same. This means that the change of the normalization corresponds to a canonical transformation.

The second statement is obvious.

For the operator of the form (21), the transformations (50) form a part of the hierarchy (46). Indeed, if we take

$$h_B = \int Tr\,(BV(x))\,dx$$

as the Hamiltonian of the transformation w.r.t. the second P.B. $\{,\}^{(0)}$ for some diagonal matrix B then we obtain a transformation of (for an appropriate B)

$$\{V_{ij}(x), h_B\}^{(0)} = [B, V(x)]_{ij} = -\delta V_{ij}(x).$$

The corresponding monodromy matrix will transform as in (50).

In application the coefficients of the operator L satisfy certain reality conditions. The most important of them are:

$$V^* = -V \tag{51}$$

(the * denotes the hermitian conjugation) or, more generally:

$$V^* = -J^{-1}VJ \tag{52}$$

for a diagonal real matrix J. The matrix iA in these cases must be real. This reduction is compatible with the hierarchy (46) (one should take real matrix R_0). Our technique of construction of canonical coordinates works also for the real case (52) (the reality restrictions for the Darboux coordinates are discussed in [8]). More complicated reduction of the hierarchy (46) is obtained imposing an additional constraint of reality

$$\bar{V} = V \tag{53}$$

The problem of separation of variables using Darboux coordinates of the algebraic-geometric type, for the operators satisfying (52) and (53) is still open.

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