Functionals of the Peierls - Frölich type
and variational principle for Whitham equations

Boris DUBROVIN

SISSA, Trieste, Italy

We start from the following elementary example. Let us consider the function

$$f_{x,t}(u) = \frac{1}{4}u^4 - 3tu^2 - xu \tag{1}$$

depending on the parameters $x,\ t$. We assume that the parameters are chosen in such a way that the function has unique minimum at the point $u = u(x,t)$. This happens if

$$t < 0 \ \text{ or } \ 0 < t \ \text{ and } \ |x| > 4t\sqrt{2t}. \tag{2}$$

Then the minimizer satisfies the equation of motion of one-dimensional ideal fluid* (Riemann wave equation)

$$u_t = 6uu_x. \tag{3}$$

It solves the Cauchy problem for (3) with the initial data

$$x = u^3 \ \text{ for } \ t = 0. \tag{4}$$

The global solution of the Cauchy problem does not exist for positive t.

More generally, for any monotonically increasing function $g(u)$ one can reduce the solution of the Cauchy problem

$$x = g(u) \ \text{ for } \ t = 0 \tag{4}$$

for the equation (3) to minimization of the function

$$f_{x,t}(u) = G(u) - 3tu^2 - xu \tag{5}$$

for

$$G'(u) = g(u).$$

This solution exists only up to the point of gradient catastrophe (x_0, t_0) where the function (5) fails to have a unique minimum. Observe that near the point of gradient catastrophe

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* In a more standard normalization $u \to -\frac{1}{6}u$ the equation reads

$$u_t + uu_x = 0.$$

In our normalization waves with positive magnitude u move to the left.

the solution of the Cauchy problem (5), after an appropriate Galilean transformation, looks like the solution of the Cauchy problem (4).

We want to extend the functions of the type (1), (5) onto certain infinite-dimensional space in such a way that it has a unique minimum on this space. This minimum will give us the solution of the problem of evolution of multivalued functions in the sense of [7], i.e. the solution of the Cauchy problem (4) for the Whitham system. I recall that Whitham system describes modulations of the parameters of fast oscillations arising in the solutions of Korteweg - de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{6}$$

(see [13], [7] for the details).

Our infinite-dimensional space M will consist of all hyperelliptic Riemann surfaces of all genera $n \geq 0$ with real branch points $u_1 < u_2 < \ldots < u_{2n+1}$, and of their degenerations. To be more precise, we construct M as limit of the spaces M_n, $n \to \infty$. We construct these spaces inductively starting from

$$M_0 = \mathbf{R}.$$

The coordinate in M_0 we denote u.

Define now

$$M_n = M_n^0 \cup_{j=1}^n M_{n-1}^1(j) \cup_{j=1}^n M_{n-1}^2(j)$$

where

$$M_n^0 = \left\{(u_1, \ldots, u_{2n+1}) \in \mathbf{R}^{2n+1} | u_1 < u_2 < \ldots < u_{2n+1}\right\}$$

and any of the spaces $M_{n-1}^{1,2}(j)$ is isomorphic to M_{n-1} assumed to be already constructed. The space $M_{n-1}^1(j)$ is attached to the component of the boundary of M_n^0 where

$$u_{2j+1} - u_{2j} \to 0;$$

the space M_{n-1}^2 is attached to the component of the boundary of M_n^0 where

$$u_{2j} - u_{2j-1} \to 0.$$

Definition. A C^k-*smooth functional* f on M is a sequence $f_n(u_1, \ldots, u_{2n+1})$ of functions defined on every M_n satisfying the following properties.

1) On the open part M_n^0 this is a C^k-smooth function of the variables $u_1 < u_2 < \ldots < u_{2n+1}$.

2) Near an inner point of $M_{n-1}^1(j)$ this function can be represented as

$$f_n(u_1, \ldots, u_{2n+1})$$

$$= f_{n-1}(u_1, \ldots, \hat{u}_{2j}, \hat{u}_{2j+1}, \ldots, u_{2n+1}) + \epsilon f_{n,j}^1(u_1, \ldots, \hat{u}_{2j}, \hat{u}_{2j+1}, \ldots, u_{2n+1}; v, \epsilon) \tag{7}$$

for a C^k-smooth function $f^1_{n,j}$ of $u_1 < \ldots < u_{2j-1} < v < u_{2j+2} < \ldots < u_{2n+1}$ and $\epsilon > 0$. Here

$$v = \frac{u_{2j} + u_{2j+1}}{2}, \quad \epsilon = \frac{1}{4}(u_{2j+1} - u_{2j})^2. \tag{8}$$

Here and below the hat above a letter means that the correspondent coordinate is omitted.

3) Near an inner point of $M^2_{n-1}(j)$ the function can be represented as

$$f_n(u_1, \ldots, u_{2n+1})$$

$$= f_{n-1}(u_1, \ldots, \hat{u}_{2j-1}, \hat{u}_{2j}, \ldots, u_{2n+1}) + \delta f^2_{n,j}(u_1, \ldots, \hat{u}_{2j-1}, \hat{u}_{2j}, \ldots, u_{2n+1}; v, \delta) \tag{9}$$

for a C^k-smooth function $f^2_{n,j}$ of $u_1 < \ldots < u_{2j-2} < v < u_{2j+1} < \ldots < u_{2n+1}$ and $\delta > 0$. Here

$$v = \frac{u_{2j-1} + u_{2j}}{2}, \quad \delta = \left[\log \frac{4}{(u_{2j} - u_{2j-1})^2}\right]^{-1}. \tag{10}$$

Remark. The inner part of M_n parametrizes isospectral classes of n-gap potentials $u(x)$ of the Sturm - Liouville operator

$$L = -\frac{d^2}{dx^2} + u(x). \tag{11}$$

Any such potential is a certain quasiperiodic analytic function of x. Genericaly it has n independent periods. The spectrum of the operator (11) in $L_2(-\infty, \infty)$ with a n-gap potential consists of the segments

$$\text{spectrum } = [u_1, u_2] \cup [u_3, u_4] \cup \ldots \cup [u_{2n+1}, \infty). \tag{12}$$

The isospectral class of the operator coincides with the real part of the Jacobi variety of the hyperelliptic Riemann surface

$$\Gamma_n = \left\{ \mu^2 = \prod_{i=1}^{2n+1} (\lambda - u_i) \right\}. \tag{13}$$

All the potentials of the isospectral class are obtained as the restriction of a certain Abelian function U_n (the second logarithmic derivative of the theta function) onto parallel straight lines on the Jacobi variety (see [6]):

$$u(x) = U_n(k_1 x + \phi^0_1, \ldots, k_n x + \phi^0_n; u_1, \ldots, u_{2n+1})$$

where $U_n(\phi_1, \ldots, \phi_n; u_1, \ldots, u_{2n+1})$ is a certain function 2π-periodic in each $\phi_1, \ldots, \phi_n$ depending on the pparameters $u_1, \ldots, u_{2n+1}$, $\phi^0_1, \ldots, \phi^0_n$ are arbitrary real phase shifts, the wave numbers $k_i = k_i(u_1, \ldots, u_{2n+1})$ are computed as certain Abelian integrals on the Riemann surface (13). The correspondent n-gap solutions of KdV have a similar form

$$u(x) = U_n(k_1 x + \omega_1 t + \phi^0_1, \ldots, k_n x + \omega_n t + \phi^0_n; u_1, \ldots, u_{2n+1}) \tag{14}$$

where the frequencies $\omega_i = \omega_i(u_1, \ldots, u_{2n+1})$ are again some Abelian integrals on (13).

The boundary components $M^1_{n-1}(j)$ are obtained by shrinking the j-th gap in the spectrum. The n-gap potentials then tend to $(n-1)$-gap ones. The boundary components $M^2_{n-1}(j)$ result from shrinking of j-th zone of the spectrum. One of the periods of the limiting potential goes to infinity. Particularly, shrinking to points all the finite segments of the spectrum and shifting u_{2n+1} to 0 we obtain in the limit n-soliton potentials of (11) [*ibid*].

We construct now the basic example of a smooth functional on M. For a point in M^0_n denote dp the Abelian differentiall of the second kind on the hyperelliptic Riemann surface (13) of the form

$$dp = \frac{P_n(\lambda)}{2\sqrt{R_n(\lambda)}} d\lambda \tag{15a}$$

where

$$R_n(\lambda) = \prod_{i=1}^{2n+1} (\lambda - u_i) \tag{15b}$$

$$P_n(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n \tag{15c}$$

and the coefficients $a_k = a_k(u_1, \ldots, u_{2n+1})$ are uniquelly determined by the normalization conditions

$$\int_{u_{2j}}^{u_{2j+1}} dp = 0, \quad j = 1, \ldots, n. \tag{15d}$$

Remark. From the point of view of the operator (11) the diffferential dp is the density of the states per unit length. If the potential $u(x)$ is periodic then dp coincides with the differential of quasimomentum (see [6]).

Proposition. *For any fixed nonreal λ the function*

$$\frac{P_n(\lambda)}{2\sqrt{R_n(\lambda)}}$$

can be extended from M^0_n to a C^∞-smooth functional on M.

Proof is based on the results of Section III of [8].

Example. The space M_1 has two boundary components M^1_0 and M^2_0. Near M^1_0 one has

$$dp = \frac{d\lambda}{2\sqrt{\lambda - u}} + \frac{\epsilon}{8(v-u)} \frac{\lambda + v - 2u}{(\lambda - v)^2 \sqrt{\lambda - u}} + O(\epsilon^2). \tag{16}$$

Near M^2_0

$$dp = \frac{d\lambda}{2\sqrt{\lambda - u}} + 2\delta \frac{v-u}{(\lambda - v)\sqrt{\lambda - u}} + O(\delta^2). \tag{17}$$

The corrections in the r.h.s. of the last formula contain also exponentially small terms like $\exp -\frac{1}{\delta}$.

Corollary. *The coefficients* $I_k = I_k(u_1, \ldots, u_{2n+1})$ *of the expansion*

$$p(\lambda) = \sqrt{\lambda} + \sum_{k=0}^{\infty} \frac{I_k}{(2\sqrt{\lambda})^{2k+1}} \tag{18}$$

are C^∞*-smooth functionals on* M.

Remark. The functions $I_k(u_1, \ldots, u_{2n+1})$ of the spectrum of a n-gap potential $u(x)$ are local functionals of $u(x)$. That means that their values can be computed by averaging certain differential polynomials (the KdV conservation laws)

$$I_k(u_1, \ldots, u_{2n+1}) = \lim_{T->\infty} \frac{1}{2T} \int_{-T}^{T} Q_k(u, u', \ldots, u^{(k-1)})\, dx. \tag{19}$$

For example,

$$Q_0 = -u, \quad Q_1 = -u^2,$$

etc.

A *curve* on M is a continuous map of an open segment $I \subset \mathbf{R}$

$$\gamma : I \to M,$$

i.e., a sequence of smooth maps

$$\gamma_n : D_n \to M_n^0$$

defined on some open subsets of I

$$\gamma_n(x) = (u_1(x), \ldots, u_{2n+1}(x)), \quad n = 0,\ 1, \ldots$$

such that

$$D_i \cap D_j = \emptyset, \quad i \neq j, \quad \overline{\cup_{n=0}^{\infty} D_n} = I. \tag{20}$$

On the intersections of the closures of $\overline{D_n} \cap \overline{D_{n+1}}$ the maps γ_n must satisfy certain boundary conditions (see below for solutions of Whitham equations).

Let us draw such a curve on the plane as it is shown on Fig.1. We will call such a map *multivalued function*, and the curve like in Fig.1 *graph* of the multivalued function. We say that a multivalued function is C^k-smooth if the graph of it is a C^k-smooth plane curve.

To formulate our main result we need to define the Whitham system. This is a sequence of systems of the first order quasilinear evolutionary PDEs of the form

$$\partial_t u_i = \sum_{j=1}^{2n+1} A_{n,\ i\ j}(u_1, \ldots, u_{2n+1}) \partial_x u_j, \quad i = 1, \ldots, 2n+1 \tag{21_n}$$

defined for each $n = 0, 1, \ldots$. For $n = 0$ this coincides with the Riemann wave equation (3). For any $n > 0$ the system (21_n) is obtained from the KdV equation by applying the Whitham averaging procedure [16] to the family of n-gap invariant tori (see [7]). It describes slow modulations of the parameters $u_1, \ldots, u_{2n+1}$ of n-gap quasiperiodic solutions of KdV. That means that the formula

$$u(x,t) \simeq U_n \left(\epsilon^{-1} S_1(\epsilon x, \epsilon t), \ldots, \epsilon^{-1} S_n(\epsilon x, \epsilon t); u_1(\epsilon x, \epsilon t), \ldots, u_{2n+1}(\epsilon x, \epsilon t)\right)$$

describes the leading term of the asymptotics with $\epsilon \to 0$ in the region $|x| < \epsilon^{-1}$, $|t| < \epsilon^{-1}$ of the oscillating solution of KdV equation with slowly varying parameters $u_1, \ldots, u_{2n+1}$. The small parameter ϵ measures the ratio of the period of oscillations in (14) to the characteristic scale of modulations of the parameters. The phase functions $S_1(x,t), \ldots, S_n(x,t)$ are found by quadratures

$$dS_i(x,t) = k_i \left(u_1(x,t), \ldots, u_{2n+1}(x,t)\right) dx + \omega_i \left(u_1(x,t), \ldots, u_{2n+1}(x,t)\right) dt.$$

Solutions of the Whitham equations (21_n) for a given n typically exist only within certain domains of the (x,t)-plane. The main problem of the theory of Whitham equations is to glue together these solutions for different n in order to produce a C^1-smooth in the space M depending smoothly on t. In general it is not known what are to be the boundary conditions to glue together solutions of the systems (21_n) and (21_{n+1}) for an arbitrary n. These boundary conditions were found for $n = 0$ in [1].

Particularly, if for $t = 0$ and all $-\infty < x < \infty$ such a curve sits in $M_0 \subset M$ and is given by the equation

$$x = h(u) \tag{22}$$

for some smooth monotonic function $h(u)$ then it is called *solution of the Cauchy problem* for the Whitham system (21). Conjecturally this describes the weak limit of the solutions of the Cauchy problem (22) for the KdV equation with small dispersion (see [12] and references therein).

The explicit construction of the Whitham system (21_n) for any n can be found in [9], [7].

Let us now consider the functional

$$f_{x,t,c_2,\ldots,c_{2N+1},g} = xI_0 + 3tI_1 - \sum_{k=2}^{2N+1} c_k I_k - \int_{\text{spectrum}} \Phi(\lambda) dp(\lambda) \tag{23}$$

where the functionals I_k are defined from the expansion (18), $c_2, \ldots, c_{2N+1}$ are arbitrary real constants, $c_{2N+1} > 0$, and the function $\Phi(\lambda)$ is parametrized by another function $g(u)$ via the formula

$$\Phi(\lambda) = \frac{4}{\pi} \int_\lambda^\infty \frac{g(u)}{\sqrt{u - \lambda}} du. \tag{24}$$

We assume that the function $g(u)$ rapidly decrease at infinity. From Proposition it follows that (23) is a C^∞-smooth functional on M depending on the parameters $x, t, c_2, \ldots, c_{2N+1}$, $g(u)$ (we will often keep explicitly only the dependence of the functional on the parameters $x,\ t$).

Theorem. *1) The minimizer of the functional (23) is a C^1-smooth multivalued function of x that also depends C^1-smoothly on the parameters $t, c_2, \dots, c_{2N+1}$.*

2) For those values of the parameters when the minimizer $(u_1(x,t), \dots, u_{2n+1}(x,t))$ belongs to M_n^0 it satisfies the n-th Whitham system (21_n).

3) Let us assume that the minimizer of the functional (23) for $t = 0$ and $g = 0$ belongs to M_0. Then for sufficiently small function $g(u)$ the minimizer is a solution of the Cauchy problem

$$x = \left[\sum_{k=2}^{2N+1} 2^k \frac{(2k-1)!!}{k!} c_k u^k + g(u) \right]_{t=0} \tag{25}$$

(assuming monotonicity of the r.h.s.) for the Whitham equations.

We will call (23) *functionals of the Peierls - Fröhlich type*. Properties of such functionals studied in [3, 4, 11] play an important role in the proof of Theorem.

Conjecture. *The curve $\gamma(x,t)$ given by the minimizer of the functional of the Peierls - Fröhlich type describes the weak limit of the solutions of the Cauchy problem (25) for the KdV equation with small dispersion.*

It would be interesting to compare our variational principle for the Whitham system with the maximization problem of Lax and Levermore describing the weak limit of the Cauchy problem.

Remark. The dependence of the minimizer on the parameters c_k is governed by the equations of *Whitham hierarchy*. The latter is obtained by application of the averaging procedure to the KdV hierarchy. See [5] where a variational principle for solving Whitham-type hierarchies was proposed in a more general setting of hierarchies arising in 2D topological field theories.

Example. Let us study minima of the functional

$$f_{x,t} = xI_0 + 3tI_1 - \frac{1}{20} I_3. \tag{26}$$

Restriction of the functional onto M_0 has the form (1). Thus the minimizer

$$x + 6ut = u^3 \tag{27}$$

of the restriction solves the Riemann wave equation (i.e., the Whitham equation for $n = 0$). Let us find for which $x,\ t$ the solution of (27) is also minimal along the directions transversal to $M_0 \subset M$.

First we consider embedding M_0 as the component M_0^1 of the boundary of the space M_1. In other words, we study what happens with the functional (26) when a small gap of the width $2\sqrt{\epsilon}$ opens in the spectrum near the point v for some $v > u$. From the formula (16) one obtains

$$f_{x,t} = -xu - 3tu^2 + \frac{1}{4}u^4$$

$$+\epsilon \frac{-30tu + u^3 + 5x + 60tv + 2u^2v + 8uv^2 - 16v^3}{u - v} + O(\epsilon^2). \tag{28}$$

Near the minimizer (27) the coefficient in front of ϵ can be rewritten in the form

$$\frac{1}{5}\left(3u^2 + 4uv + 8v^2 - 30t\right). \tag{29}$$

It can be easily seen that for

$$x < -12\sqrt{3}t^{3/2}$$

the polynomial (29) takes positive values for an arbitrary v. So for these x near the point (27) the value of the functional (26) increases when a small gap opens at any place of the spectrum (recall that $\epsilon > 0$). On the curve

$$x = -12\sqrt{3}t^{3/2}$$

from (27) we find

$$u = -2\sqrt{3t}.$$

The polynomial (29) then has a double root

$$v = \frac{1}{2}\sqrt{3t}.$$

For

$$x > -12\sqrt{3}t^{3/2}$$

the polynomial (29) has real roots v, so $f_{x,t}$ does not have minimum at (27) for these values of (x, t). The minimizer moves into the inner part of M_1. That means that for $x = -12\sqrt{3}t^{3/2}$ a small gap opens in the spectrum around the point $-\frac{1}{2}\sqrt{3t}$.

Let us consider now behaviour of $f_{x,t}$ near $M_0 = M_0^2 \subset M_1$ (a point v of discrete spectrum is added). From the formula (17) one obtains

$$f_{x,t} - xu - 3u^2t + \frac{1}{4}u^4 + \frac{8}{35}\delta(v-u)$$

$$\times\left[-70tu + 5u^3 - 35x - 140tv + 6u^2v + 8uv^2 + 16v^3\right] + O(\delta^2). \tag{30}$$

As above, near the point (27) one can rewrite the coefficient in front of δ as

$$\frac{8}{35}(v-u)^2\left[30u^2 + 24uv + 16v^2 - 140t\right]. \tag{31}$$

Again, it can be easily shown that for

$$x > \frac{4}{3}\sqrt{\frac{5}{3}}t^{3/2}$$

the solution (27) is minimal along the directions transversal to $M_0 = M_0^2 \subset M_1$ (the polynomial (31) has no real roots $v < u$). The stability of the minimizer fails starting from

$$x = \frac{4}{3}\sqrt{\frac{5}{3}} t^{3/2}$$

For this value of x a point of discrete spectrum borns at

$$v = -\frac{3}{2}\sqrt{\frac{5}{3}t}$$

(then $u = 2\sqrt{\frac{5}{3}t}$). For smaller x this point inflates into a zone of the spectrum of a one-gap potential.

The restriction of the functional (26) onto M_1 can be written as the function

$$f_{x,t}(u_1, u_2, u_3) = \frac{c^4}{5184} + \frac{c^2\, g_2}{72} + \frac{3\, {g_2}^2}{140} - \frac{c\, g_3}{15} - \frac{c^3\, \eta}{108\, \omega} - \frac{c\, \eta\, g_2}{15\, \omega}$$

$$+ \frac{4\, \eta\, g_3}{35\, \omega} - \frac{c^2\, t}{12} - g_2\, t + \frac{2\, c\, \eta\, t}{\omega} - \frac{c\, x}{6} + \frac{2\, \eta\, x}{\omega} \tag{32}$$

Here $\eta = \eta(g_2, g_3)$, $\omega = \omega(g_2, g_3)$ are the standard functions (see [2]) defined for the elliptic curve

$$w^2 = 4z^3 - g_2 z - g_3 \tag{33}$$

for arbitrary parameters g_2, g_3, c satisfying

$$g_2^2 - 27 g_3^2 > 0$$

and

$$u_i = \frac{c}{6} - e_i, \quad i = 1, 2, 3$$

where $e_1 > e_2 > e_3$ are the roots of the polynomial in the r.h.s. of (33). From our Theorem and results of Potemin [14] it follows that for any positive t and

$$-12\sqrt{3} < \frac{x}{t^{3/2}} < \frac{4}{3}\sqrt{\frac{5}{3}} \tag{34}$$

the function (32) has a unique minimum on M_1. It can be shown that the minimizer is stable also in the transversal directions w.r.t. various embeddings of M_1 as components of the boundary M_2. As a consequence we conclude that, for any sufficiently small function $g(u)$ the minimizer of the functional (26) for any x, t belongs to M_1 or to the boundary of it. Recall that, according to the theorem, the minimizer gives the solution of the Cauchy problem for Whitham system with the initial data

$$x = u^3 + g(u) \tag{35}$$

for $t = 0$.

The solution of the Cauchy problem

$$x = u^3 \quad \text{for} \quad t = 0$$

for Whitham system was first studied by Gurevich and Pitaevski in 1973-74 in their papers [10] on dispersive analogue of shock waves. They found this solution numerically. Analytically this solution was found in [14]. In numerical experiments Avilov, Krichever, and Novikov [1] discovered remarkable stability of the solution of Gurevich and Pitaevski w.r.t. small perturbations $g(u)$. Our variational formulation of the Whitham system provides a simple explanation of this stability.

Remark. In a recent paper [15] Fey Rey Tian proved that the solution of the Cauchy problem (35) globally belongs to M_1 not only for small $g(u)$ under an additional assumption that the monotonically increasing function $u^3 + g(u)$ has only one inflection point.

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