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FLAT PENCILS OF METRICS AND FROBENIUS MANIFOLDS

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Abstracts This paper is based on the author's talk at 1997 Taniguchi Symposium "Integrable Systems and Algebraic Geometry". We consider an approach to the theory of Frobenius manifolds based on the geometry of flat pencils of contravariant metrics. It is shown that, under certain homogeneity assumptions, these two objects are identical. The flat pencils of contravariant metrics on a manifold M appear naturally in the classification of bihamiltonian structures of hydrodynamics type on the loop space $L(M)$. This elucidates the relations between Frobenius manifolds and integrable hierarchies.

Introduction

Let M be n-dimensional smooth manifold.

Definition 0.1. A symmetric bilinear form $(\ ,\ )$ on T^*M is called *contravariant metric* if it is invertible on an open dense subset $M_0 \subset M$.

In local coordinates $x^1, \ldots, x^n$ a contravariant metric is specified by the components (a $(2,0)$-tensor)

$$\left(dx^i, dx^j\right) = g^{ij}(x),\ i,\ j = 1, \ldots, n. \tag{0.1}$$

On M_0 the inverse matrix $(g_{ij}(x)) = \left(g^{ij}(x)\right)^{-1}$ determines a metric in the usual sense

$$ds^2 = g_{ij}(x)dx^i dx^j \tag{0.2}$$

(not necessarily positive definite). Here and below a summation over repeated indices is assumed.

Definition 0.2. The *contravariant Levi-Civita connection* for the metric $(\ ,\ )$ is determined by a collection of n^3 functions $\Gamma^{ij}_k(x)$ defined for any coordinate patch on M such that on M_0

$$\Gamma^{ij}_k(x) = -g^{is}(x)\Gamma^j_{sk}(x) \tag{0.3}$$

where $\Gamma^j_{sk}(x)$ is the Levi-Civita connection for the metric (0.2).

Lemma 0.1. *The coefficients $\Gamma^{ij}_k(x)$ of the contravariant Levi-Civita connection are determined uniquely on M_0 from the system of linear equations*

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s \tag{0.4}$$

$$\Gamma^{ij}_k + \Gamma^{ji}_k = \partial_k g^{ij}. \tag{0.5}$$

Here

$$\partial_k = \frac{\partial}{\partial x^k}.$$

Proof. On M_0 the symmetry condition $\Gamma^k_{ij} = \Gamma^k_{ji}$ of the Levi-Civita connection reads in the form (0.4). The equation $\nabla(\ ,\ ) = 0$ coincides with (0.5). Proof now follows from the wellknown theorem of existence and uniqueness of the Levi-Civita connection.

We emphasize, however, the assumption of the contravariant connection to be defined on all M but not only on M_0 (the linear system (0.4), (0.5) for Γ^{ij}_k degenerates on $M \setminus M_0$).

Having a contravariant connection one can define the operators ∇^u of covariant derivatives along any 1-form $u \in T^*M$. For example, for a covector field $v = v_i dx^i$ we obtain

$$\nabla^u v = \left(g^{ij}u_i\partial_j v_k + \Gamma^{ij}_k u_i v_j\right) dx^k. \tag{0.6}$$

Particularly, for $u = dx^i$ the operators $\nabla^i := \nabla^u$ are related with the usual covariant derivatives defined on M_0 by raising of the index

$$\nabla^i = g^{is}\nabla_s. \tag{0.7}$$

If the covector v_j is defined globally on M then $\nabla^i v_j$ will be globally defined (1,1)-tensor on M. From this it easily follows the correctness of the definition of contravariant connection.

Definition 0.3. A function $f(x)$ is called *flat coordinate* of $(\ ,\ )$ if the differential $\xi = df$ is covariantly constant w.r.t. the Levi-Civita connection

$$g^{is}\partial_s \xi_j + \Gamma^{is}_j \xi_s = 0,\ i,\ j = 1, \dots, n, \tag{0.8}$$

$$\xi_j = \partial_j f.$$

Definition 0.4. A contravariant metric is said to be *flat iff* on M_0 there locally exist n independent flat coordinates.

Choosing a system of flat coordinates one reduces the matrix (0.1) of the metric to a constant form and the coefficients Γ^{ij}_k of the Levi-Civita connection to zero.

Lemma 0.2. *The contravariant metric is flat* iff *the Riemann curvature tensor*

$$R^{ijk}_l := g^{is}\left(\partial_s \Gamma^{jk}_l - \partial_l \Gamma^{jk}_s\right) + \Gamma^{ij}_s \Gamma^{sk}_l - \Gamma^{ik}_s \Gamma^{sj}_l \tag{0.9}$$

identically vanishes.

This is a standard fact of differential geometry (see, e.g., [DFN]). It is important that our formula for the curvature involves only contravariant components of the metric and of the connection.

We give now our main

Definition 0.5. Two contravariant metrics $(\ ,\ )_1$ and $(\ ,\ )_2$ form a *flat pencil* if:

1) The linear combination

$$(\ ,\ )_1 - \lambda(\ ,\ )_2 \tag{0.10}$$

for any λ is a contravariant metric on M.

2) If ${\Gamma_1}^{ij}_k$ and ${\Gamma_2}^{ij}_k$ are the contravariant Levi-Civita connections for these two metrics then for any λ the linear combination

$${\Gamma_1}^{ij}_k - \lambda {\Gamma_2}^{ij}_k$$

is the contravariant Levi-Civita connection for the metric (0.10).

3). The metric (0.10) is flat for any λ.

We say that the flat pencil of metrics is *quasihomogeneous of the degree d* if there exists a function τ on M such that the vector fields

$$E := \nabla_1 \tau, \quad E^i = g_1^{is}\partial_s \tau \tag{0.11a}$$

$$e := \nabla_2 \tau, \quad e^i = g_2^{is}\partial_s \tau \tag{0.11b}$$

satisfy the following properties

$$[e, E] = e \tag{0.12}$$

$$\mathcal{L}_E(\ ,\ )_1 = (d-1)(\ ,\ )_1 \tag{0.13}$$

$$\mathcal{L}_e(\ ,\ )_1 = (\ ,\ )_2 \tag{0.14}$$

$$\mathcal{L}_e(\ ,\ )_2 = 0. \tag{0.15}$$

Definition 0.6. A *Frobenius algebra* is a pair $(A, \ <\ ,\ >)$ where A is a commutative associative algebra (over $\mathbf{R}$ or $\mathbf{C}$) with a unity and $<\ ,\ >$ stands for a symmetric non-degenerate *invariant* bilinear form on A. The invariance means validity of the following identity

$$< a\, b, c >=< a, b\, c > \tag{0.16}$$

for arbitrary 3 vectors $a,\ b,\ c \in A$.

Definition 0.7. The Frobenius algebra is called *graded* if a linear operator $Q : A \to A$ and a number d are defined such that

$$Q(a\, b) = Q(a)b + aQ(b), \tag{0.17a}$$

$$< Q(a), b > + < a, Q(b) >= d < a, b > \tag{0.17b}$$

for any $a,\ b \in A$. The operator Q is called *grading operator* and the number d is called *charge* of the Frobenius algebra. In the case of diagonalizable grading operators we may assign degrees to the eigenvectors e_α of Q

$$\deg(e_\alpha) = q_\alpha \ \text{ if } \ Q(e_\alpha) = q_\alpha e_\alpha. \tag{0.18}$$

Then the usual property of the degree of the product of homogeneous elements of the algebra holds true

$$\deg{(ab)} = \deg{a} \deg{b}.$$

Besides, $< a, b >$ can be nonzero only if $\deg a + \deg b = d$ where d is the charge.

We will consider also graded Frobenius algebras $(A, <\ ,\ >)$ over graded commutative associative rings R. In this case we have two grading operators $Q_R : R \to R$ and $Q_A : A \to A$ satisfying the properties

$$\begin{aligned}
Q_R(\alpha\beta) &= Q_R(\alpha)\beta + \alpha Q_R(\beta),\ \alpha,\ \beta \in R && (0.19a)\\
Q_A(ab) &= Q_A(a)b + aQ_A(b),\ a,\ b \in A && (0.19b)\\
Q_A(\alpha a) &= Q_R(\alpha)a + \alpha Q_A(a),\ \alpha \in R,\ a \in A && (0.19c)\\
Q_R < a, b > +d\ < a, b > &=< Q_A(a), b > + < a, Q_A(b) >,\ a,\ b \in A. && (0.19d)
\end{aligned}$$

As above the number d is called *the charge* of the graded Frobenius algebra over the graded ring.

Definition 0.8. (Smooth, analytic) *Frobenius structure* on the manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, <\ ,\ >_t)$ depending (smoothly, analytically) on the point t. This structure must satisfy the following axioms.

FM1. The metric on M induced by the invarint bilinear form $< \, , \, >_t$ is flat. Denote ∇ is the Levi-Civita connection for the metric $< \, , \, >_t$. The unity vector field e must be covariantly constant,

$$\nabla e = 0. \tag{0.20}$$

As above we use here the word 'metric' as a synonim of a symmetric nondegenerate bilinear form on TM, not necessarily of a positive one. Flatness of the metric, i.e., vanishing of the Riemann curvature tensor, means that locally a *system of flat coordinates* $(t^1, \ldots, t^n)$ exists such that the matrix $< \partial_\alpha, \partial_\beta >$ of the metric in these coordinates becomes constant.

FM2. Let c be the following symmetric trilinear form on TM

$$c(u, v, w) :=< u \cdot v, w > . \tag{0.21}$$

The four-linear form

$$(\nabla_z c)(u, v, w), \; u, v, w, z \in TM \tag{0.22}$$

must be also symmetric.

Before formulating the last axiom we observe that the space $Vect(M)$ of vector fields on M acquires a structure of a Frobenius algebra over the algebra $Func(M)$ of (smooth, analytic) functions on M.

FM3. A linear *Euler vector field* $E \in Vect(M)$ must be fixed on M, i.e.,

$$\nabla\nabla\, E = 0. \tag{0.23}$$

The operators

$$\begin{aligned} Q_{Func(M)} &:= E \\ Q_{Vect(M)} &:= \mathrm{id} + \mathrm{ad}_E \end{aligned} \tag{0.24}$$

introduce in $Vect(M)$ a structure of graded Frobenius algebra of a given charge d over the graded ring $Func(M)$.

We will now spell out the requirements of Definition 0.8 in the flat coordinates t^1, ..., t^n of the metric $< \, , \, >$. Denote

$$\eta_{\alpha\beta} :=< \partial_\alpha, \partial_\beta > \tag{0.25}$$

(a constant symmetric nondegenerate matrix),

$$\partial_\alpha \cdot \partial_\beta|_t = c_{\alpha\beta}^\gamma(t)\partial_\gamma. \tag{0.26}$$

Then the components of the trilinear form (0.21) can be locally represented as the triple derivatives of a function $F(t)$

$$c\,(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma F(t) \qquad (0.27).$$

Associativity of the multiplication in the Frobenius algebra implies the following *WDVV associativity equations* for the function $F(t)$

$$\partial_\alpha\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(t) = \partial_\delta\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(t), \; \alpha, \, \beta, \, \gamma, \, \delta = 1, \ldots, n. \tag{0.28}$$

The axiom **FM3** in these coordinates can be recasted into the following equivalent form

$$\mathcal{L}_E c_{\alpha\beta}^{\gamma} = c_{\alpha\beta}^{\gamma} \tag{0.29}$$

$$\mathcal{L}_E \eta_{\alpha\beta} = (2-d)\eta_{\alpha\beta}. \tag{0.30}$$

From (0.29) one obtains the following quasihomogeneity equation for the function $F(t)$

$$\mathcal{L}_E F(t) = (3-d)F(t) + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C \tag{0.31}$$

with some constants $A_{\alpha\beta}$, B_α, C.

Finally, **FM1** means that the unity vector field is constant in the coordinates t. Usually the flat coordinates are chosen in such a way that

$$e = \partial/\partial t^1. \tag{0.32}$$

The equation (0.23) means that the matrix ∇E is constant in the flat coordinates t.

The main aim of the present paper is to prove that any Frobenius manifold carries a natural quasihomogeneous linear pencil of metrics and, under certain nondegeneracy assumption, to prove also the converse statement.

1. From Frobenius manifolds to flat pencils

We put $(\ ,\ )_2 = <\ ,\ >$ (as a bilinear form on the cotangent bundle) and we define a new bilinear form on the cotangent bundle

$$(\omega_1, \omega_2)_1 \equiv (\omega_1, \omega_2) = i_E(\omega_1 \cdot \omega_2). \tag{1.1}$$

Here i_E is the operator of inner product (i.e., the contraction of the vector field E with a 1-form).This metric was found in [Du2] to give bihamiltonian structure of the integrable hierarchies of [Du1] describing coupling of a given matter sector of a 2D topological field theory to topological gravity. It was called in [Du4] *intersection form* of the Frobenius manifold.

Theorem 1.1. *The metrics $(\ ,\ )$ and $<\ ,\ >$ on a Frobenius manifold form a flat pencil quasihomogeneous of the degree d.*

Proof essentially follows [Du4] (we only relax the assumptions about the Frobenius manifold not requiring diagonalizability of the tensor ∇E).

In the flat coordinates $t^1, \ldots, t^n$ for $<\ ,\ >$ the components of the bilinear form (1.1) are given by the formula

$$\begin{aligned} g^{\alpha\beta}(t) &= \left(dt^\alpha, dt^\beta\right) = E^\epsilon(t) c_\epsilon^{\alpha\beta}(t) \\ &= R_\epsilon^\alpha F^{\epsilon\beta}(t) + F^{\alpha\epsilon}(t) R_\epsilon^\beta + A^{\alpha\beta} \end{aligned} \tag{1.2}$$

where

$$R_\beta^\alpha = \frac{d-1}{2}\delta_\beta^\alpha + (\nabla E)_\beta^\alpha \tag{1.3}$$

$$
\begin{aligned}
c_\gamma^{\alpha\beta}(t) &= \eta^{\alpha\epsilon} c_{\epsilon\gamma}^\beta(t) \\
F^{\alpha\beta}(t) &= \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t) \\
A^{\alpha\beta} &= \eta^{\alpha\lambda}\eta^{\beta\mu} A_{\lambda\mu}
\end{aligned} \tag{1.4}
$$

and the matrix $A_{\alpha\beta}$ was defined in (0.31).

From the first line in (1.2) it follows that

$$
g^{\alpha\beta}(t) = \eta^{\alpha\beta} t^1 + \tilde{g}^{\alpha\beta}(t^2, \ldots, t^n).
$$

So

$$
g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = \eta^{\alpha\beta}(t^1 - \lambda) + \tilde{g}^{\alpha\beta}(t^2, \ldots, t^n)
$$

for any λ does not degenerate on an open dense subset in M.

Lemma 1.1. *The contravariant Levi-Civita connection for the metric* $(\ ,\ ) - \lambda < ,\ >$ *is given by the formula*

$$
\Gamma_\gamma^{\alpha\beta}(t) = c_\gamma^{\alpha\epsilon}(t) R_\epsilon^\beta. \tag{1.5}
$$

Proof. Differentiating (1.2) w.r.t. t^γ we obtain

$$
\partial_\gamma g^{\alpha\beta} = R_\epsilon^\alpha c_\gamma^{\epsilon\beta} + c_\gamma^{\alpha\epsilon} R_\epsilon^\beta = \Gamma_\gamma^{\beta\alpha} + \Gamma_\gamma^{\alpha\beta}.
$$

This proves (0.5). The second equation (0.4) follows immediately from associativity of the multiplication on T^*M. Lemma is proved.

To finish the proof of Theorem it remains to show that the curvature of the pencil of the metrics vanishes identically in λ. First observe that the terms with derivatives of Γ in

$$
(g^{\alpha\epsilon} - \lambda\eta^{\alpha\epsilon})\left(\partial_\epsilon \Gamma_\delta^{\beta\gamma} - \partial_\delta \Gamma_\epsilon^{\beta\gamma}\right) + \Gamma_\epsilon^{\alpha\beta}\Gamma_\delta^{\epsilon\gamma} - \Gamma_\epsilon^{\alpha\gamma}\Gamma_\delta^{\epsilon\beta}
$$

vanish due to equality of mixed derivatives

$$
\partial_\epsilon c_\delta^{\beta\gamma} = \partial_\delta c_\epsilon^{\beta\gamma}.
$$

The remaining terms vanish due to associativity.

Finally, we put

$$
\tau = \eta_{1\alpha} t^\alpha \tag{1.6}
$$

assuming that the coordinate t^1 is chosen as in (0.32). Then (0.11) immediately follows. The equations (0.12) – (0.15) follow from (0.29), (0.30) and from

$$
\mathcal{L}_E e = -e \tag{1.7}
$$

(a consequence of (0.29)). Theorem is proved.

2. From flat pencils to Frobenius manifolds

We begin with simple

Lemma 2.1. *The functions*

$$\Delta^{ijk}(x) = g_2^{js}\Gamma_{1s}^{ik} - g_1^{is}\Gamma_{2s}^{jk} \tag{2.1}$$

are components of a rank three tensor (i.e., of a trilinear form on T^*M). Two flat metrics g_1^{ij} and g_2^{ij} can be simultaneously reduced to constant form iff $\Delta^{ijk}=0$.

Proof. It is wellknown that the difference of usual Christoffel coefficients of two affine connections

$$\Gamma_{1st}^{k} - \Gamma_{2st}^{k} \tag{2.2}$$

is a tensor. Contracting this with $g_1^{is}g_2^{jt}$ we obtain the tensor (2.1). Two metrics are simultaneously reducible to a constant form *iff* the difference (2.2) vanishes. Lemma is proved.

We will also consider a (2,1)-tensor

$$\Delta_i^{jk} = g_{2is}\Delta^{sjk} \tag{2.3}$$

defined on the open subset $M_0 \subset M$ where the contravariant metric $(\ ,\ )_2$ does not degenerate. The tensor (2.3) defines a bilinear operation

$$T^*M_0 \times T^*M_0 \to T^*M_0 \tag{2.4a}$$

$$\begin{aligned}&(u,v) \mapsto \Delta(u,v),\\ &\Delta(u,v)_k = u_i v_j \Delta_k^{ij}(x) \text{ for } u,\ v \in T_x^*M_0.\end{aligned} \tag{2.4b}$$

Lemma 2.2. *For a flat pencil of metrics the tensor (2.3) satisfies the following properties*

$$(\Delta(u,v),w)_1 = (u,\Delta(w,v))_1 \tag{2.5}$$

$$(\Delta(u,v),w)_2 = (u,\Delta(w,v))_2 \tag{2.6}$$

$$\Delta(\Delta(u,v),w) = \Delta(\Delta(u,w),v) \tag{2.7}$$

$$\nabla_2^u\Delta(v,w) - \nabla_2^v\Delta(u,w) = \Delta\left(\nabla_2^u v - \nabla_2^v u, w\right). \tag{2.8}$$

Here ∇_2^u are the covariant derivative operators (0.6) for the second metric $(\ ,\ )_2$, u, v, w are arbitrary 1-forms on M_0.

Quasihomogeneity of the flat pencil is not assumed. Note that, due to (2.7), the algebra structure on T^*M_0 will not be associative but *right-symmetric.*

Proof (see [Du4], Appendix D). Let us choose a system of flat coordinates $x^1, \ldots, x^n$ for the metric $(\ ,\ )_2$. In these coordinates we have

$$\Gamma_{2k}^{ij} = 0,\ \Gamma_{1k}^{ij} = \Delta_k^{ij}.$$

From the definition it follows that Δ^{ij}_k will also coincide with the contravariant Levi-Civita connection for all the linear pencil $(\ ,\ )_1 - \lambda(\ ,\ )_2$ with an arbitrary λ. Writing the symmetry condition (0.4)

$$\left(g_1^{is} - \lambda g_2^{is}\right) \Delta_s^{jk} = \left(g_1^{js} - \lambda g_2^{js}\right) \Delta_s^{ik}$$

we obtain (2.5) and (2.6). Vanishing of the curvature (0.9) of the pencil gives the equations

$$\left(g_1^{is} - \lambda g_2^{is}\right) \left(\partial_s \Delta_l^{jk} - \partial_l \Delta_s^{jk}\right) + \Delta_s^{ij} \Delta_l^{sk} - \Delta_s^{ik} \Delta_l^{sj} = 0 \text{ for any } i,\ j,\ k,\ l.$$

Vanishing of the linear in λ term implies

$$\partial_s \Delta_l^{jk} - \partial_l \Delta_s^{jk} = 0.$$

This coincides with (2.8). Vanishing of the remaining terms gives (2.7). Lemma is proved.

Lemma 2.3. *For a quasihomogeneous flat pencil the following equations hold true*

$$\nabla_2 \nabla_2 \tau = 0 \tag{2.9}$$

$$\nabla_2 \nabla_2 E = 0. \tag{2.10}$$

Proof. We have

$$0 = \mathcal{L}_e g_{1ij} = 2 \nabla_{2i} \nabla_{2j} \tau.$$

This proves (2.9). From (0.12) and (0.13) it follows

$$\mathcal{L}_E(\ ,\ )_2 = (d-2)(\ ,\ )_2. \tag{2.11}$$

So the vector field E generates the one-parameter group of linear conformal transformations of the metric $(\ ,\ )_2$. This proves (2.10).

Corollary 2.1. *The eigenvalues of the matrix*

$$\nabla_{2i} E^j(x) \tag{2.12}$$

do not depend on the point of the manifold.

Definition 2.1. A quasihomogeneous flat pencil is said to be *regular* if the (1,1)-tensor

$$R_i^j = \frac{d-1}{2} \delta_i^j + \nabla_{2i} E^j \tag{2.13}$$

does not degenerate on M.

Theorem 2.1. *Let M be a manifold carrying a regular quasihomogeneous flat pencil. Denote $M_0 \subset M$ the subset of M where the metric $(\ ,\ )_2$ is invertible. Define the multiplication of 1-forms on M_0 putting*

$$u \cdot v := \Delta\left(u, R^{-1} v\right). \tag{2.14}$$

Then there exists a unique Frobenius structure on M such that

$$< \, , \, >= (\, , \,)_2, \tag{2.15}$$

the multiplication of tangent vectors is $< \, , \, >$-dual to the product (2.14), the unity and the Euler vector fields have the form (0.11a) and (0.11b) resp., and the intersection form is equal to $(\, , \,)_1$.

Proof. Let us choose flat coordinates $t^1, \ldots, t^n$ for the metric $(\, , \,)_2$. The components of the metric in these coordinates are given by a constant symmetric invertible matrix

$$\eta^{\alpha\beta} := \left(dt^\alpha, dt^\beta\right)_2. \tag{2.16}$$

We also denote

$$g^{\alpha\beta}(t) := \left(dt^\alpha, dt^\beta\right)_1$$

and

$$K_\alpha^\beta := \partial_\alpha E^\beta. \tag{2.17}$$

This matrix is constant due to Lemma 2.3. The components of the contravariant Levi-Civita connection for the metric $g^{\alpha\beta}$ in these coordinates we denote $\Gamma_\gamma^{\alpha\beta}$. Recall that in this coordinate system

$$\Delta_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta}. \tag{2.18}$$

Lemma 2.4. *The vector field e is constant in the coordinates t^α. It is an eigenvector of the operator (2.17) with the eigenvalue $1-d$.*

Proof. Constancy of e follows from (2.9). Let us normalize the choice of the flat coordinates requiring that

$$e^\alpha = \eta^{\alpha\, n}. \tag{2.19}$$

In these coordinates

$$\tau = t^n + \text{const}. \tag{2.20}$$

So

$$E^\alpha(t) = g^{\alpha\, n}(t), \;\; \alpha = 1, \ldots, n. \tag{2.21}$$

From (0.12) we obtain

$$\eta^{n\,\epsilon} K_\epsilon^\alpha = \eta^{\alpha\, n}.$$

Using (2.14) we obtain

$$\eta^{n\,\epsilon} K_\epsilon^\alpha + \eta^{\alpha\epsilon} K_\epsilon^n = (2-d)\eta^{n\,\alpha}.$$

Hence

$$\eta^{\alpha\epsilon} K_\epsilon^n = (1-d)\eta^{n\,\alpha}.$$

Lowering the index α we prove Lemma.

We will use also below the choice (2.19) of the flat coordinate t^n. Then

$$K_\alpha^n = (1-d)\delta_\alpha^n. \tag{2.22}$$

Lemma 2.5. *In the coordinates* t^α

$$\Delta_\beta^{\alpha\, n} = \frac{1-d}{2}\delta_\beta^\alpha \tag{2.23}$$

$$\Delta_\beta^{n\,\alpha} = \frac{d-1}{2}\delta_\beta^\alpha + K_\beta^\alpha \tag{2.24}$$

Proof. From (0.13) it follows that

$$\mathcal{L}_E g_{\alpha\beta} = (1-d)g_{\alpha\beta}.$$

Using Christoffel formula one obtains

$$\begin{aligned}\Gamma_{\alpha\beta}^n &= \frac{1}{2}g^{n\,\epsilon}\left(\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\alpha\epsilon} - \partial_\epsilon g_{\alpha\beta}\right)\\ &= -\frac{1}{2}\mathcal{L}_E g_{\alpha\beta} = \frac{d-1}{2}g_{\alpha\beta}.\end{aligned}$$

Raising the index we obtain

$$\Gamma_\beta^{\alpha\, n} = \frac{1-d}{2}\delta_\beta^\alpha.$$

Due to (2.18) this proves (2.23). Using the equation (0.5)

$$\Gamma_\gamma^{n\,\alpha} + \Gamma_\gamma^{\alpha\, n} = \partial_\gamma g^{\alpha\, n} = K_\gamma^\alpha$$

we obtain (2.24). Lemma is proved.

Lemma 2.6.

$$\mathcal{L}_E \Delta_\gamma^{\alpha\beta} = (d-1)\Delta_\gamma^{\alpha\beta} \tag{2.25}$$

$$\mathcal{L}_e \Delta_\gamma^{\alpha\beta} = 0. \tag{2.26}$$

Proof. Denote

$$\begin{aligned}\tilde\Gamma_\gamma^{\alpha\beta} &:= \mathcal{L}_E \Delta_\gamma^{\alpha\beta} + (1-d)\Delta_\gamma^{\alpha\beta}\\ &\equiv \partial_E \Gamma_\gamma^{\alpha\beta} - K_\epsilon^\alpha \Gamma_\gamma^{\epsilon\beta} - \Gamma_\gamma^{\alpha\epsilon}K_\epsilon^\beta + K_\gamma^\epsilon \Gamma_\epsilon^{\alpha\beta} + (1-d)\Gamma_\gamma^{\alpha\beta}\end{aligned}$$

Differentiating the equations

$$\begin{aligned}\Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} &= \partial_\gamma g^{\alpha\beta}\\ g^{\alpha\epsilon}\Gamma_\epsilon^{\beta\gamma} &= g^{\beta\epsilon}\Gamma_\epsilon^{\alpha\gamma}\end{aligned} \tag{2.27}$$

along E we obtain, after simple calculations,

$$\begin{aligned}\tilde\Gamma_\gamma^{\alpha\beta} + \tilde\Gamma_\gamma^{\beta\alpha} &= 0\\ g^{\alpha\epsilon}\tilde\Gamma_\epsilon^{\beta\gamma} &= g^{\beta\epsilon}\tilde\Gamma_\epsilon^{\alpha\gamma}.\end{aligned} \tag{2.28}$$

Since the system (2.27) has unique solution for given $g^{\alpha\beta}$, the correspondent linear homogeneous system (2.28) has only trivial solution $\tilde\Gamma^{\alpha\beta}_\gamma = 0$. This proves (2.25). The equation (2.26) can be proved in a similar way.

Corollary 2.2. *Let u, v be two 1-forms covariantly constant w.r.t. ∇_2. Then the multiplication*

$$(u, v) \mapsto \Delta(u, v)$$

on T^*M satisfies the equations

$$\Delta(u, v) + \Delta(v, u) = \mathrm{d}(u, v) \tag{2.29}$$

$$\Delta(R(u), v) + \Delta(u, R(v)) = \mathrm{d}(u, R(v)). \tag{2.30}$$

We denote by Roman 'd' the differential of a function on M to avoid confusion with the charge d in the axiom **FM3**.

Proof. The first equation is due to the first line in (2.27) together with (2.18). The second one is a spelling of (2.25).

Let us now fix a point $t \in M_0$. We denote

$$V = T^*_{t_0}M.$$

The linear operator

$$\Lambda : V \to V, \quad \Lambda = \frac{d-2}{2}\mathbf{1} + K = -\frac{1}{2}\mathbf{1} + R \tag{2.31}$$

is skew-symmetric w.r.t. $(\ ,\ )_2 = <\ ,\ >$

$$< \Lambda u, v > + < u, \Lambda v >= 0. \tag{2.32}$$

Let

$$V = \oplus_\lambda V_\lambda \tag{2.33}$$

be the root decomposition of the space V w.r.t. the root subspaces of the operator Λ. The following elementary statement is wellknown

Lemma 2.7. *The root subspaces V_λ and V_μ are $<\ ,\ >$-orthogonal if $\lambda + \mu \neq 0$. The pairing*

$$<\ ,\ >: V_\lambda \times V_{-\lambda} \to \mathbf{C} \tag{2.34}$$

does not degenerate.

By the moment we have not used the regularity condition

$$\det R \neq 0. \tag{2.35}$$

If this condition holds true then

$$V_{-\frac{1}{2}} = 0. \tag{2.36}$$

Particularly, this implies that

$$d \neq 1. \tag{2.37}$$

Lemma 2.8. *The multiplication (2.14) on V is commutative.*

Proof. From (2.30) we derive the folowing property of the multiplication

$$u \cdot R(v) + R(u) \cdot v = \mathrm{d}(u, v). \tag{2.38}$$

To take the differential in the r.h.s. of the equation we continue the covectors u, v in a small neighbourhood of the point t_0 as ∇_2-constant 1-forms. It suffices to prove Lemma for $u \in V_\lambda$, $v \in V_\mu$.

Case 1: $\lambda + \mu + 1 \neq 0$. Let first u and v be the eigenvectors of Λ with the eigenvalues λ and μ resp. Then

$$R(u) = \left(\frac{1}{2} + \lambda\right) u, \quad R(v) = \left(\frac{1}{2} + \mu\right) v.$$

From (2.38) we obtain

$$(1 + \lambda + \mu) u \cdot v = \mathrm{d}(u, v). \tag{2.39}$$

This proves that $u \cdot v = v \cdot u$. Let $u^{(k)}$, $v^{(l)}$ be the adjoint vectors for the eigenvectors u and v of the heights k, l resp., i.e.,

$$\Lambda(u^{(k)}) = \lambda u^{(k)} + u^{(k-1)}, \ \Lambda(v^{(l)}) = \mu v^{(l)} + v^{(l-1)},$$

$$u^{(0)} = u, \ v^{(0)} = v, \ u^{(-1)} = v^{(-1)} = 0.$$

We use induction w.r.t. the sum of the heights $k + l$. Substituting in (2.38) $u \mapsto u^{(k)}$, $v \mapsto v^{(l)}$ we obtain

$$(1 + \lambda + \mu) u^{(k)} \cdot v^{(l)} + u^{(k)} \cdot v^{(l-1)} + u^{(k-1)} \cdot v^{(l)} = \mathrm{d}(u^{(k)}, v^{(l)}). \tag{2.40}$$

By induction

$$u^{(k)} \cdot v^{(l-1)} = v^{(l-1)} \cdot u^{(k)}, \ u^{(k-1)} \cdot v^{(l)} = v^{(l)} \cdot u^{(k-1)}.$$

This proves commutativity of $u^{(k)}$ and $v^{(l)}$.

Case 2. $\lambda + \mu + 1 = 0$. Again we use induction w.r.t. the sum of the weights. Let u, v be two eigenvectors. From (2.39) one obtains

$$\mathrm{d}(u, v) = 0.$$

Using (2.29) we conclude that

$$\Delta(u, v) + \Delta(v, u) = 0.$$

Hence

$$u \cdot v = \frac{\Delta(u, v)}{\frac{1}{2} + \lambda} = -\frac{\Delta(v, u)}{\frac{1}{2} + \lambda} = \frac{\Delta(v, u)}{\frac{1}{2} + \mu} = v \cdot u.$$

Now we prove commutativity of adjoint vectors. From the definition we have

$$u^{(k)} \cdot v^{(l)} = \frac{\Delta\left(u^{(k)}, v^{(l)}\right)}{\frac{1}{2}+\mu} - \frac{u^{(k)} \cdot v^{(l-1)}}{\frac{1}{2}+\mu}$$
$$v^{(l)} \cdot u^{(k)} = \frac{\Delta\left(v^{(l)}, u^{(k)}\right)}{\frac{1}{2}+\lambda} - \frac{v^{(l)} \cdot u^{(k-1)}}{\frac{1}{2}+\lambda}.$$

So

$$u^{(k)} \cdot v^{(l)} - v^{(l)} \cdot u^{(k)} = -\frac{\mathrm{d}\left(u^{(k)}, v^{(l)}\right)}{\frac{1}{2}+\lambda} + \frac{u^{(k)} \cdot v^{(l-1)} + v^{(l)} \cdot u^{(k-1)}}{\frac{1}{2}+\lambda}.$$

Applying (2.40) we derive that

$$\mathrm{d}\left(u^{(k)}, v^{(l)}\right) = u^{(k)} \cdot v^{(l-1)} + v^{(l)} \cdot u^{(k-1)}.$$

Lemma is proved.

End of the proof of Theorem. We obtained a symmetric multiplication on the cotangent planes T_t^*M

$$\left(dt^\alpha, dt^\beta\right) \mapsto dt^\alpha \cdot dt^\beta =: c_\gamma^{\alpha\beta}(t) dt^\gamma \tag{2.41}$$

where the coefficients $c_\gamma^{\alpha\beta}(t)$ are defined by this equation. The 1-form $dt^n = d\tau$ is the unity of this multiplication. Indeed, due to (2.23)

$$\Delta(dt^\alpha, dt^n) = \frac{1-d}{2} dt^\alpha.$$

But the 1-form dt^n is an eigenvector of Λ with the eigenvalue $-d/2$ (this follows from (2.22)). So

$$dt^\alpha \cdot dt^n = dt^\alpha$$

for any α. Associativity of the multiplication follows from the right-symmetry property (2.7) and from the commutativity.

By duality we obtain a commutative associative multiplication on T_tM

$$\partial_\alpha \cdot \partial_\beta = c_{\alpha\beta}^\gamma(t) \partial_\gamma$$

with

$$c_{\alpha\beta}^\gamma(t) = \eta_{\alpha\epsilon} c_\beta^{\epsilon\gamma}(t).$$

The vector e of the form (2.19) will be the unity of this multiplication. From commutativity of the multiplication and from (2.6) it follows that the tensor

$$< \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$$

is symmetric w.r.t. α, β, γ. From this and from (2.8) it follows that the gradient

$$\partial_\delta < \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$$

is symmetric w.r.t. all the four indices. This proves **FM2**.

The equation (2.10) implies (0.23). From the definition of K it follows that

$$\mathcal{L}_E K_\alpha^\beta = 0.$$

Hence

$$\mathcal{L}_E c_\gamma^{\alpha\beta} = (d-1) c_\gamma^{\alpha\beta}.$$

Lowering the index α we obtain

$$\mathcal{L}_E c_{\alpha\gamma}^\beta = c_{\alpha\gamma}^\beta.$$

This proves (0.29). The equation (0.30) follows from (2.11). So, we obtained a Frobenius structure on M_0. Finally, comparing the equation (2.38) with the second line in the formula (1.2) for the entries of the intersection form we conclude that the metric $g^{\alpha\beta}$ coincides with the intersection form of the Frobenius manifold. Theorem is proved.

Remark. In some cases the regularity assumption of non degenerateness of the operator $R = \frac{d-1}{2}\mathbf{1} + \nabla_2 E$ can be relaxed. For example, for $d = 1$ the operator R is always degenerate since

$$R(d\tau) = 0.$$

However, Theorem 2.1 remains valid under the assumption that the root subspace $V_{-\frac{1}{2}}$ (see Lemma 2.7 above) is exactly one-dimensional. Indeed, using the above construction we arrive at multiplication $u \cdot v$ defined for an arbitrary 1-form u and for any 1-form v that belongs to the image of R. The only 1-form not belonging to the image is $d\tau$. However, from (2.24) we obtain

$$\Delta(d\tau, v) = R(v).$$

So $d\tau$ is *left unity* of the multiplication. Defining $v \cdot d\tau = v$ we obtain the needed Frobenius structure (cf. [DZ1], proof of Theorem 2.1).

Example 2.1. Let W be an irreducible finite Coxeter group acting in the Euclidean space $\mathbf{R}^n$. Denote (,) the W-invariant Euclidean inner product on $\mathbf{R}^n$. According to Arnold [Arn] there exists a unique contravariant metric $(\ , \ )_1$ on the orbit space

$$M = \mathbf{C}^n / W$$

such that for any two W-invariant polynomials $p(x)$, $q(x)$

$$(dp, dq)_1 = (dp(x), dq(x)). \tag{2.42}$$

Here we consider dp, dq as 1-forms on the orbit space. The bilinear form degenerates on the discriminant $\Sigma \subset M$ consisting of all nonregular orbits.

The Euler vector field is defined by

$$E = \frac{1}{h} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \tag{2.43}$$

where $x_1, \ldots, x_n$ are Euclidean coordinates in $\mathbf{R}^n$ and h is the Coxeter number of W. Recall [Bour] that h is the maximum of the degrees of basic invariant polynomials $p_1(x)$, $\ldots$, $p_n(x)$, i.e., such homogeneous polynomials that

$$\mathbf{C}[x_1, \ldots, x_n]^W = \mathbf{C}[p_1, \ldots, p_n]$$

(Chevalley theorem). Let $\deg p_1(x) = h$. Introduce a vector field on the orbit space

$$e = \frac{\partial}{\partial p_1}. \tag{2.44}$$

It is well-defined up to multiplication by a nonzero constant factor. It was proved by K.Saito [Sa] (see also [SYS]) that the metric

$$(\ ,\ )_2 := \mathcal{L}_e(\ ,\ )_1 \tag{2.45}$$

is flat and it does not degenerate globally on M. The flat coordinates of this metric give a distinguished system of generators in the ring of W-invariant polynomials on $\mathbf{R}^n$ first discovered in [SYS]. In [Du3] it was shown (see also [Du4]) that the metrics $(\ ,\ )_1$ and $(\ ,\ )_2$ form a flat quasihomogeneous regular pencil of the degree

$$d = 1 - \frac{2}{h}. \tag{2.46}$$

The vector fields E and e have the form (2.43), (2.44), the function τ is

$$\tau = \frac{1}{2\,h}(x, x). \tag{2.47}$$

This produces a polynomial Frobenius structure on the orbit space [*ibid.*].

This construction was generalized in [DZ1] to produce a Frobenius structure on orbit spaces of certain extensions of affine Weyl groups. In this case $d = 1$ but the arguments of the above Remark work.

3. Flat pencils and bihamiltonian structures on loop spaces

We define loop space $L(M)$ of all smooth maps

$$S^1 \to M.$$

In a local coordinate system $x^1, \ldots, x^n$ any such a map is given by a 2π-periodic smooth vector-function $(x^1(s), \ldots, x^n(s))$. We will consider *local functionals*

$$I[x] = \frac{1}{2\pi} \int_0^{2\pi} P\left(x; \dot{x}, \ddot{x}, \ldots, x^{(m)}\right) ds \tag{3.1}$$

$$x = \left(x^1, \ldots, x^n\right),\ \dot{x} = \left(\dot{x}^1, \ldots, \dot{x}^n\right) = \frac{dx}{ds},\ \ldots$$

as “functions” on the loop space. Here $P\left(x;\dot x,\ddot x,\dots,x^{(m)}\right)$ is a polynomial in $\dot x$, $\ddot x$, ... with the coefficients smooth functions of x. The nonnegative integer m may depend on the functional.

A *local translation invariant Poisson bracket* on the loop space is, by definition, a Lie algebra structure on this space of functionals

$$(I_1, I_2) \mapsto \{I_1, I_2\}$$

with the Poisson bracket of the following form

$$\{I_1, I_2\} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I_1}{\delta x^i(s)} A^{ij} \frac{\delta I_2}{\delta x^j(s)}\, ds \tag{3.2}$$

where A^{ij} is a linear differential operator of some finite order N of the form

$$A^{ij} = \sum_{k=0}^{N} a_k^{ij}\left(x;\dot x,\ddot x,\dots,x^{(m_k)}\right)\frac{d^k}{ds^k}. \tag{3.3}$$

The variational derivatives are the functions defined by the usual rule

$$I[x+\delta x] - I[x] = \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta I}{\delta x^i(s)}\delta x^i(s)\, ds + O\left(||\delta x||^2\right). \tag{3.4a}$$

For the local functionals of the form (3.1) the variational derivatives are obtained by applying Euler - Lagrange operator

$$\frac{\delta I}{\delta x^i(s)} = \frac{\partial P}{\partial x^i} - \frac{d}{ds}\frac{\partial P}{\partial \dot x^i} + \frac{d^2}{ds^2}\frac{\partial P}{\partial \ddot x^i} - \dots. \tag{3.4b}$$

The coefficients $a_k^{ij}\left(x;\dot x,\ddot x,\dots,x^{(m_k)}\right)$ of the Poisson bracket must be polynomials in the derivatives $\dot x$, $\ddot x$ ... with smooth in x coefficients.

In computations with local Poisson brackets it is convenient to use the formalism of δ-functions introducing the matrix of distributions

$$\{x^i(s_1), x^j(s_2)\} = \sum_{k=0}^{N} a_k^{ij}\left(x(s_1);\dot x(s_1),\ddot x(s_1),\dots,x^{(m_k)}(s_1)\right)\delta^{(k)}(s_1-s_2). \tag{3.5}$$

Here $\delta(s)$ is the δ-function on the circle defined by the identity

$$\frac{1}{2\pi}\int_0^{2\pi} f(s)\delta(s)\, ds = f(0)$$

for any smooth 2π-periodic function $f(s)$. The derivatives of δ-function are defined in the standard way by the equations

$$\frac{1}{2\pi}\int_0^{2\pi} f(s_2)\delta^{(k)}(s_1-s_2)\, ds_2 = f^{(k)}(s_1).$$

The formula (3.2) for the Poisson bracket can be recasted into the form

$$\{I_1, I_2\} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\delta I_1}{\delta x^i(s_1)}\{x^i(s_1), x^j(s_2)\}\frac{\delta I_2}{\delta x^j(s_2)}\, ds_1\, ds_2. \tag{3.6}$$

To make the Poisson bracket independent on the choice of the local coordinates on M the coefficients a_k^{ij} must transform in an appropriate way with the changes of coordinates $y = y(x)$. The transformation law of the coefficients is determined by the Leibnitz identity for the Poisson bracket (3.5)

$$\{y^p(s_1), y^q(s_2)\} = \frac{\partial y^p}{\partial x^i}(s_1)\frac{\partial y^q}{\partial x^j}(s_2)\{x^i(s_1), x^j(s_2)\} \tag{3.7}$$

together with the following identities for the derivatives of δ-function

$$f(s_2)\delta^{(k)}(s_1 - s_2) = \sum_{l=0}^{k}\binom{k}{l} f^{(l)}(s_1)\delta^{(k-l)}(s_1 - s_2). \tag{3.8}$$

The constraints for the coefficients a_k^{ij} imposed by skew symmetry and by Jacobi identity for the Poisson bracket (3.2) can be written as a finite system of equations of degree one and two resp. for these coefficients and their derivatives.

Let us assign degrees to the derivatives putting

$$\deg \frac{d^k x^i}{ds^k} = k,\ \ k = 1,\, 2,\, \dots \tag{3.9a}$$

and we put

$$\deg\ f(u) = 0 \tag{3.9b}$$

for any function independent on the derivatives.

Definition 3.1. We say that the bilinear operation (3.2) (or (3.5)) is *graded homogeneous of the degree D* if the coefficients are graded homogeneous polynomials in the derivatives of the degrees

$$\deg\ a_k^{ij}\left(x; \dot{x}, \ddot{x}, \dots, x^{(m)}\right) = D - k,\ \ k = 0,\, 1,\, \dots \tag{3.10}$$

Clearly the order N of (3.5) cannot be greater than the degree D.

Lemma 3.1. *The degree D does not depend on the choice of local coordinates x^1, $\dots$, x^n.*

Proof easily follows from the transformation property (3.7) together with (3.8).

Example 3.1. The graded homogeneous Poisson bracket of degree 0 has the form

$$\{x^i(s_1), x^j(s_2)\} = h^{ij}(x)\delta(s_1 - s_2) \tag{3.11}$$

where $h^{ij}(x)$ is a usual (i.e., a finite-dimensional one) Poisson bracket on the manifold M.

Example 3.2. The graded homogeneous Poisson bracket of degree 1 has the form

$$\{x^i(s_1), x^j(s_2)\} = g^{ij}\left(x(s_1)\right)\dot\delta(s_1 - s_2) + \Gamma^{ij}_k(x)\dot x^k \delta(s_1 - s_2). \tag{3.12}$$

The coefficients $g^{ij}(x)$ and $\Gamma^{ij}_k(x)$ are some functions on M depending on the choice of local coordinates. This class of Poisson brackets is called *Poisson brackets of hydrodynamics type*. It was first introduced and studied in [DN1]. The following main result was proved in this paper (see also [DN3]).

Let us assume that the matrix $g^{ij}(x)$ does not degenerate on an open dense subset of M. (This assumption does not depend on the choice of local coordinates on M.) In this case we call the bracket (3.12) *nondegenerate*.

Theorem 3.1. *The graded homogeneous nondegenerate Poisson brackets of the degree 1 on the loop space $L(M)$ are in 1-to-1 correspondence with flat contravariant metrics $g^{ij}(x)$ on M. The coefficients $\Gamma^{ij}_k(x)$ in (3.12) must be the Levi-Civita contravariant connection of this metric.*

Remark 3.1. The flat coordinates $t^1, \ldots, t^n$ of the flat metric give the densities of Casimirs of the Poisson bracket (3.12)

$$C^\alpha = \frac{1}{2\pi}\int_0^{2\pi} t^\alpha(s)\, ds,\ \ \alpha = 1, \ldots, n \tag{3.13a}$$

$$\{I, C^\alpha\} = 0 \text{ for any functional } I. \tag{3.13b}$$

Doing a change of the dependent variables

$$x^i(s) \mapsto t^\alpha(x(s))$$

we rewrite the Poisson bracket (3.12) in the following constant form

$$\{t^\alpha(s_1), t^\beta(s_2)\} = \eta^{\alpha\beta}\dot\delta(s_1 - s_2) \tag{3.14}$$

where the constant coefficients $\eta^{\alpha\beta}$ are the entries of the matrix of the metric in the flat coordinates t.

The coefficients of graded homogeneous Poisson brackets of the degree $D > 1$ are also certain differential-geometric objects on the manifold M. These Poisson brackets were first introduced in [DN2] under the name *homogeneous differential-geometric Poisson brackets* (see also [DN3]).

We recall now the general definition of a compatible pair of Poisson brackets [Mag].

Definition 3.2. Two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ on the same space are called *compatible* if the linear combination

$$\{\ ,\ \}_1 - \lambda\{\ ,\ \}_2 \tag{3.15}$$

is a Poisson bracket for any λ.

Given a compatible pair of Poisson brackets one can construct certain family of commuting Hamiltonians. The Hamiltonians $H^{\alpha,p}$ are determined by the recursion relations

$$\{\,.\,, H^{\alpha,p}\}_1 = \{\,.\,, H^{\alpha,p+1}\}_2, \quad p = 0,\, 1,\, \ldots,\; \alpha = 1,\, \ldots, n \tag{3.16a}$$

starting from the Casimirs

$$H^{\alpha,0} = C^\alpha, \;\; \alpha = 1, \ldots, n \tag{3.16b}$$

of the second Posson bracket. By the construction the correspondent evolutionary systems admit a bi-hamiltonian structure

$$\frac{\partial x^i}{\partial T^{\alpha,p}} = \{x^i(s), H^{\alpha,p}\}_2 = \{x^i(s), H^{\alpha,p-1}\}_1, \quad \alpha = 1,\, \ldots, n, \quad p = 1,\, 2,\, \ldots.$$

In some cases it is possible to prove complete integrability of the bi-hamiltonian systems.

We prove now the following simple

Theorem 3.2. *Two graded homogeneous nondegenerate Poisson brackets of the degree 1 on the loop space $L(M)$ are compatible* iff *the correspondent flat metrics form a flat pencil.*

Proof. The linear combination (3.15) of two Poisson brackets of the form (3.12) reads

$$\begin{aligned}&\{x^i(s_1), x^j(s_2)\}_1 - \lambda\{x^i(s_1), x^j(s_2)\}_2 \\ &= \left[g_1^{ij}(x(s_1)) - \lambda g_2^{ij}(x(s_1))\right] \dot\delta(s_1 - s_2) + \left[{\Gamma_1}_k^{ij}(x) - \lambda {\Gamma_2}_k^{ij}(x)\right]\, \dot x^k \delta(s_1 - s_2).\end{aligned} \tag{3.17}$$

Now the proof immediately follows from Theorem 3.1.

Corollary 3.1. *The loop space $L(M)$ of any Frobenius manifold M carries a graded homogeneous of degree 1 nondegenerate bi-hamiltonian structure.*

This follows from Theorem 1.1.

Observe that, for $d \neq 1$, the variable

$$T(s) := \frac{2}{1-d}\tau(s) \tag{3.18}$$

where the flat coordinate τ was defined in (1.6) has the Poisson bracket with itself of the form

$$\{T(s_1), T(s_2)\}_1 = [T(s_1) + T(s_2)]\, \dot\delta(s_1 - s_2). \tag{3.19}$$

This coincides with the Poisson bracket on the dual space to the Lie algebra of one-dimensional vector fields (i.e., the Virasoro algebra with zero central charge). Other Poisson brackets of $T(s)$ are of the form, due to (2.21), (2.23)

$$\{t^\alpha(s_1), T(s_2)\}_1 = \frac{2}{1-d} E^\alpha(t(s_1))\dot\delta(s_1 - s_2) + \dot t^\alpha \delta(s_1 - s_2). \tag{3.20}$$

Recall that $E(t)$ depends linearly on t.

From the results of Section 2 above it follows that, under the assumption of quasihomogeneity and regularity, bihamiltonian structures (3.17) on the loop space $L(M)$ are in 1-to-1 correspondence with Frobenius structures on M.

The role of the quasihomogeneity condition in the theory of the degree 1 bihamiltonian structures on $L(M)$ could seem more motivated from the point of view of a general differential-geometric approach to classical W-algebras outlined in [DZ2]. In this approach we consider Poisson brackets of the form of formal series in an independent variable ϵ

$$\{x^i(s_1), x^j(s_2)\} = \sum_{k\geq 0} \epsilon^{2k} \{x^i(s_1), x^j(s_2)\}^{(k)} \tag{3.21}$$

where the k-th coefficient $\{\ ,\ \}^{(k)}$ must be a graded homogeneous operation of the degree $2k+1$. The skew symmetry and Jacobi identity for the bracket (3.21) must fulfill as an identity for formal power series in ϵ^2. The main requirement is that the Poisson bracket (3.21) must be reducible to the constant form (3.14) by a transformation

$$x^i = x^i(t) + \sum_{k\geq 1} \epsilon^k Q_k^i(t; \dot{t}, \ddot{t}, \dots) \tag{3.22}$$

where the coefficients $Q_k^i(t; \dot{t}, \ddot{t}, \dots)$ must be graded homogeneous polynomials of the degree k in the derivatives $\dot{t}$, $\ddot{t}$, ... Particularly, the leading term $\{\ ,\ \}^{(0)}$ of (3.21) is a graded homogeneous Poisson bracket of degree 1. Validity of Jacobi identity for (3.21) implies that $\{\ ,\ \}^{(0)}$ is a Poisson bracket. So, under the nondegeneracy condition for this bracket, the leading term in (3.22) is given by the flat coordinates of the correspondent contravariant metric.

We also bring attention of the reader to the construction of [DN1] (justified in a recent paper [Mal]) of "averaged Poisson brackets" used to describe Hamiltonian structure of Whitham equations. Particularly, according to this construction, the leading term in the small dispersion expansion of an arbitrary local Poisson bracket posessing of a sufficiently rich family of commuting local Hamiltonians is always given by a degree 1 graded homogeneous Poisson bracket. So, (3.21) can be considered as the full small dispersion expansion of the original Poisson bracket.

W-algebras were discovered by A.Zamolodchikov [Za] in order to describe additional symmetries of conformal field theories with spin greater than 1/2. It was realized by Fateev and Lukyanov [FL] that the semiclassical limit of W-algebras coincides with the second Poisson bracket of Gelfand - Dickey integrable hierarchy. These semiclassical limits of W-algebras were constructed for all simple Lie groups using Drinfeld - Sokolov construction of the corresponding integrable hierachies [DS]. They were called *classical W-algebras* (see also [DIZ], [Bouw]). The role of the first Poisson bracket of the hierarchy looked not to be relevant in the construction. However, it will be important in our differential-geometric approach to classical W-algebras and their generalization. Recall that any Poisson bracket (3.21) by the assumption has no invariants w.r.t. the transformations of the form (3.22).

By our definition (see [DZ2]) a classical W algebra is a pair of Poisson brackets of the form (3.21) such that the linear combination $\{\ ,\ \}_1 - \lambda\{\ ,\ \}_2$ for any λ is again a Poisson bracket satisfying the above reducibility condition. We also require validity of certain quasihomogeneity conditions for the coefficients of the Poisson brackets. We begin with the leading terms $\{\ ,\ \}_1^{(0)}$ and $\{\ ,\ \}_2^{(0)}$. The requirement is that this compatible pair of the degree 1 Poisson brackets corresponds to a quasihomogeneous regular pencil of metrics on M. To motivate this requirement we recall that, according to (3.19), (3.20) the Poisson bracket $\{\ ,\ \}_1^{(0)}$ is a nonlinear chiral extension of the conformal Virasoro algebra with the central charge 0. The nonzero central charge will arrive with the ϵ^2-correction (see below).

Using Theorem 2.2 we see that the leading term in the ϵ^2-expansion of a classical W-algebra is determined by a Frobenius structure on M. Let E, e, d be resp. the Euler and the unity vector fields and the charge of the Frobenius manifold. We will write the quasihomogeneity conditions for the coefficients of the Poisson brackets using the flat coordinates t^α on the Frobenius manifold.

Let

$$\{t^\alpha(s_1), t^\beta(s_2)\}_1^{(k)} = \sum_l a_{k,\,l}^{\alpha\beta}(t; \dot{t}, \ddot{t}, \dots)\delta^{(l)}(s_1 - s_2) \tag{3.23}$$

$$\{t^\alpha(s_1), t^\beta(s_2)\}_2^{(k)} = \sum_l b_{k,\,l}^{\alpha\beta}(t; \dot{t}, \ddot{t}, \dots)\delta^{(l)}(s_1 - s_2). \tag{3.24}$$

1. We require that

$$\mathcal{L}_e a_{k,\,l}^{\alpha\beta} = b_{k,\,l}^{\alpha\beta} \tag{3.25}$$

$$\mathcal{L}_e b_{k,\,l}^{\alpha\beta} = 0, \tag{3.26}$$

2. Let us introduce the prolungated vector field

$$\mathcal{E} = E - \sum_{m\geq 1}\sum_{\alpha,\,\beta}\left(m\,\delta_\beta^\alpha + K_\beta^\alpha\right) t^{\beta(m)}\frac{\partial}{\partial t^{\alpha(m)}} \tag{3.27}$$

(the matrix K_β^α was introduced in (2.17)). Then we require that

$$\mathcal{L}_{\mathcal{E}} a_{k,\,l}^{\alpha\beta} = (k(d-3)+l)a_{k,\,l}^{\alpha\beta} + \Lambda_\epsilon^\alpha a_{k,\,l}^{\epsilon\beta} + a_{k,\,l}^{\alpha\epsilon}\Lambda_\epsilon^\beta \tag{3.28}$$

$$\mathcal{L}_{\mathcal{E}} b_{k,\,l}^{\alpha\beta} = (k(d-3)+l-1)b_{k,\,l}^{\alpha\beta} + \Lambda_\epsilon^\alpha b_{k,\,l}^{\epsilon\beta} + b_{k,\,l}^{\alpha\epsilon}\Lambda_\epsilon^\beta. \tag{3.29}$$

3. The first Poisson bracket of the field $T(s)$ given by (3.18) has the Virasoro form

$$\{T(s_1), T(s_2)\}_1 = [T(s_1) + T(s_2)]\dot{\delta}(s_1 - s_2) + \epsilon^2\frac{c}{12}\delta^{(3)}(s_1 - s_2) + O(\epsilon^4). \tag{3.30}$$

The number c is called *central charge* of the classical W-algebra.

For the clasical W-algebras corresponding to the simple Lie groups the sums in (3.21) are finite. All the coefficients $a_{k,\,l}^{\alpha\beta}$ are polynomials also in t. (Observe that in our notations the first and the second Poisson structures are interchanged.) The central charge c is equal [FL] to

$$c = 12\rho^2 \tag{3.31}$$

where ρ is half of the sum of positive roots of the root system of the Lie algebra.

In [DZ2] it was shown that for any semisimple Frobenius manifold there exists a germ of order 1 (i.e., the first two terms in (3.21)) of a classical W algebra with the central charge

$$c = \frac{12}{(1-d)^2}\left[\frac{n}{2} - 2\mathrm{tr}\,\Lambda^2\right]. \tag{3.32}$$

Remarkably, for the Frobenius manifolds on the orbit spaces of simply-laced Weyl groups (see Example 2.1 above) the formulae (3.31) and (3.32) give the same result!

The corrections $\{\ ,\ \}_1^{(1)}$ and $\{\ ,\ \}_2^{(1)}$ are uniquely determined by the axioms of Dijkgraaf - Witten [DW] and of Getzler [Ge] from two-dimensional topological field theory (see explicit formulae in [DZ2]). The structure of higher order corrections to these brackets remains unknown. Understanding of this structure could clarify the eventual role of Frobenius manifolds in the problem of classification of integrable hierarchies. It will also solve the problem of the genus expansion in topological field theories (see discussion of this problem in [DZ2]).

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