ON ALMOST DUALITY FOR FROBENIUS MANIFOLDS

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Dedicated to Sergei Petrovich Novikov
on the occasion of his 65th birthday.

Abstract. We present a universal construction of almost duality for Frobenius manifolds. The analytic setup of this construction is described in details for the case of semisimple Frobenius manifolds. We illustrate the general considerations by examples from the singularity theory, mirror symmetry, the theory of Coxeter groups and Shephard groups, from the Seiberg - Witten duality.

1. Introduction

The beauty of the theory of Frobenius manifold is not only in multiple connections of it with other branches of mathematics, such as quantum cohomology, singularity theory, the theory of integrable systems. Even more amazing is that, some properties discovered in the study of particular classes of Frobenius manifolds often turn out to become *universal structures* of the theory thus proving to be important also for other classes of Frobenius manifolds.

In this paper we describe one of these universal structures. We call it almost duality; it associates to a given Frobenius manifold a somewhat different creature we called almost Frobenius manifold. Its crucial part is still in the WDVV associativity equations; however the properties of the unity and Euler vector fields are to be modified.

For the Frobenius structures on the base of universal unfoldings of isolated hypersurface singularities the duality is based on the wellknown correspondence between complex oscillatory integrals and periods of closed forms over vanishing cycles. For quantum cohomologies the duality seems to be closely related to the mirror construction; at least this is the case in simple examples. In the setting of the theory of integrable systems the duality generalizes the property of the classical Miura transformation between KdV and modified KdV equations.

The paper is organized as follows. In Section 2 we recall necessary information from the general theory of Frobenius manifolds. In 3 we introduce the almost duality and prove the Reconstruction Theorem that inverts the duality. The main results of Section 4 are devoted to the analytic properties of the so-called deformed flat coordinates on the almost dual to a semisimple Frobenius manifold describing them in terms of the monodromy data of the latter. Finally in Section 5 we consider examples and applications of almost duality.

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2. A brief introduction into Frobenius manifolds

In this section we will collect necessary definitions and geometric constructions of the theory of Frobenius manifolds. We will closely follow [12, 14].

2.1. **Deformed flat connection, deformed flat coordinates, and spectrum of a Frobenius manifold.** A *Frobenius algebra* is a pair $(A, < \, , \, >)$ where A is a commutative associative algebra with a unity over a field k (we will consider only the cases $k = \mathbb{R}, \mathbb{C}$) and $< \, , \, >$ is a k-bilinear symmetric nondegenerate *invariant* form on A, i.e.,

$$< x \cdot y, z >=< x, y \cdot z >$$

for arbitrary vectors x, y, z in A.

Definition 1. *Frobenius structure* of the charge d on the manifold M is a structure of a Frobenius algebra on the tangent spaces $T_tM = (A_t, < \, , \, >_t)$ depending (smoothly, analytically etc.) on the point $t \in M$. It must satisfy the following axioms.

FM1. The metric $< \, , \, >_t$ on M is flat (but not necessarily positive definite). Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be flat,

$$\nabla e = 0. \tag{2.1}$$

FM2. Let c be the 3-tensor $c(x, y, z) :=< x \cdot y, z >$, x, y, $z \in T_tM$. The 4-tensor $(\nabla_w c)(x, y, z)$ must be symmetric in x, y, z, $w \in T_tM$.

FM3. A linear vector field $E \in Vect(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$, such that

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y$$
$$E < x, y > - < [E, x], y > - < x, [E, y] >= (2 - d) < x, y > .$$

The last condition means that the derivations $Q_{Func(M)} := E$, $Q_{Vect(M)} := \mathrm{id} + \mathrm{ad}_E$ define on the space $Vect(M)$ of vector fields on M a structure of graded Frobenius algebra over the graded ring of functions $Func(M)$ (see details in [14]).

Flatness of the metric $< \, , \, >$ implies local existence of a system of *flat coordinates* $t^1, \ldots, t^n$ on M. We will denote $\eta_{\alpha\beta}$ the constant Gram matrix of the metric in these coordinates

$$\eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} \right\rangle.$$

The inverse matrix $\eta^{\alpha\beta}$ defines the inner product on the cotangent planes

$$< dt^\alpha, dt^\beta >= \eta^{\alpha\beta}.$$

The flat coordinates will be chosen in such a way that the unity e of the Frobenius algebra coincides with $\partial/\partial t^1$

$$e = \frac{\partial}{\partial t^1}.$$

In these flat coordinates on M the structure constants of the Frobenius algebra $A_t = T_tM$

$$\frac{\partial}{\partial t^\alpha} \cdot \frac{\partial}{\partial t^\beta} = c_{\alpha\beta}^\gamma(t) \frac{\partial}{\partial t^\gamma} \tag{2.2}$$

can be locally represented by third derivatives of a function $F(t)$,

$$c_{\alpha\beta}^{\gamma}(t) = \eta^{\gamma\epsilon} \frac{\partial^3 F(t)}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}$$

also satisfying

$$\frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta} \equiv \eta_{\alpha\beta}. \tag{2.3}$$

The function $F(t)$ is called *potential* of the Frobenius manifold. It is defined up to adding of an at most quadratic polynomial in $t^1, \ldots, t^n$. It satisfies the following system of *WDVV associativity equations*

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F(t)}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\alpha} \tag{2.4}$$

for arbitrary $1 \leq \alpha, \beta, \gamma, \delta \leq n$.

The vector field E is called *Euler vector field.* In the flat coordinates it must have the form

$$E = \left(a_\beta^\alpha t^\beta + b^\alpha\right) \frac{\partial}{\partial t^\alpha}$$

for some constants a_β^α, b^α satisying

$$a_1^\alpha = \delta_1^\alpha, \ b^1 = 0.$$

The potential $F(t)$ is a quasihomogeneous function in the following sense

$$E\,F = (3-d)F + \frac{1}{2} A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C \tag{2.5}$$

where $A_{\alpha\beta}$, B_α, C are some constants and d is the charge of the Frobenius manifold.

One of the main geometrical structures of the theory of Frobenius manifolds is the *deformed flat connection.* This is a symmetric affine connection $\tilde\nabla$ on $M \times \mathbb{C}^*$ defined by the following formulae

$$\tilde\nabla_x\, y = \nabla_x y + z\, x \cdot y, \ \ x, y \in TM, \ \ z \in \mathbb{C}^* \tag{2.6}$$

$$\tilde\nabla_{\frac{d}{dz}} y = \partial_z y + E \cdot y - \frac{1}{z} \mathcal{V} y$$

$$\tilde\nabla_x \frac{d}{dz} = \tilde\nabla_{\frac{d}{dz}} \frac{d}{dz} = 0 \tag{2.7}$$

where

$$\mathcal{V} := \frac{2-d}{2} - \nabla E \tag{2.8}$$

is an antisymmetric operator on the tangent bundle TM w.r.t. $<\ ,\ >$,

$$< \mathcal{V}\, x, y >= - < x, \mathcal{V}\, y > .$$

Observe that the unity vector field e is an eigenvector of this operator with the eigenvalue

$$\mathcal{V} e = -\frac{d}{2} e.$$

Vanishing of the curvature of the connection $\tilde\nabla$ is essentially equivalent to the axioms of Frobenius manifold.

Definition 2. A function $f = f(t;z)$ defined on an open subset in $M \times \mathbb{C}^*$ is called $\tilde\nabla$-*flat* if

$$\tilde\nabla df = 0. \tag{2.9}$$

Introducing the row vector $\xi = (\xi_1, \dots, \xi_n)$ where

$$\xi_\alpha := \frac{\partial f}{\partial t^\alpha}$$

we represent the equations for $\tilde\nabla$-flatness in the form

$$\partial_\alpha \xi = z\, \xi\, C_\alpha(t) \tag{2.10}$$

$$\partial_z \xi = \xi \left(\mathcal{U}(t) - \frac{\mathcal{V}}{z} \right). \tag{2.11}$$

Here we introduce the matrix $\mathcal{U}(t)$ of multiplication by the Euler vector field

$$\mathcal{U}^\alpha_\beta(t) := E^\epsilon(t) c^\alpha_{\epsilon\beta}(t) \tag{2.12}$$

and the matrix $C_\alpha(t)$ of multiplication by $\partial_\alpha := \partial/\partial t^\alpha$

$$(C_\alpha(t))^\beta_\gamma := c^\beta_{\alpha\gamma}(t). \tag{2.13}$$

The same system can be rewritten for the column vector $y = (y^1, \dots, y^n)^T$,

$$y^\alpha = \eta^{\alpha\beta}\xi_\beta$$

in the form

$$\partial_\alpha y = z\, C_\alpha(t) y \tag{2.14}$$

$$\partial_z y = \left(\mathcal{U}(t) + \frac{\mathcal{V}}{z} \right) y. \tag{2.15}$$

Because of vanishing of the torsion and curvature of the connection $\tilde\nabla$ there locally exist, on $M \times \mathbb{C}^*$ n independent flat functions $\tilde t^1(t;z), \dots, \tilde t^n(t;z)$. They are called *deformed flat coordinates* on a Frobenius manifold. The analytic properties of deformed flat coordinates as multivalued functions of $z \in \mathbb{C}^*$ can be used for describing moduli of semisimple Frobenius manifolds (see the next Section). Here we briefly describe the behaviour near the regular singularity $z = 0$ of a particular basis of deformed flat coordinates (see details in [14]).

To fix a system of the deformed flat coordinates we are to choose a basis in the space of solutions to the system (2.14), (2.15). Such a basis corresponds to a choice of a representative in the equivalence class of *normal forms* of the system (2.15) near $z = 0$ (see details in [14]). The parameters of such a normal form are called *spectrum* of the Frobenius manifold. Let us first recall the description of the parameters.

Definition 3. The *spectrum* of a Frobenius manifold is a quadruple $(V, < \ , \ >, \hat\mu, R)$ where V is a n-dimensional linear space over $\mathbb{C}$ equipped with a symmetric non-degenerate bilinear form $< \ , \ >$, semisimple antisymmetric linear operator $\hat\mu : V \to V$, $< \hat\mu\, a, b > + < a, \hat\mu\, b >= 0$ and a nilpotent linear operator $R : V \to V$ satisfying the following properties. First,

$$R^* = -e^{\pi i \hat\mu} R e^{-\pi i \hat\mu} \tag{2.16}$$

Observe the following consequence of (2.16)

$$R e^{2\pi i \hat\mu} = e^{2\pi i \hat\mu} R. \tag{2.17}$$

In particular, R leaves invariant the eigensubspaces of the operator $e^{2\pi i \hat\mu}$. It *does not* leave invariant the eigensubspaces V_μ of the operator $\hat\mu$. However,

$$R\, V_\mu \subset \oplus_{m\in\mathbb{Z}} V_{\mu+m} \tag{2.18}$$

The crucial condition in the definition of the spectrum is that, the operator R must also be *$\hat\mu$-nilpotent*, i.e., in the decomposition (2.18) only nonnegative integers m are present. We define the components of the operator R

$$\begin{aligned} &R = R_0 + R_1 + R_2 + \dots \\ &R_m V_\mu \subset V_{\mu+m}, \text{ for any } \mu \in \operatorname{Spec} \hat\mu. \end{aligned} \tag{2.19}$$

By the construction the operator R satisfies

$$z^{\hat\mu} R z^{-\hat\mu} = R_0 + R_1 z + R_2 z^2 + \dots. \tag{2.20}$$

Observe also the following useful identity

$$[\hat\mu, R_k] = k\, R_k,\ \ k = 0, 1, \dots. \tag{2.21}$$

Any polynomial of the matrices R_k can be uniquely decomposed as follows

$$P(R_0, R_1, \dots) = [P(R_0, R_1, \dots)]_0 + [P(R_0, R_1, \dots)]_1 + \dots \tag{2.22}$$

$$z^{\hat\mu} [P(R_0, R_1, \dots)]_m z^{-\hat\mu} = z^m [P(R_0, R_1, \dots)]_m. \tag{2.23}$$

The last restriction for the spectrum is that, the eigenvector $e \in V$ of $\hat\mu$ must satisfy $R_0\, e = 0$.

We will now explain how to associate a 5-tuple $(V, < \ ,\ >, \hat\mu, R, e)$ to a Frobenius manifold. The linear space V with a symmetric nondegenerate bilinear form $< \ ,\ >$ and a vector $e \in V$ have already been constructed above. Denote $\hat\mu : V \to V$ the semisimple part of the operator $\mathcal{V}$, i.e.,

$$\hat\mu := \oplus_{\mu\in Spec\, \mathcal{V}} \mu\, P_\mu \tag{2.24}$$

where $P_\mu : V \to V_\mu$ is the projector of V onto the root subspace of $\mathcal{V}$

$$V = \oplus_{\mu\in Spec\, \mathcal{V}} V_\mu,\ \ V_\mu := \operatorname{Ker} (\mathcal{V} - \mu\cdot 1)^n,$$

$P_\mu(V_{\mu'}) = 0$ for $\mu \neq \mu'$, $P_\mu|_{V_\mu} = \mathrm{id}_{V_\mu}$. Clearly the operator $\hat\mu$ is antisymmetric, $\hat\mu^* = -\hat\mu$. Denote R_0 the nilpotent part of $\mathcal{V}$

$$\mathcal{V} = \hat\mu + R_0.$$

Other operators R_1, R_2, ... are not determined by $\mathcal{V}$ only. They appear only in presence of resonances, i.e., pairs of eigenvalues μ, μ' of $\mathcal{V}$ such that $\mu - \mu' \in \mathbb{Z}_{>0}$ (see details in [14]).

Let us choose a basis $e_1, \dots, e_n$ in V such that $e_1 = e$. The matrices of the linear operators $\hat\mu$ and R we will denote by the same symbols.

Theorem 2.1. *For a sufficiently small ball $B \in M$ there exists a fundamental matrix of solutions to the system (2.14), (2.15) of the form*

$$y(t;z) = \Theta(t;z) z^{\hat{\mu}} z^R = \sum_{p \geq 0} \Theta_p(t) z^{p+\hat{\mu}} z^R \tag{2.25}$$

such that the matrix valued function $\Theta(t;z) : V \to V$ is analytic on $B \times \mathbb{C}$ satisfying

$$\Theta(t;0) \equiv 1 \tag{2.26}$$

$$\Theta^*(t;-z)\Theta(t;z) \equiv 1. \tag{2.27}$$

By the construction the columns of the fundamental matrix $Y(t;z)$ are gradients of a certain system of deformed flat coordinates

$$(\tilde{t}(t;z), \dots, \tilde{t}(t;z)) = (\theta_1(t;z), \dots, \theta_n(t;z)) z^{\hat{\mu}} z^R \tag{2.28}$$

where

$$\theta_\alpha(t;z) = \sum_{p=0}^{\infty} \theta_{\alpha,p}(v) z^p, \ \alpha = 1, \dots, n. \tag{2.29}$$

Definition 4. We will call (2.28) *Levelt system of deformed flat coordinates* on M at $z = 0$.

The ambiguity in the choice of the Levelt basis of deformed flat coordinates is described in [14].

2.2. **Canonical coordinates on semisimple Frobenius manifolds. Definition 5.** The Frobenius manifold M is called *semisimple* if the family of n-dimensional algebras (2.2) is semisimple for any $t = (t^1, \dots, t^n) \in M_{ss}$ for an open dense subset $M_{ss} \subset M$.

In this Section we summarize, following [10, 12] a very efficient technique of working with semisimple Frobenius manifolds based on introduction of *canonical coordinates.*

Let M be a semisimple Frobenius manifold. Denote $M_{ss} \subset M$ the open dense subset in M consisting of all points $t \in M$ s.t. the operator $\mathcal{U}(t)$ of multiplication by the Euler vector field

$$\mathcal{U}(t) = E(t) \cdot \ : T_t M \to T_t M$$

has simple spectrum. (Actually, the subset $M_{ss} \subset M$ could be slightly smaller than the set of all points of semisimplicity of the algebra (2.2). It can be shown however that, for an analytic Frobenius manifold this is still an open dense subset.) Denote $u_1(t), \dots, u_n(t)$ the eigenvalues of this operator, $t \in M_{ss}$.

Theorem 2.2. *The mapping*

$$M_{ss} \to \left(\mathbb{C}^n \setminus \cup_{i<j} (u_i = u_j)\right)/S_n, \ t \mapsto (u_1(t), \dots, u_n(t))$$

is an unramified covering. Therefore one can use the eigenvalues as local coordinates on M_{ss}. In these coordinates the multiplication table of the algebra (2.2) becomes

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}. \tag{2.30}$$

The basic idempotents $\partial/\partial u_i$ are pairwise orthogonal

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle = 0, \ i \neq j.$$

Observe that we violate the indices convention labelling the canonical coordinates by subscripts. We will *never* use summation over repeated indices when working in the canonical coordinates.

Choosing locally branches of the square roots

$$\psi_{i1}(u) := \sqrt{< \partial/\partial u_i, \partial/\partial u_i >}, \ i = 1, \dots, n \tag{2.31}$$

we obtain a transition matrix $\Psi = (\psi_{i\alpha}(u))$ from the basis $\partial/\partial t^\alpha$ to the orthonormal basis

$$f_1 = \psi_{11}^{-1}(u) \frac{\partial}{\partial u_1}, \ f_2 = \psi_{21}^{-1}(u) \frac{\partial}{\partial u_2}, \dots, \ f_n = \psi_{n1}^{-1}(u) \frac{\partial}{\partial u_n} \tag{2.32}$$

of the normalized idempotents

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \frac{\psi_{i\alpha}(u)}{\psi_{i1}(u)} \frac{\partial}{\partial u_i}. \tag{2.33}$$

Equivalently, the Jacobi matrix has the form

$$\frac{\partial u_i}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}. \tag{2.34}$$

The matrix $\Psi(u)$ satisfies orthogonality condition

$$\Psi^*(u)\Psi(u) \equiv 1, \ \Psi^* := \eta^{-1}\Psi^T\eta, \ \eta = (\eta_{\alpha\beta}), \ \eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} \right\rangle. \tag{2.35}$$

In this formula Ψ^T stands for the transposed matrix. The lengths (2.31) coincide with the first column of this matrix. So, the metric $< \ , \ >$ in the canonical coordinates reads

$$< \ , \ > = \sum_{i=1}^n \psi_{i1}^2(u) du_i^2. \tag{2.36}$$

The inverse Jacobi matrix can be computed using (2.35). This gives

$$\frac{\partial t_\alpha}{\partial u_i} = \psi_{i1}\psi_{i\alpha}, \quad t_\alpha := \eta_{\alpha\beta} t^\beta. \tag{2.37}$$

Denote $V(u) = (V_{ij}(u))$ the matrix of the antisymmetric operator $\mathcal{V}$ (2.8) w.r.t. the orthonormal frame (2.32)

$$V(u) := \Psi(u)\, \mathcal{V}\, \Psi^{-1}(u). \tag{2.38}$$

The matrix $V(u)$ satisfies the following system of commuting time-dependent Hamiltonian flows on the Lie algebra $so(n)$ equipped with the standard Lie - Poisson bracket

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V; u)\}, \ i = 1, \dots, n \tag{2.39}$$

with quadratic Hamiltonians

$$H_i(V;u) = \frac{1}{2}\sum_{j\neq i} \frac{V_{ij}^2}{u_i - u_j}. \tag{2.40}$$

The matrix $\Psi(u)$ satisfies

$$\frac{\partial \Psi}{\partial u_i} = V_i(u)\Psi, \; V_i(u) := \mathrm{ad}_{E_i}\mathrm{ad}_U^{-1}(V(u)). \tag{2.41}$$

Here

$$U = \mathrm{diag}\,(u_1, \dots, u_n) = \Psi\,\mathcal{U}\Psi^{-1}, \tag{2.42}$$

the matrix unity E_i has the entries

$$(E_i)_{ab} = \delta_{ai}\delta_{ib}. \tag{2.43}$$

The system (2.39) coincides with the equations of isomonodromy deformations of the following linear differential operator with rational coefficients

$$\frac{dY}{dz} = \left(U + \frac{V}{z}\right) Y. \tag{2.44}$$

The latter is nothing but the last component of the deformed flat connection (2.7) written in the orthonormal frame (2.32). Namely, the solutions to (2.44) are related to the gradients of the $\tilde\nabla$-flat functions satisfying (2.15) by

$$Y = \Psi^{-1} y. \tag{2.45}$$

The first part (2.14) after the gauge transformation (2.45) reads

$$\partial_i Y = (z\,E_i + V_i)Y. \tag{2.46}$$

In particular the fundamental matrix (2.25) gives the fundamental matrix of solutions to (2.44), (2.46) of the form

$$\begin{aligned} Y_0(u;z) &= \Psi^{-1}(u)\Theta(t(u);z)z^{\hat\mu}z^R \\ &= \sum_{p\geq 0} \Theta_p(t(u))z^{p+\hat\mu}z^R. \end{aligned} \tag{2.47}$$

The integration of (2.39), (2.41) and, more generally, the reconstruction of the Frobenius structure can be reduced to a solution of certain Riemann - Hilbert problem [14].

2.3. **Intersection form, discriminant, and periods of a Frobenius manifold.** We now recall some important points of the theory of intersection form on a Frobenius manifold and of the corresponding period mapping (see [12, 13, 14]). Intersection form is a symmetric bilinear form on T^*M. It is defined by the formula

$$(\omega_1, \omega_2) = i_E(\omega_1 \cdot \omega_2), \; \omega_1, \, \omega_2 \in T_t^*M. \tag{2.48}$$

The product $\omega_1 \cdot \omega_2$ of 1-forms is induced from the product of vectors in T_tM by the isomorphism

$$<\,,\;>: T_tM \to T_t^*M.$$

This means that, in the flat coordinates for $<\ ,\ >$,

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta}(t)dt^\gamma,\ c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\nu}\frac{\partial^3 F(t)}{\partial t^\lambda \partial t^\nu \partial t^\gamma} = \eta^{\alpha\lambda}c_{\lambda\gamma}^\beta(t). \tag{2.49}$$

For the Gram matrix

$$g^{\alpha\beta}(t) := \left(dt^\alpha, dt^\beta\right)$$

of the bilinear form (2.48) one obtains

$$\begin{aligned} g^{\alpha\beta}(t) &= E^\epsilon(t)c_\epsilon^{\alpha\beta}(t) \\ &= F^{\alpha\beta}(t) - \mathcal{V}_\rho^\alpha F^{\rho\beta}(t) - F^{\alpha\rho}(t)\mathcal{V}_\rho^\beta + A^{\alpha\beta} \end{aligned} \tag{2.50}$$

Here

$$F^{\alpha\beta}(t) = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial^2 F(t)}{\partial t^{\alpha'}\partial t^{\beta'}},$$

$$A^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}A_{\alpha'\beta'}$$

and the constant matrix $(A_{\alpha\beta})$ was defined in (2.5).

Proposition 2.3. *For any complex parameter λ denote $\Sigma_\lambda \subset M$ the subset*

$$\Sigma_\lambda := \{t \in M \mid \det\left[(\ ,\ )_t - \lambda <\ ,\ >_t\right] = 0\}. \tag{2.51}$$

It is a proper analytic subset (i.e., $\det\left(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}\right)$ does not vanish identically on M). On $M \setminus \Sigma_\lambda$ the inverse matrix

$$g_{\alpha\beta}(t;\lambda) := \left(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}\right)^{-1}$$

defines a flat metric that we denote $(\ ,\ )_\lambda$. The Levi-Civita connection of this metric has the Christoffel coefficients $\Gamma_{\beta\gamma}^\alpha(t;\lambda)$ of the form

$$\Gamma_{\beta\gamma}^\alpha(t;\lambda) = -g_{\beta\rho}(t;\lambda)\Gamma_\gamma^{\rho\alpha}(t) \tag{2.52}$$

where

$$\Gamma_\gamma^{\alpha\beta}(t) = c_\gamma^{\alpha\epsilon}(t)\left(\frac{1}{2} - \mathcal{V}\right)_\epsilon^\beta. \tag{2.53}$$

The functions $\Gamma_\gamma^{\alpha\beta}(t)$ are defined everywhere on M (but not only on $M \setminus \Sigma_\lambda$). We will call them *contravariant Christoffel coefficients* of the metric $(\ ,\ )_\lambda$.

Remark 1. The formulae of Proposition imply that the metrics $(\ ,\ )$ and $<\ ,\ >$ on T^*M form *flat pencil* [12, 13]. This means that:

1). The contravariant Christoffel coefficients for an arbitary linear combination $(\ ,\ ) - \lambda <\ ,\ >$ on T^*M are

$$\Gamma^{\cdot\cdot}_{\cdot\,(\ ,\ )} - \lambda\Gamma^{\cdot\cdot}_{\cdot\,<\ ,\ >} \tag{2.54}$$

where $\Gamma^{\cdot\cdot}_{\cdot\,(\ ,\ )}$ and $\Gamma^{\cdot\cdot}_{\cdot\,<\ ,\ >}$ are the contravariant Christoffel coefficients of the Levi-Civita connections $\nabla^{(\ ,\ )}$ and $\nabla^{<\ ,\ >}$ for the metrics $(\ ,\ )$ and $<\ ,\ >$ resp.

2). The linear combination $(\ ,\ ) - \lambda <\ ,\ >$ is a flat metric on T_t^*M, $t \in M \setminus \Sigma_\lambda$ for any $\lambda \in \mathbb{C}$.

3). The flat pencil is said to be *quasihomogeneous of the charge* d if a function f exists such that the Lie derivatives of the metrics along the vector fields

$$E := \nabla^{(\ ,\ )} f,\ e := \nabla^{<\ ,\ >} f \tag{2.55}$$

have the form

$$Lie_E(\ ,\ ) = (d-1)(\ ,\ ),\ Lie_e(\ ,\ ) = <\ ,\ >, \tag{2.56}$$

$$Lie_E <\ ,\ > = (d-2) <\ ,\ >,\ Lie_e <\ ,\ > = 0 \tag{2.57}$$

for some constant $d \in \mathbb{C}$ and the commutator of the vector fields equals

$$[e, E] = e. \tag{2.58}$$

In the case of Frobenius manifolds the connection $\nabla^{<\ ,\ >}$ coincides with ∇, the connection $\nabla^{(\ ,\ )}$ defined by (2.53) for $\lambda = 0$ will be denoted ∇_*. The function f has the form

$$f = t_1 \equiv \eta_{1\alpha} t^\alpha.$$

As it was shown in [11] (see also [12, 13]) existence of a flat quasihomogeneous pencil with certain restrictions for the eigenvalues of the operator $\nabla^{<\ ,\ >}E$ can be used for an alternative axiomatization of Frobenius manifolds.

Definition 6. A function $p = p(t;\lambda)$ is called λ-*period* of the Frobenius manifold if it satisfies

$$(\nabla_* - \lambda \nabla) dp = 0. \tag{2.59}$$

Due to Proposition 2.3, on the universal covering of

$$M \times \mathbb{C} \setminus \cup_\lambda \lambda \times \Sigma_\lambda$$

there exist n independent λ-periods $p^1(t;\lambda), \ldots, p^n(t;\lambda)$. They give a system of flat coordinates for the metric $(\ ,\ ) - \lambda <\ ,\ >$. More precisely,

Corollary 2.4. *The flat coordinates $p^1(t;\lambda), \ldots, p^n(t;\lambda)$ of the metric $(\ ,\ ) - \lambda <\ ,\ >$ on a sufficiently small domain in $M \setminus \Sigma_\lambda$ can be determined from the system of linear differential equations*

$$\begin{gathered}(g^{\alpha\epsilon}(t) - \lambda\, \eta^{\alpha\epsilon})\, \partial_\beta \xi_\epsilon + c^{\alpha\rho}_\beta(t) \left(\frac{1}{2} - \mathcal{V}\right)^\epsilon_\rho \xi_\epsilon = 0,\ \alpha,\, \beta = 1, \ldots, n \\ \xi_\epsilon = \frac{\partial x(t;\lambda)}{\partial t^\epsilon},\ \epsilon = 1, \ldots, n.\end{gathered} \tag{2.60}$$

A full system of independent coordinates $p^a = p^a(t;\lambda)$ is obtained from a fundamental system of solutions $\xi^a(t;\lambda) = (\xi^a_\epsilon(t;\lambda))$, $a = 1, \ldots, n$, of the linear system (2.60). In these coordinates the Gram matrix of the bilinear form (2.48) is a constant one

$$G^{ab} \equiv \left(dp^a, dp^b\right) - \lambda < dp^a, dp^b > = \left[g^{\alpha\beta}(t) - \lambda\, \eta^{\alpha\beta}\right] \xi^a_\alpha(t;\lambda) \xi^b_\beta(\lambda, t) \tag{2.61}$$

The dependence of the new flat coordinates on t, λ can be chosen in such a way that the partial derivatives $\xi_\epsilon(t;\lambda)$ satisfy also a system of linear differential equations with rational coefficients

$$(g^{\alpha\epsilon}(t) - \lambda\, \eta^{\alpha\epsilon}) \frac{\partial \xi_\epsilon}{\partial \lambda} - \eta^{\alpha\rho} \left(\frac{1}{2} - \mathcal{V}\right)^\epsilon_\rho \xi_\epsilon = 0,\ \alpha = 1, \ldots, n. \tag{2.62}$$

The corresponding flat coordinates have the form

$$p^a(t;\lambda) = \hat p^a\left(t^1 - \lambda, t^2, \dots, t^n\right) \tag{2.63}$$

where $\hat p^a(t)$ are flat coordinates of the metric $(\ ,\ )$, $t \in M \setminus \Sigma_0$.

We will omit the hat over $p^a(t)$ in the notations for the flat coordinates of the intersection form.

Let us rewrite the system (2.60), (2.62) in matrix notations. The equations (2.60), (2.62) for the row-vector $\xi = (\partial_1 p, \partial_2 p, \dots, \partial_n p)$ read

$$\partial_\alpha \xi \cdot (\mathcal{U} - \lambda) = \xi\left(\mathcal{V} - \frac{1}{2}\right) C_\alpha, \tag{2.64}$$

$$\partial_\lambda \xi \cdot (\mathcal{U} - \lambda) = \xi\left(\frac{1}{2} - \mathcal{V}\right). \tag{2.65}$$

The matrices $\mathcal{U}(t)$, $C_\alpha(t)$ were defined in (2.12), (2.13). Observe that, for $d \neq 1$, one can reconstruct the λ-period $p(t;\lambda)$ knowing its gradient $\xi(t;\lambda)$ using

Lemma 2.5. *Let $\xi = (\xi_\alpha(t;\lambda))$ be an arbitrary solution of the system (2.64), (2.65), and $d \neq 1$. Then the function*

$$p(t;\lambda) = \frac{2}{1-d} i_{E-\lambda\, e}\xi \equiv \frac{2}{1-d}\left[E^\epsilon(t)\xi_\epsilon(t;\lambda) - \lambda\, \xi_1(t;\lambda)\right] \tag{2.66}$$

satisfies (2.60).

Proof Multiplying (2.64) by $E^\alpha(t) - \lambda\, \delta_1^\alpha$ and taking the sum over α one obtains

$$\left(E^\alpha(t) - \lambda\, \delta_1^\alpha\right) \partial_\alpha \xi = \xi\left(\mathcal{V} - \frac{1}{2}\right).$$

Using

$$\nabla E = \frac{1-d}{2} + \frac{1}{2} - \mathcal{V}$$

and closedness of the 1-form ξ we rewrite the above equation in the form

$$d\,\left(i_{E-\lambda\, e} dp\right) = \frac{1-d}{2} dp.$$

So

$$i_{E-\lambda\, e} dp = \frac{1-d}{2} p + \text{const}.$$

Doing a shift along p we kill the constant if $d \neq 1$. Lemma is proved.

Definition 7. The subset

$$\Sigma := \Sigma_0 \subset M$$

is called *discriminant* of the Frobenius manifold M. Any function $p = p(t)$ on $M \setminus \Sigma$ satisfying (2.64), (2.65) with $\lambda = 0$ is called *period* of the Frobenius manifold. A system of n independent periods $p^1(t), \dots, p^n(t)$ gives flat coordinates of the intersection form. They determine a local isometry of the complement $M \setminus \Sigma$ to the complex Euclidean space $\mathbb{C}^n$ equipped with the quadratic form $G = \left(G^{ab}\right) = \left((dp^a, dp^b)\right)$. This map is called *period mapping* of the Frobenius manifold.

Sometimes we will call it *even* period mapping to distinguish from the odd one to be introduced in Section 2.

Analytic continuation of the period mapping $\mathbf{p}(t) := (p^1(t), \dots, p^n(t))$ is a single-valued analytic vector-function on the universal covering of $M \setminus \Sigma$. Continuing this function along a closed loop γ on $M \setminus \Sigma$ one obtains a new system of flat coordinates

$$\mathbf{p}(t) \mapsto \mathbf{p}(t)\, M_\gamma + a_\gamma \tag{2.67}$$

where M_γ is a $n \times n$ matrix satisfying

$$M_\gamma G\, M_\gamma^T = G$$

and a_γ is a constant vector. The matrix M_γ and the vector a_γ depend only on the homotopy class of the loop $[\gamma] \in \pi_1(M \setminus \Sigma)$. For $d \neq 1$, due to the quasihomogeneity (2.66) of the components of the period mapping, one may assume that $a_\gamma = 0$. We will mainly consider here only the case $d \neq 1$.

The following simple statement will be useful later on.

Lemma 2.6. *If, for $d \neq 1$, the flat coordinates $p^1(t)$, ..., $p^n(t)$ of the intersection form are chosen in such a way that*

$$Lie_E p = \frac{1-d}{2} p \tag{2.68}$$

and

$$(dp^a, dp^b) = G^{ab},\ (G_{ab}) = (G^{ab})^{-1},$$

then

$$t_1 := \eta_{1\alpha} t^\alpha = \frac{1-d}{4} G_{ab} p^a p^b. \tag{2.69}$$

Proof From (2.48) we have the following expression for the gradient of the function t_1 w.r.t. the metric (,)

$$\nabla_* t_1 = E. \tag{2.70}$$

Equation (2.68) implies that

$$E = \frac{1-d}{2} p^a \frac{\partial}{\partial p^a}. \tag{2.71}$$

Rewriting (2.70), in the flat coordinates p^1, ..., p^n we obtain

$$G^{ab} \frac{\partial t_1}{\partial p^b} = \frac{1-d}{2} p^a.$$

This proves Lemma.

The representation (for $d \neq 1$)

$$\pi_1(M \setminus \Sigma) \to O(\mathbb{C}^n, G),\ \gamma \mapsto M_\gamma^{-1} \tag{2.72}$$

is called *monodromy representation* of the Frobenius manifold.

3. Dual (almost) Frobenius manifold and its deformed flat connection

Here we will construct a transformation associating to an arbitrary Frobenius manifold M a new structure on $M^* = M \setminus \Sigma$ satisfying all the axioms of a Frobenius manifold but constancy of the unity (cf. [12], Remark 4.2).

For $t \in M \setminus \Sigma$ we define a new multiplication of tangent vectors $u, v \in T_tM$ by

$$u * v := \frac{u \cdot v}{E}. \tag{3.1}$$

Proposition 3.1. *The multiplication (3.1) together with the intersection form $(\ ,\ )$, the unity = the Euler vector field = E satisfies all the axioms of Frobenius manifold but (2.1).*

Proof Associativity of the multiplication is obvious. As it follows from the definition, the superposition

$$f : T_tM \xrightarrow{(\ ,\ )^{-1}} T_t^*M \xrightarrow{<\ ,\ >} T_tM$$

(here $(\ ,\ )$, $<\ ,\ >$ are considered as bilinear forms on T_t^*M) establishes an isomorphism of the algebras

$$f(u * v) = f(u) \cdot f(v).$$

In other words, on the cotangent planes both the multiplications are given by the same formula (2.49). From this we derive invariance of the intersection form w.r.t. the new multiplication, i.e., the symmetry of the expression

$$g_\epsilon^{\alpha\beta} g^{\epsilon\gamma} = i_E \left(dt^\alpha \cdot dt^\beta \cdot dt^\gamma \right)$$

w.r.t. α, β, γ.

Next, we are to prove the symmetry of the covariant derivatives

$$\nabla_*^\gamma c_\rho^{\alpha\beta} = g^{\gamma\epsilon} \partial_\epsilon c_\rho^{\alpha\beta} - \Gamma_\epsilon^{\gamma\alpha} c_\rho^{\epsilon\beta} - \Gamma_\epsilon^{\gamma\beta} c_\rho^{\alpha\epsilon} + \Gamma_\rho^{\gamma\epsilon} c_\epsilon^{\alpha\beta}$$

w.r.t. α, β, γ. Here ∇_* is the Levi-Civita connection for the intersection form. Rewriting the first term in the r.h.s. as

$$g^{\gamma\epsilon} \partial_\epsilon c_\rho^{\alpha\beta} = g^{\gamma\epsilon} \partial_\rho c_\epsilon^{\alpha\beta} = \partial_\rho \left(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma} \right) - c_\epsilon^{\alpha\beta} \left(\Gamma_\rho^{\gamma\epsilon} + \Gamma_\rho^{\epsilon\gamma} \right)$$

(here we use the condition $\nabla_* g^{\alpha\beta} = 0$), we obtain

$$\nabla_*^\gamma c_\rho^{\alpha\beta} = \partial_\rho \left(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma} \right) - \Gamma_\epsilon^{\gamma\alpha} c_\rho^{\epsilon\beta} - \Gamma_\epsilon^{\gamma\beta} c_\rho^{\alpha\epsilon} - c_\epsilon^{\alpha\beta} \Gamma_\rho^{\epsilon\gamma}$$

$$= \partial_\rho \left(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma} \right) - c_\epsilon^{\gamma\lambda} c_\rho^{\epsilon\beta} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\alpha - c_\epsilon^{\gamma\lambda} c_\rho^{\alpha\epsilon} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\beta - c_\epsilon^{\alpha\beta} c_\rho^{\epsilon\lambda} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\gamma.$$

Using associativity we recast the last expression into the form

$$= \partial_\rho \left(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma} \right) - \left[c_\epsilon^{\gamma\beta} c_\rho^{\epsilon\lambda} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\alpha + c_\epsilon^{\alpha\gamma} c_\rho^{\epsilon\lambda} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\beta + c_\epsilon^{\alpha\beta} c_\rho^{\epsilon\lambda} \left(\frac{1}{2} - \mathcal{V} \right)_\lambda^\gamma \right].$$

Using the symmetry of the first term we clearly see that the expression is symmetric in α, β, γ.

It remains to prove that E is the Euler vector field also for the new algebra structure, and that it also plays the role of the unity for the new multiplcation. The first statement follows from the formulae for the Lie derivatives

$$Lie_E g^{\alpha\beta} = (d-1)g^{\alpha\beta},\ Lie_E c_\gamma^{\alpha\beta} = (d-1)c_\gamma^{\alpha\beta}.$$

The second statement is clear since the isomorphism

$$(\ ,\ ) : T^*M \to TM$$

maps the unity $dt_1 = \eta_{1\alpha}dt^\alpha$ of the multiplication (2.49) to the Euler vector field E. Proposition is proved.

Corollary 3.2. *Let $p^1(t), \ldots, p^n(t)$ be a system of flat coordinates of the intersection form defined locally on $M \setminus \Sigma$. Then there exists a function $F_*(p)$ such that*

$$\frac{\partial^3 F_*(p)}{\partial p^i \partial p^j \partial p^k} = G_{ia}G_{jb}\frac{\partial t^\gamma}{\partial p^k}\frac{\partial p^a}{\partial t^\alpha}\frac{\partial p^b}{\partial t^\beta}c_\gamma^{\alpha\beta}(t). \tag{3.2}$$

Here

$$(G_{ij}) = \left(G^{ij}\right)^{-1}$$

and a constant symmetric nondegenerate matrix G^{ij} is defined by

$$G^{ij} = \frac{\partial p^i}{\partial t^\alpha}\frac{\partial p^j}{\partial t^\beta}g^{\alpha\beta}(t).$$

The function $F_*(p)$ satisfies the following associativity equations

$$\frac{\partial^3 F_*(p)}{\partial p^i \partial p^j \partial p^a}G^{ab}\frac{\partial^3 F_*(p)}{\partial p^b \partial p^k \partial p^l} = \frac{\partial^3 F_*(p)}{\partial p^l \partial p^j \partial p^a}G^{ab}\frac{\partial^3 F_*(p)}{\partial p^b \partial p^k \partial p^i},\ i,\ j,\ k,\ l = 1, \ldots, n. \tag{3.3}$$

For $d \neq 1$ $F_*(p)$ satisfies the homogeneity condition

$$\sum_a p^a \frac{\partial F_*}{\partial p^a} = 2\, F_* + \frac{1}{1-d}\sum G_{ab}p^a p^b. \tag{3.4}$$

Observe that the definition of the function F_* can be rewritten in the following way

$$d\left(\frac{\partial F_*}{\partial p^a \partial p^b}\right) = dp_a \cdot dp_b. \tag{3.5}$$

We have constructed an *almost Frobenius structure* on the complement $M_* = M \setminus \Sigma$. By definition, this means that all the axioms of Frobenius manifold hold true but the constancy of the unity. We will called this object *dual (almost) Frobenius manifold*. At the end of this Section we describe the deformed flat coordinates of the dual Frobenius manifold. Denote ν the parameter of the deformation (it was z for the original Frobenius manifold). By definition the deformed flat coordinates are independent flat functions $\tilde{p}(p;\nu)$ for the deformed flat connection $\tilde{\nabla}_*$ defined on $M_* \times \mathbb{C}$ by the formulae similar to (2.6). In the flat coordinates $p^1, \ldots, p^n$ for the intersection form they satisfy the system

$$\frac{\partial \xi_a}{\partial p^b} = \nu \overset{*}{c}{}_{ab}^{c}(p)\xi_c,\ \xi_a = \frac{\partial \tilde{p}(p;\nu)}{\partial p^a} \tag{3.6}$$

where

$$\overset{*}{c}{}^{c}_{ab}(p) := G^{cd}\frac{\partial^3 F_*(p)}{\partial p^d \partial p^a \partial p^b}. \tag{3.7}$$

Definition 8. Any function $\tilde p = \tilde p(p;\nu)$ satisfying (3.6) is called *twisted period* of the Frobenius manifold.

As usual one has an invariant pairing on twisted periods defined by the formula

$$(\xi(\nu), \xi(-\nu)) := \xi_i(p;\nu) G^{ij} \xi_j(p;-\nu). \tag{3.8}$$

The above expression does not depend on $p \in M_*$

Let us rewrite the system (3.6) in the original coordinates $t^1, \ldots, t^n$.

Proposition 3.3. *The gradients $\xi = (\xi_1, \ldots, \xi_n)$ of the deformed flat coordinates on $M_* \times \mathbb{C}$*

$$\xi_\alpha := \partial_\alpha \tilde p(t;\nu), \ \ \alpha = 1, \ldots, n \tag{3.9}$$

satisfy the following system of linear differential equations

$$\partial_\alpha \xi \cdot \mathcal{U} = \xi \cdot \left(\mathcal{V} + \nu - \frac{1}{2}\right) C_\alpha. \tag{3.10}$$

The notations are the same as above.

Proof By definition the differential equations for the deformed flat coordinates have the form

$$\partial_\alpha \xi_\beta - \Gamma^\gamma_{\alpha\beta}\xi_\gamma - \nu\, \hat c^\gamma_{\alpha\beta}\xi_\gamma = 0.$$

Here $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel coefficients of the Levi-Civita connection for the metric 2.48), $\hat c^\gamma_{\alpha\beta}$ are the structure constants of the dual Frobenius manifold. Multiplying this equation by $\mathcal{U}^\alpha_\rho$ and using (2.53) and the definition (3.1) of the multiplication law on the dual Frobenius manifold we arrive at (3.10). □

For $\nu = 0$ we obtain the original periods

$$\tilde p(p;0) \equiv p$$

i.e., the flat functions for the metric (,). The twisted periods

$$\varpi(t) = \tilde p(t;\nu = 1/2) \tag{3.11}$$

will be called *odd periods*. This name is motivated by the following construction.

Let us introduce a Poisson bracket on the Frobenius manifold by the formula

$$\{f, g\} := < df, \mathcal{V}\, dg > . \tag{3.12}$$

The flat coordinates $t^1, \ldots, t^n$ are Darboux coordinates for this Poisson bracket:

$$\{t^\alpha, t^\beta\} = \eta^{\alpha\gamma}\mathcal{V}^\beta_\gamma. \tag{3.13}$$

The Poisson structure does not degenerate *iff* the operator $\mathcal{V}$ is invertible.

Proposition 3.4. *Let $\varpi_1(t)$, $\varpi_2(t)$ be any two odd periods. Then their Poisson bracket (3.12) is constant.*

Proof is given by a straightforward calculation of the derivatives $\partial_\alpha\{\varpi_1, \varpi_2\}$ using (3.10) for $\nu = \frac{1}{2}$.

Corollary 3.5. *A system of n independent odd periods $\varpi_1(t), \ldots, \varpi_n(t)$ gives another system of Darboux coordinates for the Poisson bracket (3.12).*

The generating function of the canonical transformation

$$\left(t^1, \ldots, t^n\right) \mapsto (\varpi_1(t), \ldots, \varpi_n(t))$$

between the two systems of Darboux coordinates will play an important role in the constructions of Section 4.

Let us now consider the dependence of the twisted periods on ν. It turns out that, instead of the differential equation with rational coefficients given by (2.7) the crucial role in the theory of almost Frobenius manifolds plays a *difference equation in ν*.

Lemma 3.6. *If $\xi = \xi(t; \nu)$ is a solution to (3.10) then $\partial_1\xi$ is a solution to the same system with a shift $\nu \mapsto \nu - 1$.*

We will represent the claim of the lemma in the form

$$\partial_1\xi(t;\nu) = \xi(t; \nu - 1) \tag{3.14}$$

and, respectively for the twisted periods

$$\frac{\partial}{\partial t^1}\tilde{p}(t;\nu) = \tilde{p}(t; \nu - 1). \tag{3.15}$$

Proof Dependence of $\xi(t;\nu)$ on t^1 is determined, due to (3.10), by the system

$$\dot{\xi} \cdot \mathcal{U} = \xi \cdot \left(\mathcal{V} + \nu - \frac{1}{2}\right), \ \dot{\xi} := \partial_1\xi. \tag{3.16}$$

So one can rewrite the equation (3.10) as

$$\partial_\alpha\xi \cdot \mathcal{U} = \dot{\xi} \cdot C_\alpha\mathcal{U}$$

where the commutativity $\mathcal{U}C_\alpha = C_\alpha\mathcal{U}$ of operators of multiplication has been used. Because of invertibility of the operator $\mathbf{U}$ on M_* one has

$$\partial_\alpha\xi = \dot{\xi} \cdot C_\alpha.$$

Differentiation (3.10) w.r.t. t^1 and using $\partial_1\mathcal{U}(t) = \mathrm{id}$, $\partial_1 C_\alpha = 0$ yields

$$\partial_\alpha\dot{\xi} \cdot \mathcal{U} + \partial_\alpha\xi = \dot{\xi} \cdot \left(\mathcal{V} + \nu - \frac{1}{2}\right) C_\alpha.$$

Using the above equation we arrive at

$$\partial_\alpha\dot{\xi} = \dot{\xi} \cdot \left(\mathcal{V} + \nu - \frac{3}{2}\right) C_\alpha.$$

This equation coincides with (3.10) up to shift $\nu \mapsto \nu - 1$. The Lemma is proved. $\square$

Using (3.10) one can rewrite the shift equation (3.14) in the form

$$\xi(t;\nu-1) = \xi(t;\nu)\left[\mathcal{A}(t)\nu + \mathcal{B}(t)\right], \ \mathcal{A}(t) = \mathcal{U}^{-1}, \ \mathcal{B}(t) = \left(\mathcal{V} + \nu - \frac{1}{2}\right)\mathcal{U}^{-1}. \tag{3.17}$$

Observe that the shift operator (3.17) *changes the sign* of the invariant bilinear form (3.8): if $\xi(t;\nu-1)$ and $\xi(t;\nu)$ are related by (3.17) then

$$(\xi(\nu-1), \xi(-\nu+1)) = -(\xi(\nu), \xi(-\nu)). \tag{3.18}$$

We are now ready to give a precise definition of almost Frobenius manifold and prove a Reconstruction Theorem inverting the above "almost duality"

$$M \mapsto M_*.$$

For simplicity we will only consider the case $d \neq 1$. Moreover we will present the definition of almost Frobenius manifolds only in the flat coordinates leaving the coordinate-free formulation, along the lines of Section 2, as an exercise for the reader.

Definition 9. *Almost Frobenius structure* of the charge $d \neq 1$ on the manifold M_* is a structure of a Frobenius algebra on the tangent planes $T_pM_* = (\ * \ , (\ ,\ )_p)$ depending (smoothly, analytically etc.) on the point $p \in M_*$. It must satisfy the following axioms.

AFM1. The metric $(\ ,\ )_p$ is flat.

AFM2. In the flat coordinates $p^1, \ldots, p^n$ for the metric,

$$(dp^i, dp^j) = G^{ij} \tag{3.19}$$

the structure constants of the multiplication

$$\frac{\partial}{\partial p^i} * \frac{\partial}{\partial p^j} = \overset{*}{c}{}^k_{ij}(p)\frac{\partial}{\partial p^k} \tag{3.20}$$

can be locally represented in the form

$$\overset{*}{c}{}^k_{ij}(p) = G^{kl}\frac{\partial^3 F_*(p)}{\partial p^l \partial p^i \partial p^j} \tag{3.21}$$

for some function $F_*(p)$. The function satisfies the homogeneity equation

$$\sum_{i=1}^n p^i \frac{\partial F_*(p)}{\partial p^i} = 2\,F_*(p) + \frac{1}{1-d}(p,p). \tag{3.22}$$

The Euler vector field

$$E = \frac{1-d}{2}\sum_{i=1}^n p^i \frac{\partial}{\partial p^i} \tag{3.23}$$

is the unity of the Frobenius algebra.

AFM3. There exists a vector field

$$e = e^k(p)\frac{\partial}{\partial p^k} \tag{3.24}$$

being an invertible element of the Frobenius algebra T_pM_* for every $p \in M_*$ such that the operator

$$\tilde{p} \mapsto e\tilde{p}$$

acts by a shift $\nu \mapsto \nu - 1$ on the solutions of the deformed flat connection equations

$$\frac{\partial^2 \tilde p}{\partial p^i \partial p^j} = \nu \overset{*}{c}{}^k_{ij}(p) \frac{\partial \tilde p}{\partial p^k}. \tag{3.25}$$

Theorem 3.7. *Let us define on an almost Frobenius manifold. M_* a new multiplication $\cdot$ and a new bilinear form $<\ ,\ >$ by*

$$u \cdot v := \frac{u * v}{e} \tag{3.26}$$

$$< u, v > := \left(u, \frac{v}{e} \right). \tag{3.27}$$

These multiplication and bilinear form gives a Frobenius structure of the charge d on M_* with the unity e and the Euler vector field E.

Proof Using (3.25) we can rewrite the shift operator $\nu \mapsto \nu - 1$ as follows

$$\xi(p; \nu - 1) = \xi(p; \nu)(A(p)\nu + B(p)),\ \xi(p; \nu) = (\xi_i(p; \nu)),\ \xi_i(p; \nu) = \frac{\partial \tilde p(p; \nu)}{\partial p^i} \tag{3.28}$$

(cf. (3.17) above) where

$$\begin{aligned} A(p) &= (A^i_j(p)),\ A^i_j(p) = e^k(p) \overset{*}{c}{}^i_{kj}(p) \\ B(p) &= (B^i_j(p)),\ B^i_j(p) = \partial_j e^i(p). \end{aligned} \tag{3.29}$$

Since the operator (3.28) changes the sign of the pairing (3.8), the operators $A = A(p)$, $B = B(p)$ satisfy

$$(Au, v) = (u, Av),\ (Bu, v) + (u, Bv) = -(u, Av)$$

for any two tangent vectors u, v. The second of these two equations implies

$$\partial^i e^j + \partial^j e^i = -e^k \overset{*}{c}{}^{ij}_k \tag{3.30}$$

(all raising and lowering of Latin indices in this Section is done by means of the matrix (G^{ij}) and its inverse (G_{ij})).

Let us first prove that the metric $<\ ,\ >$ is flat. It is convenient to work with the contravariant metric on T^*M_*. The Gram matrix of it in the coordinates p^i is given by

$$\eta^{ij}(p) := < dp^i, dp^j > = e^k(p) \overset{*}{c}{}^{ij}_k(p). \tag{3.31}$$

Let us compute the Christoffel coefficients of the Levi-Civita connection for the metric $<\ ,\ >$. Let us denote these coefficients $\gamma^k_{ij}(p)$; we put

$$\gamma^{ij}_k(p) := -\eta^{is}(p)\gamma^j_{sk}(p) \tag{3.32}$$

(cf. (2.52)).

Lemma 3.8. *The Christoffel coefficients (3.32) of the Levi-Civita connection for the metric (3.31) are given by one of the following two equivalent formulae*

$$\gamma^{ij}_k = -\partial^i \partial_k e^j \tag{3.33}$$

$$\gamma^{ij}_k = \overset{*}{c}{}^i_{ks} \partial^s e^j. \tag{3.34}$$

Proof Let us first check equality of (3.33) and (3.34). Indeed, from (3.15) it follows that the functions $\tilde{p} = e^1(p), \ldots, \tilde{p} = e^n(p)$ give a basis in the space of solutions of the equation (3.25) for $\nu = -1$. Spelling this equation out yields the needed equality.

We are now to check that the Christoffel coefficients γ_k^{ij} satisfy the following two equations (see [12], equations (3.26) and (3.27)) uniquely determining the Levi-Civita connection

$$\partial_k \eta^{ij} = \gamma_k^{ij} + \gamma_k^{ji} \tag{3.35}$$

$$\eta^{is}\gamma j k_s = \eta^{js}\gamma_s^{ik}. \tag{3.36}$$

Indeed, the first of these two equations follows from (3.33) and from the identity

$$\eta^{ij} = -\partial^i e^j - \partial^j e^i \tag{3.37}$$

equivalent to (3.30). The second one follows from (3.34) and from associativity of the multiplication $*$. The Lemma is proved. □

We will now prove that the metric (3.31) is flat. Because of (3.36) the connection is torsion-free. It remains to prove

Lemma 3.9. *The curvature of the connection (3.33), (3.34) vanishes.*

Proof We are to prove the following identity (see [12], formula (3.33))

$$\eta^{is}\left(\partial_s \gamma_l^{jk} - \partial_l \gamma_s^{jk}\right) + \gamma_s^{ij}\gamma_l^{sk} - \gamma_s^{ik}\gamma_l^{si} = 0.$$

To this end we are to substitute the formula (3.33) in the first bracket and also the expressions (3.31) and (3.34) in the remaining terms. After such substitution the first bracket will be vanishing because of equality of mixed derivatives, and the remaining terms will be equal to zero because of associativity of the product $*$. The Lemma is proved. □

We will now proceed to studying the properties of the multiplication (3.26).

Lemma 3.10. *The multiplication (3.26) is commutative and associative. The vector* e *is the unity of it. The bilinear form* $< \, , \, >$ *is symmetric nondegenerate and invariant w.r.t. the multiplication.*

Proof Rewriting the formulae (3.26) and (3.27) in the form

$$u \cdot v = u * v * e^{-1}$$

$$< u, v >= (u * v, e^{-1})$$

we obtain

$$< u \cdot v, w >= (u * v * w, e^{-2}).$$

The statements of the Lemma easily follow from the above formulae. □

We are now to check the main property **FM2** of the multiplication (3.26). Let us denote ∇_k the covariant derivatives w.r.t. the Levi-Civita connection for the metric

$<\ ,\ >$. Introduce the operators

$$\nabla^k := \eta^{ks}\nabla_s.$$

It suffices to prove

Lemma 3.11. *The coefficients* $c_k^{ij} = c_k^{ij}(p)$ *defined by*

$$dp^i \cdot dp^j = c_k^{ij}(p) dp^k$$

satisfy

$$\nabla^l c_k^{ij} = \nabla^i c_k^{lj}. \tag{3.38}$$

Proof The main trick in the proof is the coincidence of the two multiplications $*$ and $\cdot$ on the *cotangent* spaces T^*M_*. So c_k^{ij} in (3.38) can be replaced by $\overset{*}{c}{}_k^{ij}$. So we are to prove symmetry in i, j, l of the covariant derivative

$$\nabla^l \overset{*}{c}{}_k^{ij} = \eta^{ls}\partial_s \overset{*}{c}{}_k^{ij} - \gamma_q^{li} \overset{*}{c}{}_k^{qj} - \gamma_q^{lj} \overset{*}{c}{}_k^{iq} + \gamma_k^{lq} \overset{*}{c}{}_q^{ij}.$$

The derivative $\partial_s \overset{*}{c}{}_k^{ij}$ is symmetric in s, k due to (3.21). Using this symmetry and also the equation (3.35) we recast the r.h.s. of the last expression into the form

$$= \partial_k \left(\eta^{ls} \overset{*}{c}{}_s^{ij} \right) - \gamma_q^{li} \overset{*}{c}{}_k^{qj} - \gamma_q^{lj} \overset{*}{c}{}_k^{qi} - \gamma_k^{ql} \overset{*}{c}{}_q^{ij}.$$

In the first term we can replace back $\overset{*}{c}{}_s^{ij}$ by c_s^{ij}. After such a replacement the symmetry of the first term in i, j, l becomes evident. The remaining three terms can be rewritten, using (3.34) as the k-th component of the following one-form

$$-\left(dp^l * dp^j * de^i + dp^l * dp^i * de^j + dp^i * dp^j * de^l \right).$$

The needed symmetry becomes obvious. The Lemma is proved. □

Lemma 3.12. *The unity vector field e satisfies (2.1).*

Proof Using the formula (3.31) and also the expression (3.34) for the Christoffel coefficients we compute

$$\nabla^i e^k = \eta^{is}\partial_s e^k - \gamma_s^{ik} e^s = e^l \overset{*}{c}{}_l^{is} \partial_s e^k - \overset{*}{c}{}_{sl}^{i} \partial^l e^k e^s = 0.$$

The Lemma is proved. □

It remains to settle the quasihomogeneity property **FM3**. To this end the following quasihomogeneity of the deformed flat coordinates $\tilde{p}(p;\nu)$ will be useful

$$Lie_E \tilde{p}(p;\nu) = \left[\frac{1-d}{2} + \nu \right] \tilde{p}(p;\nu). \tag{3.39}$$

In particular, the components $e^k(p)$ of the vector field e satisfy

$$Lie_E e^k = -\frac{1+d}{2} e^k, \ k = 1, \ldots, n \tag{3.40}$$

(see the proof of Lemma 3.8).

Lemma 3.13. *The vector field E is linear in the flat coordinates for the metric $<\,,\,>$. It satisfies*

$$[e, E] = e. \tag{3.41}$$

Proof Using (3.34) and also

$$E^k \overset{*}{c}{}^i_{kj} = \delta^i_j \tag{3.42}$$

(see Axiom **AFM2**) we obtain the following expression for the first covariant derivative of the vector field E

$$\nabla^i E^j = \frac{1-d}{2}\eta^{ij} - \partial^i e^j.$$

For the second covariant derivative $\nabla^k \nabla^i E^j$ we obtain, using (3.35) and also equality of (3.33) and (3.34)

$$\nabla^k \nabla^i E^j = \frac{1-d}{2}(\gamma^{ij}_s + \gamma^{ji}_s) + \eta^{ks}\gamma^{ij}_s - \gamma^{ki}_q \left[\frac{1-d}{2}\eta^{qj} - \partial^q e^j\right] - \gamma^{kj}_q \left[\frac{1-d}{2}\eta^{iq} - \partial^i e^q\right].$$

After substitution of the expression (3.34) for the Christoffels we rewrite the last equation as follows

$$= \frac{3-d}{2}\eta^{ks} \overset{*}{c}{}^{iq}_s \partial_q e^j + \frac{1-d}{2}\eta^{ks} \overset{*}{c}{}^{jq}_s \partial_q e^i - \frac{1-d}{2}\eta^{js} \overset{*}{c}{}^{kq}_s \partial_q e^i - \frac{1-d}{2}\eta^{is} \overset{*}{c}{}^{kq}_s \partial_q e^j$$
$$+ \overset{*}{c}{}^k_{qs} \partial^s e^i \partial^q e^j + \overset{*}{c}{}^k_{qs} \partial^s e^j \partial^i e^q.$$

Replacing $\overset{*}{c}{}^{iq}_s$, $\overset{*}{c}{}^{jq}_s$, $\overset{*}{c}{}^{kq}_s$, and $\overset{*}{c}{}^{kq}_s$ by c^{iq}_s, c^{jq}_s, c^{kq}_s, and c^{kq}_s resp. and using symmetries like

$$\eta^{ks} c^{jq}_s = \eta^{js} c^{kq}_s$$

etc. we arrive at the following expression

$$= \eta^{is} c^{kq}_s \partial_q e^j + c^{qk}_s \left(\partial^i e^s + \partial^s e^i\right) \partial_q e^j.$$

The expression in the parenthesis in the second term is equal to $-\eta^{is}$, due to (3.30). Therefore $\nabla^k \nabla^i E^j = 0$.

To prove (3.41) we are to use (3.40). So,

$$[e, E]^k = e^i \partial_i \frac{1-d}{2} p^k - E^i \partial_i e^k = \frac{1-d}{2} e^k + \frac{1+d}{2} e^k = e^k.$$

The Lemma is proved. □

The last step in the proof of the Theorem 3.7 is

Lemma 3.14. *The linear vector field E satisfies equations of Axiom* **FM3**.

Proof Let us first prove that

$$Lie_E \eta^{ij} \equiv E^k \partial_k \eta^{ij} - \partial_k E^i \eta^{kj} - \partial_k E^j \eta^{ik} = (d-2)\eta^{ij}.$$

Using (3.35) we rewrite the last equation in the form

$$= E^k(\gamma^{ij}_k + \gamma^{ji}_k) - (1-d)\eta^{ij}.$$

Substituting the expression (3.34) for the Christoffels and using (3.42) and (3.30) we arrive at

$$= \partial^i e^j + \partial^j e^i - (1-d)\eta^{ij} = (d-2)\eta^{ij}.$$

Next, we are to prove that

$$Lie_E c_k^{ij} \equiv E^s \partial_s c_k^{ij} - \partial_s E^i c_k^{sj} - \partial_s E^j c_k^{is} + \partial_k E^s c_k^{ij} = (d-1) c_k^{ij}.$$

The last three terms give $-(1-d)c_k^{ij}$. In the first term, because of equality $c_k^{ij} = \overset{*}{c}{}_k^{ij}$ and because of (3.21) we can interchange the indices s and k. After this we rewrite this term as follows

$$E^s \partial_k c_s^{ij} = \partial^k \left(E^s \overset{*}{c}{}_s^{ij} \right) - \partial_k E^s c_s^{ij} = -c_k^{ij}.$$

Here we use again (3.42). This proves the Lemma and also the Theorem 3.7. □

Example 1. Let $(A = \mathrm{span}(e_1, \dots, e_n), <\ ,\ >)$ be a Frobenius algebra with the trivial grading

$$q_\alpha = \deg e_\alpha = 0, \ \alpha = 1, \dots, n.$$

It carries a structure of a trivial Frobenius manifold $M = A$ with

$$F(t) = \frac{1}{6} < t^3, e >, \ t = t^\alpha e_\alpha \in A.$$

Here $e \in A$ is the unity. The Euler vector field is equal to $E = t^\alpha \partial_\alpha$. The dual (almost) Frobenius manifold M^* can be identified with the set of all invertible elements $x = x^\alpha e_\alpha$ of A with the metric

$$\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = < e_\alpha, e_\beta >$$

and with the multiplication defined by

$$\left(\frac{\partial}{\partial x^\alpha} * \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\gamma} \right) = 2(e_\alpha \cdot e_\beta \cdot e_\gamma, x^{-1}).$$

Integrating one obtains

$$F_*(x) = \frac{1}{2} \left(x^2, \log x^2 \right). \tag{3.43}$$

The map

$$\begin{aligned} &M^* \to M \\ &x \mapsto t = \frac{1}{4} x^2 \end{aligned} \tag{3.44}$$

transforms the metric (,) to the intersection form of the trivial Frobenius manifold M. The latter can be recast into the following bilinear form on T^*M depending linearly on the coordinates

$$\left(dt^\alpha, dt^\beta \right) = c_\gamma^{\alpha\beta} t^\gamma.$$

This metric and the flat coordinates (3.44) for it was first considered by A.Balinski and S.P.Novikov [3] in their theory of linear Poisson brackets of hydrodynamic type. The solution (3.43) to the equations of associativity was found in Appendix to [8, 9] (in a different but equivalent form).

We will end this Section with a slightly more general construction of the almost dual Frobenius manifold. Let us introduce a one-parameter family of Frobenius algebra structures on the tangent spaces T_tM by

$$\begin{aligned} u *_\lambda v &:= \frac{u \cdot v}{E - \lambda e} \\ (u, v)_\lambda &:= < u, \frac{v}{E - \lambda e} > . \end{aligned} \tag{3.45}$$

It is easy to see that, for any λ the above formulae define on $M_*(\lambda) := M \setminus \Sigma_\lambda$ a structure of almost Frobenius manifold. For $\lambda = 0$ one obtains the old definition $M_*(0) = M_*$. For $\lambda \to \infty$ after a suitable rescaling the Frobenius structure (3.45) goes to the original Frobenius structure.

4. Twisted period mapping and its monodromy

In this section we will describe the analytic properties of the twisted period mapping for an arbitrary semisimple Frobenius manifold.

Let ν be an arbitrary complex number,

$$q = e^{2\pi i \nu}.$$

We first introduce one more argument λ of the twisted periods doing a shift

$$\tilde{p}(t; \nu) \mapsto e^{-\lambda \partial_1} \tilde{p}(t; \nu) = \tilde{p}(t^1 - \lambda, t^2, \ldots, t^n; \nu)$$

The gradients $\xi_\alpha = \partial_\alpha \tilde{p}$ of these functions can be found from the following system of linear differential equations.

Lemma 4.1. *Near any point $t \in M \setminus \Sigma_\lambda$ there exist n independent functions $\tilde{p}^1(\nu; \lambda; t)$, ..., $\tilde{p}^n(\nu; \lambda; t)$ such that their gradients $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_\alpha = \partial_\alpha \tilde{p}$, satisfy the system*

$$\partial_\alpha \xi \cdot (\mathcal{U} - \lambda) = \xi \left(\mathcal{V} + \nu - \frac{1}{2} \right) C_\alpha \tag{4.1}$$

$$\partial_\lambda \xi \cdot (\mathcal{U} - \lambda) = \xi \left(-\mathcal{V} - \nu + \frac{1}{2} \right). \tag{4.2}$$

To prove Lemma it suffices to check compatibility of the system. We leave it as an exercise to the reader.

Observe that for $\nu = 0$ the system (4.1), (4.2) coincides with (2.64), (2.65). More generally, it coincides with the equations defining deformed flat coordinates on the almost dual Frobenius manifold $M_*(\lambda)$ (see (3.45) above).

Our nearest goal is to describe the monodromy of solutions of the system (4.1), (4.2). This will be done for an arbitrary semisimple Frobenius manifold.

We first rewrite the system (4.1), (4.2) for the twisted periods in the canonical coordinates. We will denote $\tilde{p}(\nu; \lambda; u)$ the function $\tilde{p}(\nu; \lambda; t(u))$ when it cannot lead to a confusion.

Let $\phi_i(\nu;\lambda;t)$ be the components of the one-form $\xi = d\tilde{p}$ in the moving frame f_1, ..., f_n,

$$\phi_i = \frac{\partial_i \tilde{p}}{\psi_{i1}}. \tag{4.3}$$

Lemma 4.2. *In the new coordinates the system (4.1), (4.2) reads*

$$\partial_i \phi = \left(\frac{E_i\left(\frac{1}{2} - \nu + V\right)}{\lambda - u_i} + V_i \right) \phi \tag{4.4}$$

$$(U-\lambda)\frac{d\phi}{d\lambda} = \left(\frac{1}{2} - \nu + V\right)\phi \tag{4.5}$$

(we write now ϕ as a column-vector). Here E_i is the matrix having all the entries zero but $(E_i)_{ii} = 1$, the skew-symmetric matrix V_i has the form

$$V_i = \mathrm{ad}_{E_i}\mathrm{ad}_U^{-1}(V).$$

Proof is similar to Proposition H.2 in [12]. We omit it.

If ϕ is a solution to the system (4.4), (4.5) then $d\phi/d\lambda$ is a solution of the same system with $\nu \mapsto \nu - 1$. So it suffices to describe the monodromy of the system assuming that

$$\mathrm{Re}\,\nu > -\frac{1}{2}.$$

Let us cover the Frobenius manifold $M_{s\,s}$ with convenient charts. We choose a real number $0 \le \varphi < 2\pi$ and we define an open subset $M_{s\,s}^0 \subset M_{s\,s}$ containing all the points $t \in M_{s\,s}$ such that their canonical coordinates $u_1(t)$, ..., $u_n(t)$ satisfy the following condition: the rays L_1, ..., L_n on the complex plane of the form

$$L_j = \left\{u_j + i\,\rho\, e^{-i\varphi} \mid 0 \le \rho < \infty\right\},\ j = 1, \dots, n \tag{4.6}$$

must not intersect. On $M_{s\,s}^0$ we can order the canonical coordinates u_1, ..., u_n in such a way that the rays L_1, ..., L_n exit from the infinite point of the complex plane in the counter-clockwise order. After this ordering we are able to define the matrix-valued functions $\Psi(u)$ and $V(u)$ as it was explained above. We can define therefore the linear differential operator with rational coefficients

$$\frac{d}{dz} - \left(U + \frac{V}{z}\right) \tag{4.7}$$

and compute it monodromy data $(\hat{\mu}, e, R, S, C)$ at each point of $M_{s\,s}^0$. Here the $n \times n$ matrices $\hat{\mu}$, R describe monodromy at the origin, e is an eigenvector of the matrix $\hat{\mu}$, S is the Stokes matrix of the operator (4.7) computed with respect to the line

$$\ell = \{\arg\, z = \varphi\} \tag{4.8}$$

with its natural orientation, C is the central connection matrix (see the definitions and the full list of constraints for the monodromy data in [14]). Observe that S is an upper triangular matrix due to the above choice of ordering of the entries u_1, ..., u_n of the diagonal matrix U. The central result of the theory of semisimple Frobenius manifolds says that the monodromy data are constant on every connected piece of $M_{s\,s}^0$. The Frobenius manifold structure on any such a piece can be reconstructed by an algebraic procedure starting from the solution of a suitable Riemann - Hilbert boundary value problem with the boundary conditions given in terms of the monodromy data. We will denote $Fr(\hat\mu, R, e, S, C)$ such a Frobenius structure on any connected component of $M^0_{s\,s}$ characterized by the monodromy data $(\hat\mu, R, e, S, C)$. As a consequence of the general theory of Riemann - Hilbert problems we derive that the image of the map

$$M^0_{s\,s} \to \mathbb{C}^n \setminus \mathrm{diag}\,,\ t \mapsto (u_1(t), \dots, u_n(t))$$

is a complement to a closed analytic subset in $\mathbb{C}^n$. The gluing of the patches $Fr(\hat\mu, R, e, S, C)$ along the boundaries

$$\arg(u_i - u_j) = \frac{\pi}{2} - \varphi \text{ for some } i \neq j$$

is given by an action of the braid group B_n on the monodromy data described in [12, 14].

In every patch $Fr(\hat\mu, R, e, S, C)$ we will construct a fundamental matrix of solutions of the Fuchsian system (4.14) depending on $u_1, \dots, u_n$ according to the equations (4.15).

Theorem 4.3. *Let $q := e^{2\pi i\nu}$ be not a root of the characteristic equation*

$$\det\left(q\, S + S^T\right) = 0. \tag{4.9}$$

Then there exist n linearly independent solutions $\phi^{(1)}, \dots, \phi^{(n)}$ of the system (4.14) analytic in $(\lambda, u_1, \dots, u_n)$ on

$$\mathbb{C} \setminus \cup_j L_j \times Fr(\hat\mu, R, e, S, C) \tag{4.10}$$

such that the monodromy transformations $M_1, \dots, M_n$ along the small loops encircling counter-clockwise the points $u_1, \dots, u_n$ are reflections

$$M_i \phi^{(j)} = \phi^{(j)} - q^{1/2}\left(\check\phi^{(i)}, \phi^{(j)}\right)_q \phi^{(i)} \tag{4.11}$$

w.r.t. the bilinear form

$$\left(\check\phi^{(i)}, \phi^{(j)}\right)_q := \left(q^{1/2} S + q^{-1/2} S^T\right)_{ij}. \tag{4.12}$$

Proof For $\mathrm{Re}\,\nu << 0$ we can construct a fundamental matrix $\Phi(\lambda)$ of solutions to (4.5) applying a Laplace-type transform to a fundamental matrix $Y(z)$ of solutions to (2.44):

$$\Phi(\lambda) = \frac{i}{\sqrt{2\pi}}(1 + q^{-1}) \int_0^\infty Y(z) e^{-\lambda z} \frac{dz}{z^{\nu + \frac12}}. \tag{4.13}$$

Technically it is more convenient, following [4, 12, 14], to use a sort of inverse transform expressing solutions to (2.44) via Laplace-type integrals applied to the solutions to the system (4.5).

We rewrite the Fuchsian system (4.5) in the standard way

$$\frac{d\phi}{d\lambda} = \sum_{i=1}^n \frac{A_i}{\lambda - u_i}\, \phi \tag{4.14}$$

where

$$A_i := E_i \left(\nu - \frac{1}{2} - V\right), \; i = 1, \dots, n. \tag{4.15}$$

The matrices E_i were defined in (2.43). This system has Fuchsian singularities at the points $\lambda = u_1, \dots, \lambda = u_n, \lambda = \infty$. The dependence on $u_1, \dots, u_n$ imposed by (4.4) reads

$$\frac{\partial \phi}{\partial u_i} = \left(V_i - \frac{A_i}{\lambda - u_i}\right) \phi. \tag{4.16}$$

We define the needed solutions $\phi^{(1)}, \dots, \phi^{(n)}$ according to their behaviour near the finite singularities of the system.

Let us first consider the generic case ν being not a half-integer.

Lemma 4.4. *For*

$$\nu \notin \mathbb{Z} + \frac{1}{2} \tag{4.17}$$

there exist n solutions of the Fuchsian system (4.14) analytic in

$$\lambda \in \mathbb{C} \setminus \cup_j L_j \tag{4.18}$$

of the form

$$\phi^{(i)} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \nu\right)} (u_i - \lambda)^{\nu - \frac{1}{2}} \left[e_i + O(u_i - \lambda)\right], \; \lambda \to u_i \tag{4.19}$$

where the expression in the square brackets is analytic in a small neighborhood of $\lambda = u_i$. Here e_i is the column-vector having the i-th component 1 and all other components zero. The solutions are determined uniquelly after chosing of a branch of the functions $(u_i - \lambda)^{\nu - \frac{1}{2}}$.

We choose the normalizing factor on (4.19) in order to meet the shift condition (3.14).

Proof The $n - 1$ eigenvalues of the matrix A_i are zeroes and one eigenvalue is equal to $\nu - 1/2$. So the eigenvalues of the monodromy transformation M_i are all equal to 1 but a simple eigenvalue $-q$. In the nonresonant case

$$\nu - \frac{1}{2} \notin \mathbb{Z}$$

the unique eigenvector of M_i with the eigenvalue $-q$ can be represented, after an appropriate normalization, in the form (4.19). Lemma is proved.

Observe that any solution ϕ of the Fuchsian system (4.14) near $\lambda = u_i$ can be uniquelly represented in the form

$$\phi = g\, \phi^{(i)} + analytic$$

for some constant g. Here *analytic* is a solution of the same system analytic at $\lambda = u_i$. Particularly, one can find a matrix of constants $G = (g^{ij})$ such that

$$\phi^{(i)} = g^{ji} \phi^{(j)} + analytic \text{ near } \lambda = u_j. \tag{4.20}$$

By definition we have

$$g^{ii} = 1, \; i = 1, \dots, n.$$

Lemma 4.5. *The action of the monodromy transformation M_j onto the solution $\phi^{(i)}$ is given by the formula*

$$M_j \phi^{(i)} = \phi^{(i)} - (q+1) g^{ji} \phi^{(j)}. \tag{4.21}$$

Proof From (4.19) we have

$$M_i \phi^{(i)} = -q\, \phi^{(i)}.$$

This gives (4.21) for $j = i$. If

$$\phi^{(i)} = g^{ji} \phi^{(j)} + \psi$$

with ψ analytic near $\lambda = u_j$ then

$$M_j \phi^{(i)} = M_j \left(g^{ji} \phi^{(j)} + \psi \right)$$

$$= -q\, g^{ji} \phi^{(j)} + \psi = \phi^{(i)} - (q+1) g^{ji} \phi^{(j)}.$$

Lemma is proved.

Similarly, continuing the solution $\phi^{(i)}$ clockwise around $\lambda = u_j$ we obtain the transformation

$$M_j^{-1} \phi^{(i)} = \phi^{(i)} - (1 + q^{-1}) g^{ji} \phi^{(j)}. \tag{4.22}$$

Let us introduce another system of branch cuts $\hat{L}_1, \dots, \hat{L}_n$ on the complex λ-plane opposite to $L_1, \dots, L_n$ resp. We introduce solutions $\hat{\phi}^{(1)}, \dots, \hat{\phi}^{(n)}$ to (4.14) of the same form (4.19) analytic on

$$\mathbb{C} \setminus \cup_j \hat{L}_j. \tag{4.23}$$

Here we choose the branches of the functions $(u_j - \lambda)^{\nu - \frac{1}{2}}$ near $\hat{L}_j$ by rotating the branches of the same functions defined near L_j in the counter-clockwise direction.

Lemma 4.6. *The result of the counter-clockwise analytic continuation of the solutions $\phi^{(1)}, \dots, \phi^{(n)}$ is related to the solutions $\hat{\phi}^{(1)}, \dots, \hat{\phi}^{(n)}$ by the transformation*

$$\left(\phi^{(1)}, \dots, \phi^{(n)} \right) = \left(\hat{\phi}^{(1)}, \dots, \hat{\phi}^{(n)} \right) \left[1 + (1 + q^{-1}) G_+ \right] \tag{4.24}$$

Similarly, for the clockwise analytic continuation of the same functions one obtains

$$\left(\phi^{(1)}, \dots, \phi^{(n)} \right) = -q^{-1} \left(\hat{\phi}^{(1)}, \dots, \hat{\phi}^{(n)} \right) \left[1 + (1 + q) G_- \right]. \tag{4.25}$$

Here the matrices $G_\pm$ are defined as follows

$$(G_+)^{ij} = \begin{cases} 0, & i \geq j \\ g^{ij}, & i < j \end{cases}$$

$$(G_-)^{ij} = \begin{cases} 0, & i \leq j \\ g^{ij}, & i > j \end{cases} \tag{4.26}$$

(4.27)

Proof Let us prove the first formula. Counter-clockwise analytic continuation of $\phi^{(1)}$ till $\hat{L}_1$ does not meet obstructions: we simply rotate the branchcut L_1 untill $\hat{L}_1$. So

$$\hat{\phi}^{(1)} = \phi^{(1)}.$$

To continue analytically $\phi^{(2)}$ till $\hat{L}_2$ we are to cross the branchcut L_1. This changes $\phi^{(2)}$ to $M_1^{-1}\phi^{(2)}$. Using (4.22) for $i = 2$, $j = 1$ we obtain for the analytic continuation

$$\hat{\phi}^{(2)} = M_1^{-1}\phi^{(2)} = \phi^{(2)} - (1+q^{-1})g^{12}\hat{\phi}^{(1)}.$$

Similarly,

$$\begin{aligned}\hat{\phi}^{(3)} &= M_1^{-1}M_2^{-1}\phi^{(3)} = M_1^{-1}\phi^{(3)} - (1+q^{-1})g^{23}M_1^{-1}\phi^{(2)} \\ &= \phi^{(3)} - (1+q^{-1})\left[\hat{\phi}^{(1)}g^{13} + g^{23}\hat{\phi}^{(2)}\right],\end{aligned}$$

etc.,

$$\hat{\phi}^{(n)} = M_1^{-1}M_2^{-1}\dots M_{n-1}^{-1}\phi^{(n)} = \phi^{(n)} - (1+q^{-1})\left[\hat{\phi}^{(1)}g^{1n} + \hat{\phi}^{(2)}g^{2n} + \dots + \hat{\phi}^{(n-1)}g^{n-1,n}\right].$$

This gives (4.24). Similarly, continuing analytically the functions $\phi^{(i)}$ in the clockwise direction and using (4.21) we derive (4.25) (here we are to take into account the change of the branches of the functions $(u_i - \lambda)^{\nu-\frac{1}{2}}$). Lemma is proved.

Let us define now vector-functions $Y_R^{(i)}(z)$ and $Y_L^{(i)}(z)$ via the following (inverse) Laplace transforms

$$\begin{aligned}Y_R^{(i)}(z) &:= \frac{i}{(2\pi)^{3/2}}\frac{q}{1+q}z^{\nu+\frac{1}{2}}\oint_{\hat{L}_i}\hat{\phi}^{(i)}(\lambda)e^{z\,\lambda}d\lambda \\ Y_L^{(i)}(z) &:= \frac{i}{(2\pi)^{3/2}}\frac{q}{1+q}z^{\nu+\frac{1}{2}}\oint_{L_i}\phi^{(i)}(\lambda)e^{z\,\lambda}d\lambda.\end{aligned} \tag{4.28}$$

Here the loop integrals are taken along the infinite cycle coming from infinity along the left shore of the branchcut $L_j/\hat{L}_j$ resp., then encircling the point $\lambda = u_j$ and returning to infinity along another shore of the same branchcut. We define the branch of $z^{\nu+\frac{1}{2}}$ doing a branchcut along the negative half of the line ℓ.

Lemma 4.7. *The vector-functions $Y_{R/L}^{(i)}(z)$ are linearly independent solutions to the system (2.44). They are analytic in the half-planes $\Pi_{R/L}$ resp. to the right/to the left of the line ℓ. i.e.,*

$$\Pi_+ = \{z \mid \varphi < \arg z < \varphi + \pi\}. \tag{4.29}$$

In these half-planes they have the asymptotic development of the form

$$Y_{R/L}^{(i)}(z) \sim (e_i + O(1/z))\, e^{z\,u_i}. \tag{4.30}$$

Corollary 4.8. *The Stokes matrix of the operator (4.7) w.r.t. the oriented line ℓ is equal to*

$$S = 1 + (1+q^{-1})G_+. \tag{4.31}$$

The transposed Stokes matrix is equal to

$$S^T = 1 + (1+q)G_-. \tag{4.32}$$

Proof of Lemma. At infinity the Fuchsian system (4.14) has a regular singularity. So the solutions $\phi^{(i)}(\lambda)$ and $\hat{\phi}^{(i)}(\lambda)$ grow at infinity not faster than some power of $|\lambda|$. This proves convergence of the integrals for $|\lambda| \to \infty$, $z \in \Pi_{R/L}$ resp. Analyticity of the integrals (4.28) in the half-planes Π_R/Π_L resp. is a standard fact of the theory of Laplace integrals (see, e.g., [7]). Integrating the convergent expansions near $\lambda = u_i$ of the form (4.19) of the functions $\phi^{(i)}(\lambda)/\hat{\phi}^{(i)}(\lambda)$ multiplied by $e^{z\,\lambda}$ we arrive at the asymptotic developments (4.30) at $z = \infty$ of the integrals. Plugging the integrals (4.28) into (2.44) and integrating by parts (we can neglect the boundary terms due to the exponential vanishing of the integrand at infinity) we prove that the integrals satisfy the system (2.44). Their linear independence follows from the independence of the principal terms of the asymptotic developments. Lemma is proved. □

Proof of Corollary. To analytically continue the integral $Y_L^{(i)}(z)$ in the clockwise direction to the half-plane Π_L through the positive part of the line ℓ one is to rotate counter-clockwise the contour L_i till it will get to the position $\hat{L}_i$. Using (4.24) we derive that in a narrow sector around the positive half-line ℓ

$$\left(Y_L^{(1)}, \dots, Y_L^{(n)}\right) = \left(Y_R^{(1)}, \dots, Y_R^{(n)}\right)\left[1 + (1+q^{-1})G_+\right].$$

This gives the formula (4.31) for the Stokes matrix. Similarly, continuing analytically $Y_L^{(i)}(z)$ in the counter-clockwise direction through the negative part of the line ℓ (here the branch of $z^{\nu+\frac{1}{2}}$ changes) and using (4.25) we arrive at

$$\left(Y_L^{(1)}, \dots, Y_L^{(n)}\right) = \left(Y_R^{(1)}, \dots, Y_R^{(n)}\right)\left[1 + (1+q)G_-\right].$$

As we know from the theory of Stokes matrices for the operator (4.7), in the narrow sector near the positive part of ℓ

$$\left(Y_L^{(1)}, \dots, Y_L^{(n)}\right) = \left(Y_R^{(1)}, \dots, Y_R^{(n)}\right) S$$

and in the narrow sector near the negative part of ℓ

$$\left(Y_L^{(1)}, \dots, Y_L^{(n)}\right) = \left(Y_R^{(1)}, \dots, Y_R^{(n)}\right) S^T$$

(see, e.g., [14]). Corollary is proved. □

Using (4.31) and (4.32) we rewrite the formula (4.21) of the monodromy transformation that we redenote $M_i(q) \in GL(n, \mathbb{C}[q, q^{-1}])$ as follows

$$M_i(q)\phi^{(j)} = \begin{cases} \phi^{(j)} - q\, s_{ij}\phi^{(i)}, & i < j \\ -\, q\, \phi^{(i)}, & i = j \\ \phi^{(j)} - s_{ji}\phi^{(i)}, & i > j \end{cases} \tag{4.33}$$

This is the reflection (4.11) w.r.t. the bilinear form (4.12).

It remains to prove linear independence of the solutions $\phi^{(1)}, \dots, \phi^{(n)}$ under the assumption (4.17). Any linear dependence

$$c_1\phi^{(1)} + \cdots + c_n\phi^{(n)} = 0$$

would give a vector invariant w.r.t. to the total monodromy operator $M_n M_{n-1} \dots M_1$ corresponding to a big counter-clockwise loop around the origin. From (4.24), (4.25) it follows that such an operator acts as follows

$$\left(\phi^{(1)}, \dots, \phi^{(n)}\right) \mapsto -q\left(\phi^{(1)}, \dots, \phi^{(n)}\right) S^{-T} S. \tag{4.34}$$

Here we denote

$$S^{-T} := \left(S^T\right)^{-1}.$$

The transformation (4.34) has an invariant vector *iff*

$$\det\left[q\, S^{-T} S + 1\right] = 0.$$

This coincides with degeneracy of the matrix (4.12). The contradiction proves independence of the solutions.

To complete the proof of Theorem in the non-resonant case we are to prove that the solutions $\phi^{(i)}$ satisfy also the equations (4.16). To this end we consider the fundamental matrix

$$\Phi(\lambda; u) := \left(\phi^{(1)}(\lambda; u), \dots, \phi^{(n)}(\lambda; u)\right) \tag{4.35}$$

(recall that the dependence of this matrix on $u = (u_1, \dots, u_n)$ is determined uniquely). Due to compatibility of the system (4.14), (4.16) the matrix

$$\partial_i \Phi - \left(\frac{E_i\left(\frac12 - \nu + V\right)}{\lambda - u_i} + V_i\right)\Phi$$

is again a matrix solution of (4.14). So it has the form

$$\partial_i \Phi - \left(\frac{E_i\left(\frac12 - \nu + V\right)}{\lambda - u_i} + V_i\right)\Phi = \Phi\, T_i$$

for some matrix T_i independent on λ. Using the expansions

$$\phi^{(i)} = (u_i - \lambda)^{\nu - \frac12}\left[e_i + \frac{e_i\,(u_i - \lambda)}{\nu + \frac12}(V_i e_i - 2H_i e_i) + O\left(u_i - \lambda\right)^2\right],\ \lambda \to u_i$$

with

$$H_i = \frac12 \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}$$

of the solutions (4.19) and

$$\psi = a + O(u_i - \lambda),\ \lambda \to u_i,\ \left(\nu - \frac12 + V\right) a = 0$$

of any solution to (4.14) analytic at $\lambda = u_i$ we prove that $T_i = 0$. This completes the proof of Theorem in the nonresonant case (4.17).

Before going further we will give an interpretation of the bilinear form $(\ ,\ )_\nu$ defined in (4.12). Let us denote $L(\nu)$ the n-dimensional space of solutions to the system (4.14), (4.16). We have a natural pairing (cf. (3.8))

$$\begin{aligned} &L(-\nu) \times L(\nu) \to \mathbb{C}, \\ &(\phi_{-\nu}, \psi_\nu) := \phi_{-\nu}^T(\lambda; u)(U - \lambda)\psi_\nu(\lambda; u),\ \psi_\nu \in L(\nu),\ \phi_{-\nu} \in L(-\nu). \end{aligned} \tag{4.36}$$

It is easy to see that the pairing does not depend on λ neither on u. Clearly, this pairing does not degenerate under the nonresonancy assumption (4.17).

Lemma 4.9. *In the nonresonant case the matrix of the pairing w.r.t. the bases $\phi_\nu^{(1)}$, ..., $\phi_\nu^{(n)}$ and $\phi_{-\nu}^{(1)}$, ..., $\phi_{-\nu}^{(n)}$ in $L(\nu)$ and $L(-\nu)$ resp. coincides, up to a scalar factor with (4.12):*

$$\left(\phi_{-\nu}^{(i)}, \phi_\nu^{(j)}\right) = \left(q^{1/2} S + q^{-1/2} S^T\right)_{ij}. \tag{4.37}$$

So, the formula for the monodromy transformations (4.21) can be recast into the form

$$M_i \phi_\nu^{(j)} = \phi_\nu^{(j)} - q^{1/2} \left(\phi_{-\nu}^{(i)}, \phi_\nu^{(j)}\right) \phi_\nu^{(i)}. \tag{4.38}$$

Proof We observe first that, for any i the solution $\phi_\nu^{(i)}$ is orthogonal to any solution $\psi \in L(-\nu)$ analytic at $\lambda = u_i$. So, using (4.20) we obtain

$$\left(\phi_{-\nu}^{(i)}, \phi_\nu^{(j)}\right) = q^{-1/2}(1+q) g^{ij}.$$

Using (4.31), (4.32) we arrive at the proof of Lemma. □

Let us now proceed to the resonant case. We consider first the particular case $\nu = \frac{1}{2}$ (actually, this is still a nonresonant case in the standard sense of the theory of linear ODEs with rational coefficients. However, the monodromy matrices appear to have nontrivial Jordan blocks). The matrices A_i are now all nilpotent,

$$A_i^2 = 0.$$

So they have just one block $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in their Jordan normal form. The monodromy matrix M_i has $n-1$ linearly independent eigenvectors with the eigenvalue 1 and, if the i-th row of the matrix V is nonzero, it has one Jordan block $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$. That means that there are $n-1$ solutions to (4.14) analytic at $\lambda = u_i$. One of these solutions belongs to the image of $M_i - \mathrm{id}$. We denote it, like above, $\phi^{(i)}$. It can be normalized in such a way that

$$\phi^{(i)}(\lambda) = e_i + O(u_i - \lambda), \ \lambda \to u_i. \tag{4.39}$$

Such a normalization determines the solution uniquely. A logarithmic solution denoted by $\chi^{(i)}(\lambda)$ corresponds to $\phi^{(i)}(\lambda)$ in the following sense

$$(M_i - \mathrm{id})\, \chi^{(i)} = 2\pi\, i\, \phi^{(i)}. \tag{4.40}$$

Near $\lambda = u_i$ it can be written as

$$\chi^{(i)}(\lambda) = \log(u_i - \lambda)\, \phi^{(i)}(\lambda) + \delta^{(i)}(\lambda) \tag{4.41}$$

where $\delta^{(i)}(\lambda)$ is a vector-function of λ analytic at $\lambda = u_i$. It is easy to see that the value of this function at $\lambda = u_i$ must satisfy the condition

$$\left[V \delta^{(i)}(\lambda = u_i)\right]_i = -1 \tag{4.42}$$

(here and below $[\ ]_i$ stands for the i-th coordinate of the vector). To obtain a basis in the space of solutions to (4.14) with $\nu = 1/2$ we are to add to $\phi^{(i)}$ and $\chi^{(i)}$, where i is fixed, $n-2$ independent solutions analytic at $\lambda = u_i$. Any such a solution $\psi(\lambda)$ must satisfy, like in (4.42), the equation

$$[V\psi(\lambda = u_i)]_i = 0.$$

We can always assume that the i-th coordinate of $\psi(u_i)$ is equal to zero.

Globally the solutions $\phi^{(1)}, \ldots, \phi^{(n)}$ are analytic in (4.6) with the same choice of the branch cuts $L_1, \ldots, L_n$ as above. We define a matrix g^{ij} in such a way that $g^{ii} = 0$ and

$$\phi^{(i)}(\lambda) = \frac{1}{2\pi i} g^{ji}\chi^{(j)}(\lambda) + \textit{analytic},\ \lambda \to u_j,\ i \neq j. \tag{4.43}$$

Like in Lemma 4.5, using the last equation we prove

Lemma 4.10. *The monodromy transformation M_i acts onto the solution $\phi^{(j)}$ as follows*

$$M_i\phi^{(j)} = \phi^{(j)} + g^{ij}\phi^{(i)}. \tag{4.44}$$

Observe that

$$M_i^{-1}\phi^{(j)} = \phi^{(j)} - g^{ij}\phi^{(i)}.$$

Let us introduce another system $\hat\phi^{(1)}, \ldots, \hat\phi^{(n)}$ of solutions to (4.14) with the same behaviour (4.39) but with the branchcuts along $\hat L_1, \ldots, \hat L_n$.

Lemma 4.11. *The results of counter-clockwise/clockwise analytic continuation of the solutions $\phi^{(1)}, \ldots, \phi^{(n)}$ are given by the following formulae*

$$\left(\phi^{(1)}, \ldots, \phi^{(n)}\right) = \left(\hat\phi^{(1)}, \ldots, \hat\phi^{(n)}\right)(1 \pm G_\pm). \tag{4.45}$$

Here G_+/G_- are the upper/lower triangular parts of the matrix $G = (g^{ij})$.

Proof is similar to that of Lemma 4.6.

We introduce also the solutions $\hat\chi^{(1)}, \ldots, \hat\chi^{(n)}$ related to $\hat\phi^{(1)}, \ldots, \hat\phi^{(n)}$ by the formulae (4.41).

Lemma 4.12. *The matrix G has the form*

$$G = S - S^T. \tag{4.46}$$

Proof We construct solutions to the system (2.44) using the following Laplace integrals

$$\begin{aligned} Y_L^{(i)}(z) &= \frac{z}{2\pi i}\oint_{L_i}\chi^{(i)}(\lambda)e^{\lambda z}d\lambda = -z\int_{L_i}\phi^{(i)}(\lambda)e^{\lambda z}d\lambda \\ Y_R^{(i)}(z) &= \frac{z}{2\pi i}\oint_{\hat L_i}\hat\chi^{(i)}(\lambda)e^{\lambda z}d\lambda = -z\int_{\hat L_i}\phi^{(i)}(\lambda). \end{aligned} \tag{4.47}$$

Like in Lemma 4.7 we prove that these are the columns of the matrix solutions $Y_{L/R}(z)$ to (2.44) having the asymptotic development (4.30) in $\Pi_{L/R}$ resp. From (4.45) we conclude that

$$S = 1 + G_+,\ S^T = 1 - G_-.$$

This proves Lemma. □

We define now an antisymmetric bilinear form on $L(1/2)$ by one of the following three expressions:
(4.48)
$$(\phi,\psi)_{1/2} := -2\pi\,\phi^T(\lambda)\,V\,\psi(\lambda) = 2\pi\,\frac{d\phi^T}{d\lambda}\,(U-\lambda)\,\psi(\lambda) = -2\pi\,\phi^T(\lambda)\,(U-\lambda)\,\frac{d\psi}{d\lambda}.$$
It is induced by the pairing $L\left(-\frac{1}{2}\right)\times L\left(\frac{1}{2}\right)\to\mathbb{C}$ and by the shift operator $L\left(-\frac{1}{2}\right)\to L\left(\frac{1}{2}\right)$ defined by (3.14).

Lemma 4.13. *The matrix of the bilinear form is*
$$\left(\phi^{(i)},\phi^{(j)}\right)_{1/2} = -i\left(S-S^T\right)_{ij}. \tag{4.49}$$

Proof It is easy to see that $\left(\phi^{(i)},\psi\right)_{1/2}=0$ for any solution $\psi(\lambda)$ analytic at $\lambda=u_i$. Using (4.42) we obtain that $\left(\phi^{(i)},\chi^{(i)}\right)_{1/2}=2\pi$. From these two facts and from (4.43) we derive that
$$\left(\phi^{(i)},\phi^{(j)}\right)_{1/2} = -i\,g^{ij}.$$
From Lemma 4.12 we obtain (4.49). Lemma is proved. □

As a consequence we obtain that, under the assumption (4.9), (i.e., the assumption of nondegeneracy of the matrix $S-S^T$) the solutions $\phi^{(1)}, \dots, \phi^{(n)}$ form a basis in the space $L(1/2)$, and that the monodromy transformations act in this basis as in (4.11):
$$M_i\phi^{(j)} = \phi^{(j)} - i\left(\phi^{(i)},\phi^{(j)}\right)_{1/2}\phi^{(i)}.$$
As above we prove that the functions $\phi^{(1)}(\lambda;u), \dots, \phi^{(n)}(\lambda;u)$ satisfy also the equations (4.16).

We have proved Theorem for $\nu=1/2$. To obtain the proof for $\nu=-m+\frac{1}{2}$ with a positive integer m we just use the isomorphism
$$\frac{d^m}{d\lambda^m}\,:\,L(1/2)\to L(-m+1/2).$$
This produces a basis in $L(-m+1/2)$ with the needed monodromy. Finally, for $\nu = m+\frac{1}{2}$ with a positive integer m we use the non-degenerate pairing (4.36) to construct a basis in $L(m+1/2)$ dual to the basis in $L(-m+1/2)$. This completes the proof of Theorem. □

We will now use the above Theorem in order to describe the geometry of the discriminant Σ w.r.t. the geometry induced by the intersection form. We first recall that, in any chart $Fr(\hat\mu,R,e,S,C)$ on M_{ss} the intersection $\Sigma\cap M_{ss}$ splits into the union of hypersurfaces
$$\Sigma\cap M_{ss} = \cup_{k=1}^n\{u_k=0\},\quad u_i\neq u_j\quad\text{for } i\neq j. \tag{4.50}$$
We will describe the behaviour of the flat coordinates $p^1, \dots, p^n$ of the intersection form on (4.50). Within any connected and simply connected domain in the coordinate patch

$Fr(\hat{\mu}, R, e, S, C) \setminus \Sigma$ one can choose, for $d \neq 1$ the particular system of coordinates by putting

$$p^a(u) = \frac{2}{1-d} \sum_{i=1}^n u_i \psi_{i1}(u) \phi_i^{(a)}(u; \lambda = 0; \nu = 0), \quad a = 1, \dots, n \tag{4.51}$$

(cf. the formula (2.66)). Here the basis $\phi^{(a)} = (\phi_1^{(a)}, \dots, \phi_n^{(a)})^T$ of solutions to the Fuchsian system (4.14) is chosen as in (4.19). Note that, due to (4.37) (for $\nu = 0$) one has

$$(dp^a, dp^b) =: G^{ab} = \left(S + S^T\right)^{ab}. \tag{4.52}$$

We will describe the limiting behaviour of the basis of the periods (4.51) near (4.50). The following statement is a refinement of Lemma G.2 in [12].

Theorem 4.14. *Let M be a semisimple Frobenius manifold satisfying $d \neq 1$ and*

$$\det(S + S^T) \neq 0. \tag{4.53}$$

Let $\mathbf{D} \subset Fr(\hat{\mu}, R, e, S, C) \setminus \Sigma \subset M_{ss} \setminus \Sigma$ be a connected simply connected domain such that $\Sigma \subset \overline{\mathbf{D}}$. Then the functions $p^1(u)$, ..., $p^n(u)$ can be extended continuously up to Σ. With respect to this continuation the component

$$u_k = 0$$

of Σ becomes an affine hyperplane

$$p^k = 0. \tag{4.54}$$

Proof It is technically more convenient to compute the limiting behaviour of the λ-periods

$$\tilde{p}^a(u; \lambda) = \frac{2}{1-d} \sum_{i=1}^n (u_i - \lambda) \psi_{i1}(u) \phi_i^{(a)}(u; \lambda; \nu = 0), \quad a = 1, \dots, n \tag{4.55}$$

on Σ_λ. Because of (4.52), 4.53) the functions (4.55) are independent. Note that the k-th component of the intersection of $\Sigma_\lambda \cap M_{ss}$ are the hypersurfaces

$$u_k = \lambda.$$

Near $u_k = \lambda$ one has

$$\phi_i^{(k)} = \frac{\sqrt{2}}{\sqrt{u_k - \lambda}} \left(\delta_i^k + O(u_k - \lambda)\right) \tag{4.56}$$

and, for $l \neq k$

$$\phi_i^{(k)} = \frac{G^{kl}}{\sqrt{u_l - \lambda}} \left(\delta_i^l + O(\sqrt{u_l - \lambda})\right). \tag{4.57}$$

So, near $u_k = \lambda$

$$\tilde{p}^k(u; \lambda) = \frac{2\sqrt{2}}{1-d} \psi_{k1}(u) \sqrt{u_k - \lambda} + O(u_k - \lambda) \tag{4.58}$$

and, for $l \neq k$

$$\tilde{p}^l(u;\lambda) = \tilde{p}^l_0 + \frac{\sqrt{2}}{1-d} G^{kl}\psi_{k1}(u)\sqrt{u_k - \lambda} + O(u_k - \lambda) \tag{4.59}$$

where $p^l_0 = p^l_0(u_1, \ldots, \hat{u}_k, \ldots, u_n; \lambda)$ is an analytic function on

$$Fr(\hat{\mu}, R, e, S, C) \setminus \{u_k = \lambda\}.$$

From these formulae the continuity of the functions (4.55) up to Σ_λ readily follows. In particular, for $\lambda = 0$ one obtains continuity of the periods (4.51) up to Σ. Also (4.54) readily follows from (4.58). The Theorem is proved. □

Observe that the angles between the hyperplanes (4.54) can be computed from the Gram matrix (4.52).

We will now apply the formulae (4.58), (4.59) in order to describe the analytic properties of the dual almost Frobenius manifold near Σ.

Theorem 4.15. *Under the assumptions $d \neq 1$ and (4.53) the structure coefficients $\overset{*}{c}{}^c_{ab}(p)$ of the dual Frobenius manifold are continuous up to the hyperplane (4.54) for all a, b, c except*

$$\overset{*}{c}{}^a_{kk} = \frac{1}{1-d}\frac{1}{p^k} G^{ka} + \text{regular terms}, \quad p^k \to 0. \tag{4.60}$$

Proof Let us first derive a formula for $\overset{*}{c}{}^c_{ab}(u)$. We will use the coordinates (4.55) and we will set $\lambda = 0$ at the end of the computation.

Lemma 4.16. *At any point $u = (u_1, \ldots, u_n) \in M_{ss} \setminus \Sigma_\lambda$ one has*

$$\frac{\partial}{\partial p^b} * \frac{\partial}{\partial p^c} = \overset{*}{c}{}^a_{bc}(u)\frac{\partial}{\partial p^c}$$

where

$$\overset{*}{c}{}^a_{bc}(u) = G_{cd}G_{bf}\sum_{i=1}^n \frac{u_i - \lambda}{\psi_{i1}(u)}\phi^{(a)}_i(u;\lambda)\phi^{(f)}_i(u;\lambda)\phi^{(d)}_i(u;\lambda). \tag{4.61}$$

Proof From (4.3) it follows that

$$d\tilde{p}^a = \sum_{i=1}^n \psi_{i1}(u)\phi^{(a)}_i(u;\lambda)du_i.$$

From the definition of the canonical coordinates it follows that

$$du_i \cdot du_j = \delta_{ij}\psi^{-2}_{i1}du_i.$$

So

$$d\tilde{p}^a \cdot d\tilde{p}^b = \sum_{i=1}^n \phi^{(a)}_i \phi^{(b)}_i du_i$$

(we will omit writing the arguments u, λ of the functions under consideration for the sake of brevity of the formulae). Using

$$\sum_{i=1}^{n}(u_i-\lambda)\phi_i^{(a)}\phi_i^{(b)}=G^{ab}$$

we derive that

$$du_i=\psi_{i1}^{-1}(u_i-\lambda)G_{ab}\phi_i^{(a)}dp^b.$$

So

$$d\tilde{p}^a\cdot d\tilde{p}^b=G_{cd}\sum_{i=1}^{n}\psi_{i1}^{-1}(u_i-\lambda)\phi_i^{(a)}\phi_i^{(b)}\phi_i^{(d)}\,d\tilde{p}^c.$$

In other words, in the coordinates (4.55) the structure coefficients of the multiplication $\cdot$ on the cotangent bundle read

$$c_c^{ab}(u)=G_{cd}\sum_{i=1}^{n}\psi_{i1}^{-1}(u_i-\lambda)\phi_i^{(a)}\phi_i^{(b)}\phi_i^{(d)}.$$

But we already now that on the cotangent bundle

$$\overset{*}{c}{}_c^{ab}=c_c^{ab}.$$

Lowering the index

$$\overset{*}{c}{}_{bc}^{a}=G_{bf}\,\overset{*}{c}{}_c^{fa}$$

we obtain the needed formula. The Lemma is proved. □

Prove of the Theorem. Substituting the expansions (4.58), (4.59) into the formula (4.61) we obtain, near $u_k=\lambda$

$$\overset{*}{c}{}_{bc}^{a}=G_{cd}G_{bf}\sum_{i=1}^{n}\frac{u_i-\lambda}{2\sqrt{2}\psi_{i1}(u_k-\lambda)^{3/2}}G^{ak}G^{fk}G^{dk}\left[\delta_i^k+O(\sqrt{u_k-\lambda})\right]$$

$$=\delta_c^k\delta_b^k\frac{G^{ak}}{2\sqrt{2}\psi_{k1}\sqrt{u_k-\lambda}}+O(1).$$

From this formula and from (4.58) the asymptotic formula (4.60) immediately follows. □

Similarly to (4.51) one can introduce, assuming $d\neq 2$ a system of n odd periods

$$\varpi^a=\frac{2}{2-d}\sum_{i=1}^{n}u_i\psi_{i1}(u)\phi_i^{(a)}(u;\lambda=0;\nu=\frac{1}{2}),\quad a=1,\dots,n. \tag{4.62}$$

From (4.49) it follows

Corollary 4.17. *The functions ϖ^1, …, ϖ have the following constant matrix of the Poisson brackets (3.12)*

$$< d\varpi^a\mathcal{V}, d\varpi^b >= i\,(S-S^T)_{ab}. \tag{4.63}$$

We will finally describe the behaviour of the fundamental system $\left(\phi^{(1)}, \ldots, \phi^{(n)}\right)$ at $|\lambda| \to \infty$. For simplicity only the generic case (4.17) will be considered.

Before we proceed to the formulation of the Theorem, let us recall some useful formulae from the theory of Laplace integrals. First,

$$\int_0^\infty z^{s-\frac{1}{2}} e^{-\lambda z} dz = \Gamma\left(s+\frac{1}{2}\right)\lambda^{-s-\frac{1}{2}}. \tag{4.64}$$

This formula coincides with the definition of the gamma function for $\operatorname{Re} s > -\frac{1}{2}$; for other $s \notin \mathbb{Z}+\frac{1}{2}$ it is obtained by analytic continuation. Differentiating, we derive

$$\int_0^\infty z^{s-\frac{1}{2}} \log^k z e^{-\lambda z} dz = \frac{d^k}{ds^k}\Gamma\left(s+\frac{1}{2}\right)\lambda^{-s-\frac{1}{2}}. \tag{4.65}$$

We will also need a matrix analogue of the last formula.

Lemma 4.18. *The following formula holds true*

$$\int_0^\infty e^{-\lambda z} z^{\hat\mu+s-\frac{1}{2}} z^R dz = \sum_{m\geq 0}\left[e^{R\,\partial_s}\right]_m \Gamma\left(\hat\mu+s+m+\frac{1}{2}\right)\times\lambda^{-(s+m+\hat\mu+\frac{1}{2})}\lambda^{-R}. \tag{4.66}$$

In this formula the m-th component $[\;]_m$ of the matrix is defined like in (2.22).

Proof We have

$$\begin{aligned}
\int_0^\infty e^{-\lambda z} z^{s+\hat\mu-\frac{1}{2}} z^R dz &= \int_0^\infty e^{-t} t^{\hat\mu+s-\frac{1}{2}} t^{R_0+\frac{R_1}{\lambda}+\frac{R_2}{\lambda^2}+\ldots} dt\,\lambda^{-(\hat\mu+s+\frac{1}{2})}\lambda^{-R}\\
&= \sum_{m\geq 0}\sum_{k\geq 0}\int_0^\infty e^{-t} t^{\hat\mu+s-\frac{1}{2}}\lambda^{-m}\frac{[R^k]_m \log^k t}{k!} dt\,\lambda^{-(\hat\mu+s+\frac{1}{2})}\lambda^{-R}\\
&= \sum_{m\geq 0}\sum_{k\geq 0}\frac{1}{k!}\partial_s^k\Gamma(\hat\mu+s+\frac{1}{2})\frac{[R^k]_m}{\lambda^m}\lambda^{-(\hat\mu+s+\frac{1}{2})}\lambda^{-R}\\
&= \sum_{m\geq 0}\sum_{k\geq 0}\left[\frac{R^k}{k!}\right]_m \Gamma(\hat\mu+s+m+\frac{1}{2})\times\lambda^{-(\hat\mu+s+m+\frac{1}{2})}\lambda^{-R}\\
&= \sum_m [e^{R\partial_s}]_m\Gamma(\hat\mu+s+m+\frac{1}{2})\times\lambda^{-(\hat\mu+s+m+\frac{1}{2})}\lambda^{-R}.
\end{aligned}$$

The Lemma is proved. □

Theorem 4.19. *At $|\lambda| \to \infty$ within the domain (4.6) the fundamental system of solutions (4.19) has the following expansion*

$$\begin{aligned}
&\Phi = \left(\phi^{(1)}, \ldots, \phi^{(n)}\right)\\
&= \frac{i}{\sqrt{2\pi}}(q^{1/2}+q^{-1/2})^{-1}\sum_{p=0}^\infty\sum_{m\geq 0}\Theta_p(t)\left[e^{-R\,\partial_\nu}\right]_m\Gamma\left(p+m+\hat\mu-\nu+\frac{1}{2}\right)\\
&\times\lambda^{-(p+m+\hat\mu-\nu+\frac{1}{2})}\lambda^{-R}C^{-1}\left(q^{1/2}S+q^{-1/2}S^T\right).
\end{aligned} \tag{4.67}$$

Proof For $\mathrm{Re}\,\nu << 0$ let us consider the matrix

$$\tilde{\Phi}(t;\lambda) = \frac{i}{\sqrt{2\pi}}(1+q^{-1})\int_0^\infty Y_R(z;t)e^{-\lambda z}\frac{dz}{z^{\nu+\frac12}}.$$

Here the integration is to be performed along a ray on the z-plane belonging to Π_R. It is a fundamental matrix of solutions to (4.14) analytic in (4.6) satisfying the following property: the i-th column of $\Phi(t;\lambda)$ near $\lambda = u_i$ behaves

$$\left(\Phi_0(t;\lambda)\right)_i = \phi^{(i)}(t;\lambda) + \mathit{analytic}$$

and it is analytic near $\lambda = u_j$ for any $j \neq i$. Using (4.20), (4.31), (4.32) we obtain

$$\Phi_0(t;\lambda) = (1+q)\left(\phi^{(1)}(t;\lambda),\dots,\phi^{(n)}(t;\lambda)\right)\left(q\,S + S^T\right)^{-1}. \tag{4.68}$$

On the other hand, we can obtain the expansion of $\tilde{\Phi}(t;\lambda)$ near the regular singularity $\lambda = \infty$ replacing Y_L by $Y_0 C$ and integrating the series (2.47)

$$\begin{aligned}
&\frac{i}{\sqrt{2\pi}}(1+q^{-1})\int_0^\infty Y_0(z;t)e^{-\lambda z}\frac{dz}{z^{\nu+\frac12}}\\
&= \frac{i}{\sqrt{2\pi}}(1+q^{-1})\int_0^\infty \sum_p \Theta_p(t) z^{p+\hat\mu-\nu-\frac12} z^R e^{-\lambda z} dz\\
&= \frac{i}{\sqrt{2\pi}}(1+q^{-1})\sum_p\sum_{m\ge 0}\Theta_p(t)\left[e^{-R\,\partial_\nu}\right]_m \Gamma(p+m+\hat\mu-\nu+\frac12)\\
&\times \lambda^{-(p+m+\hat\mu-\nu+\frac12}\lambda^{-R}.
\end{aligned} \tag{4.69}$$

Comparing (4.68) and (4.69) we obtain (4.69). Theorem is proved. □

We will now compute the matrix of the bilinear form (4.36) in the basis given by the columns of the matrix Φ_0.

Theorem 4.20. *In the basis of the columns of the matrix Φ_0 the bilinear form (4.36) is given by the matrix*

$$\left(\Phi_0^i(-\nu),\Phi_0^j(\nu)\right) = \left(q^{1/2}+q^{-1/2}\right)^2\left[\eta\left(q^{1/2}e^{-\pi i R}e^{-\pi i\hat\mu} + q^{-1/2}e^{\pi i R}e^{\pi i\hat\mu}\right)\right]^{-1}_{ij}. \tag{4.70}$$

Proof From the proof of the previous Theorem we obtained that

$$\Phi_0(\nu) = \Phi(\nu)\,M(\nu)$$

where

$$M(\nu) = (1+q)(q\,S + S^T)^{-1}C.$$

According to the formula (4.37) it remains to compute the product of the following matrices

$$M^T(-\nu)\left(q^{1/2}S + q^{-1/2}S^T\right)M(\nu).$$

This can be easily done using

$$S = Ce^{-\pi i R}e^{-\pi i\hat\mu}\eta^{-1}C^T \tag{4.71}$$

$$S^T = Ce^{\pi i R}e^{\pi i\hat\mu}\eta^{-1}C^T. \tag{4.72}$$

The Theorem is proved. □

In particular, for $\nu = 0$ (i.e., $q = 1$) one obtains the Gram matrix of the intersection form in the basis given by the flat coordinates with good behaviour at $\lambda = \infty$. Recall that this Gram matrix was used in the free field realization of the Virasoro operators associated with the given Frobenius manifold [19]. The general formula (4.70) was used in [19] for regularization of the Virasoro operators in the resonant case.

5. Examples and applications

5.1. **Almost duality in the singularity theory.** Let us compute the twisted periods for the Frobenius structure arising, according to the K.Saito theory [40] of primitive forms (see also [39, 42, 34, 29, 38, 23]), on the base of the universal unfolding (i.e., on the parameter space of a versal deformation of the singularity) of an isolated singularity $f(x)$, $x \in B \subset \mathbb{C}^N$ for a sufficiently small ball B, $f(x) = 0$, $df(0) = 0$. Denote $f_t(x)$ the corresponding versal deformation, $t = (t^1, \ldots, t^n)$ are the *flat coordinates* [39, 23] on the base M_f of the versal deformation, n is the Milnor number of the singularity. The discriminant $\Sigma \subset M_f$ consists of those values of the parameters t for which the fiber $f_t^{-1}(0) \cap B$ is singular. The locus Σ_λ (see (2.51) is defined in a similar way by the conditions of singularity of the fiber $f_t^{-1}(\lambda) \cap B$ for λ sufficiently close to 0. The *period mapping* in the singularity theory [39, 34]

$$\begin{aligned} &\pi : M_f \setminus \Sigma \to H^{N-1}\left(f_t^{-1}(0) \cap B\right) \\ &t \mapsto [\omega_t(x)] \end{aligned} \tag{5.1}$$

is obtained by choosing a holomorphic $(N-1)$-form on $(M_f \setminus \Sigma) \times B$ closed along the fibers $f_t^{-1}(0)$. The coordinate representation of the period mapping is a multivalued vector-function

$$\pi(t) = \left(\oint_{\sigma_1} \omega_t(x), \ldots, \oint_{\sigma_n} \omega_t(x) \right) \tag{5.2}$$

where $\sigma_1, \ldots, \sigma_n \in \tilde{H}_{N-1}\left(f_t^{-1}(0) \cap B; \mathbb{Z}\right)$ is a basis of vanishing cycles. Multivaluedness of the period mapping is encoded by an action of the monodromy group W of the singularity in the space of vanishing homologies. If the differential form $\omega_t(x)$ is chosen in a clever way then π is a local diffeomorphism. Then the isomorphism

$$d\pi_* : H_{N-1}\left(f_t^{-1}(0) \cap B; \mathbb{Z}\right) \to T_t^* M_f \tag{5.3}$$

dual to the differential $d\pi$ defines a bilinear form $(\ ,\ )$ on the cotangent bundle to $M_f \setminus \Sigma$ induced by the intersection index pairing on the homologies. This bilinear form is symmetric/skew-symmetric for $N-1$ even/odd.

To identify the above period mapping and intersection form of the singularity theory with those coming from the theory of Frobenius manifolds we are to assume that: 1). $N \equiv 1 (\mathrm{mod}\, 4)$. 2). The differential form $\omega_t(x)$ must be a good primitive form in the sense of the K.Saito's theory of primitive forms [40]. For the case of simple singularities an explicit construction of a good primitive form was obtained by M.Noumi [33]. For a general hypersurface singularity existence of a good primitive form has been proved by M.Saito [42]. Recall [28] that for the case of simple singularities the period mapping

($N-1$ must be even) produces an analytic isomorphism

$$M_f \to \mathbb{C}^n/W. \tag{5.4}$$

The flat coordinates on M_f in this case are given by a certain remarkable basis of homogeneous W-invariant polynomials. Their intrinsic contruction in terms of the Weyl group and its generalization to an arbitrary finite Coxeter group was found in [41]. The construction of the Frobenius structure on the orbit spaces of finite Coxeter groups was obtained in [11] (see also [12]). Some further generalizations of this construction see in [18].

We will consider here only the case of simple singularities labelled by Weyl groups of simply-laced Lie algebras [1], i.e., by Dynkin diagrams of ADE type. The reader may have in mind the example of the A_n singularity

$$f(x) = x^{n+1},\ f_t(x) = x^{n+1} + a_1 x^{n-1} + \cdots + a_n,\ x \in \mathbb{C}. \tag{5.5}$$

Here $a_1, \ldots, a_n$ are polynomials of $t^1, \ldots, t^n$ that are constructed in the following way [41]. Let us consider the series

$$k := f_t^{\frac{1}{n+1}}(x) = x + \frac{a_1}{n+1} x^{-1} + O\left(x^{-2}\right).$$

The flat coordinates $t^\alpha = t^\alpha(a_1, \ldots, a_n)$ are defined as the first n nontrivial coefficients of the inverse function expansion

$$x = k + \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \cdots + \frac{t^1}{k^n}\right) + O\left(k^{-(n+2)}\right).$$

The discriminant Σ consists of all polynomials with multiple roots. The subspace $t \in M_{ss} \subset M_f$ consists of all polynomials $f_t(x)$ that are good Morse functions, i.e., all their critical points are nondegenerate and the critical values are pairwise distinct. The dependence $f_t(x)$ on the flat coordinates satisfies the following identities [20]

$$\begin{aligned} \phi_\alpha \phi_\beta &= c_{\alpha\beta}^\gamma \phi_\gamma + K_{\alpha\beta}^a \frac{\partial f_t}{\partial x^a} \\ \partial_\alpha \phi_\beta &= \frac{\partial K_{\alpha\beta}^a}{\partial x^a}. \end{aligned} \tag{5.6}$$

Here

$$\phi_\alpha = \phi_\alpha(x;t) := \partial_\alpha f_t, \tag{5.7}$$

$c_{\alpha\beta}^\gamma = c_{\alpha\beta}^\gamma(t)$ are the structure constants tensor of the Frobenius manifold, the polynomials $K_{\alpha\beta}^a = K_{\alpha\beta}^a(x;t)$ are defined by (5.6), we assume also a summation w.r.t. Latin indices a etc. from 1 up to N. The versal deformation can be chosen to be a quasihomogeneous one, i.e., it satisfies also the identity

$$f_t = \sum_{a=1}^N r_a x^a \frac{\partial f_t}{\partial x^a} + \sum_\alpha (1 - q_\alpha) t^\alpha \phi_\alpha \tag{5.8}$$

with some rational numbers $r_1, \ldots, r_N$ satisfying

$$\sum_a r_a = \frac{N-d}{2}, \tag{5.9}$$

$$d = 1 - \frac{2}{h},\ q_\alpha = 1 - \frac{m_\alpha + 1}{h}, \tag{5.10}$$

h being the Coxeter number and $m_1, \ldots, m_n$ the exponents of the Dynkin diagram of the singularity.

We want to show that the twisted periods can be computed as the loop integrals of the following form

$$\tilde p(t;\nu) = \oint_\gamma f_t^{\nu - \frac{N-1}{2}}(x) d^N x. \tag{5.11}$$

Here γ is a cycle in $H_N\left(B \setminus f_t^{-1}(0), \mathbf{L}(q)\right)$ where the local system $\mathbf{L}(q)$ is defined by multiplication by $(-1)^{\frac{N-1}{2}} q$ where $q := e^{2\pi i \nu}$ (see details in [21]). To this end we are to prove that the (multivalued) functions $\xi_\alpha(t;\nu)$ of $t^1, \ldots, t^n$

$$\xi_\alpha = \oint_\gamma \phi_\alpha(x;t) f_t^{\nu'-1}(x)\, d^N x, \tag{5.12}$$

$$\nu' := \nu - \frac{N-1}{2}$$

satisfy the system (3.10) We will omit the reference to the cycle γ in the computations. What we will use of the symbol of loop integral is the usual properties that the integral of a total derivative vanishes and that the derivatives of the integrals along the parameters t coincide with the integrals of the derivatives.

Proposition 5.1. *The functions (5.12) satisfy the system (3.10) for the gradients of the twisted periods of the Frobenius manifold M_f.*

Proof Multiplying (5.8) by ϕ_β and using (5.6) we obtain

$$\sum_a r_a x^a \phi_\beta \frac{\partial f_t}{\partial x^a} + \mathcal{U}_\beta^\gamma \phi_\gamma + \sum_\alpha (1 - q_\alpha) t^\alpha K^a_{\alpha\beta} \frac{\partial f_t}{\partial x^a} = \phi_\beta f_t(x).$$

Another identity we obtain differentiating (5.8) along ∂_β:

$$\sum_a r_a x^a \frac{\partial \phi_\beta}{\partial x^a} + \sum_\alpha (1 - q_\alpha) t^\alpha \frac{\partial K^a_{\alpha\beta}}{\partial x^a} = q_\beta \phi_\beta. \tag{5.13}$$

Now, differentiating (5.12) and using (5.6) we obtain

$$\partial_\epsilon \xi_\beta = (\nu' - 1) c^\gamma_{\epsilon\beta} \oint \phi_\gamma f_t^{\nu'-2} d^N x.$$

Here we have eliminated the divergence term

$$\frac{\partial K^a_{\epsilon\beta}}{\partial x^a} f_t^{\nu'-1} + (\nu'-1) K^a_{\epsilon\beta} \frac{\partial f_t}{\partial x^a} f_t^{\nu'-2} = \frac{\partial}{\partial x^a}\left[K^a_{\epsilon\beta} f_t^{\nu'-1}\right].$$

Multiplying the last equation by $\mathcal{U}^\epsilon_\alpha$ and using associativity and (5.6), (5.8) we obtain

$$\mathcal{U}^\epsilon_\alpha \partial_\beta \xi_\epsilon = (\nu'-1) c^\gamma_{\alpha\beta} \xi_\gamma - c^\gamma_{\alpha\beta} \oint \sum_a \left[\phi_\gamma r_a x^a + \sum_\epsilon (1 - q_\epsilon) t^\epsilon K^a_{\epsilon\gamma}\right] \frac{\partial f_t^{\nu'-1}}{\partial x^a}\, d^N x.$$

Integrating the last integral by parts and using (5.13) we arrive at

$$\mathcal{U}_\alpha^\epsilon \partial_\beta \xi_\epsilon = \sum_\gamma (\nu' - 1 + \sum_a r_a + q_\gamma) c_{\alpha\beta}^\gamma \xi_\gamma.$$

Due to (5.13), this coincides with (3.10). Proposition is proved.

One can construct a basis of vanishing cycles $\sigma_1, \ldots, \sigma_n$ in $\tilde{H}_{N-1}(f_t^{-1}(0) \cap B, \mathbb{Z})$ for any $t \in M_f \setminus \Sigma$. The basis varies continuously with small variations of t. It is of particular convenience to choose a so-called *distinguished basis* of vanishing cycles corresponding to the properly ordered critical values $u_1(t), \ldots, u_n(t)$ of $f_t(x)$ connected by non-intersecting paths to the origin (see, e.g., [2]). The Givental's construction [21] gives a way to associate to it a basis $\gamma_1, \ldots, \gamma_n$ in the homology $H_N\left(B \setminus f_t^{-1}(0), \mathbf{L}(q)\right)$. Taking the integrals

$$\tilde{p}_1(t;\nu) = \oint_{\gamma_1} f_t^{\nu'}(x) d^N x, \ldots, \tilde{p}_n(t;\nu) = \oint_{\gamma_n} f_t^{\nu'}(x) d^N x \tag{5.14}$$

we obtain, locally on $M_f \setminus \Sigma$, the twisted period mapping. Globally the monodromy of the mapping is described by *twisted Picard - Lefschetz formulae* found in [21]). In the next Section we will derive the analogue of these formulae for the monodromy of the twisted period mapping on an arbitrary semisimple Frobenius manifold.

5.2. **Frobenius and almost Frobenius structures on orbit spaces of finite Coxeter groups.** Let $W \subset End(\mathbb{R}^n)$ be a finite irreducible Coxeter group. In [11] it was constructed a structure of polynomial Frobenius manifold on the orbit space

$$M = \mathbb{C}^n / W. \tag{5.15}$$

A coordinate system on the orbit space is given by choosing n homogeneous W-invariant polynomials $y^1(x), \ldots, y^n(x)$ generating the ring $\mathbb{C}[x^1, \ldots, x^n]^W$ of W-invariant polynomials on $\mathbb{C}^n$. The first metric (,) on the orbit space reads

$$\left(dy^i, dy^j\right) = \sum_{a,\,b} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} G^{ab} = g^{ij}(y). \tag{5.16}$$

Here $G^{ab} = \left(dx^a, dx^b\right)$ are the contravariant components of a W-invariant Euclidean metric on $\mathbb{R}^n$. The components $g^{ij}(y)$ are polynomials in $y^1, \ldots, y^n$ (cf. [41]). The second metric [41, 40] is given by

$$< dy^i, dy^j > = \frac{\partial g^{ij}(y)}{\partial y^1} \tag{5.17}$$

assuming that $h = \deg y^1(x)$ is the maximum of the degrees of the basic invariant polynomials. The discriminant $\Sigma \subset M$ consists of all orbits containing less than $|W|$ points. Alternatively it can be described as the image of the (complexified) mirrors of all reflections in $\mathbb{R}^n \subset \mathbb{C}^n$ w.r.t. the natural projection

$$\mathbb{C}^n \to \mathbb{C}^n / W = M.$$

Outside these mirrors the projection is a local diffeomorphism onto $M \setminus \Sigma$. The period mapping

$$p^1(y^1, \ldots, y^n), \ldots, p^n(y^1, \ldots, y^n)$$

is given by inverting this local diffeomorphism, i.e., by solving the system of algebraic equations

$$y^1(p^1,\dots,p^n) = y^1,\ y^2(p^1,\dots,p^n) = y^2, \dots, y^n(p^1,\dots,p^n) = y^n.$$

The flat coordinates of the flat pencil (5.16), (5.17) are determined by the system

$$y^1(p^1,\dots,p^n) = y^1 - \lambda,\ y^2(p^1,\dots,p^n) = y^2, \dots, y^n(p^1,\dots,p^n) = y^n \tag{5.18}$$

where, we recall, the degree of the polynomial $y^1(x)$ is the maximal one.

We give now the formula for the dual potential $F_*(x)$ for the case of polynomial Frobenius manifolds defined on the orbit spaces of finite Coxeter groups (see above). Let Δ_+ be the set of all positive roots of the Coxeter group $W \subset End(\mathbb{R}^n)$. The hyperplanes

$$(\alpha, x) = 0,\ \alpha \in \Delta_+$$

are all the mirrors of the reflections in W. Let us normalize the roots by the condition

$$(\alpha, \alpha) = 2.$$

We will identify the roots α with the linear functions $x \mapsto (\alpha, x)$.

Theorem 5.2. *For any finite irreducible Coxeter group the function $F_*(x)$ defined on the universal covering of*

$$\mathbb{C}^n \setminus \cup_{\alpha\in\Delta_+} \{(\alpha, x) = 0\} \tag{5.19}$$

has the form

$$F_*(x) = \frac{h}{4} \sum_{\alpha\in\Delta_+} \alpha^2 \log \alpha^2. \tag{5.20}$$

Proof For the Frobenius manifold under consideration the discriminant Σ is a *finite* union of hyperplane in the Euclidean space $\mathbb{R}^n$. From the Theorem 4.15 we know that all the singularities of $F_*(x)$ must be on these hyperplanes only. The third derivatives of $F_*(x)$ must have singularities of the from (4.60). So, they are meromorphic functions on $\mathbb{R}^n$ with simple poles along the mirrors of the reflections of the Coxeter group. Clearly the function $F_*(x)$ is determined by these analytic properties uniquely up to adding of a quadratic factor. Let us check that the formula (5.19) satisfies the needed analytic properties.

Let $e_1, \dots, e_n$ be a basis of $\mathbb{R}^n$. As above we denote

$$G_{ab} = (e_a, e_b),\ \left(G^{ab}\right) = (G_{ab})^{-1}.$$

We also put

$$\alpha_a := (\alpha, e_a),\ a = 1, \dots, n,\ \alpha \in \Delta_+.$$

Taking the triple derivatives of $F_*(x)$ we obtain

$$\overset{*}{c}{}^{c}_{ab}(x) = \frac{h}{2} \sum_{\alpha\in\Delta_+} \frac{\alpha_a\alpha_b\alpha_d}{(\alpha, x)} G^{dc}. \tag{5.21}$$

The singular part of this formula near the mirror $(\alpha, x) = 0$ coincides with (4.60) since

$$\frac{1}{1-d} = \frac{h}{2}.$$

The Theorem is proved. □

Remark 2. Associativity of the family of algebras follows from Corollary 3.2. It can be proved also by straightforward computation [31, 46]. Remarkably, A.Veselov recently found [47] other examples of solutions to the associativity equations given by a formula similar to (5.21). In the Veselov's examples the so-called deformed root systems are used. Recall that deformed root systems were discovered in the theory of multi-dimensional integrable linear differential operators (see [6] and references therein).We do not know if Veselov's structures satisfy the Axiom **AFM3**.

Corollary 5.3. *The twisted periods as functions of the Euclidean coordinates $x^1, \ldots, x^n$ satisfy the following system of linear differential equations with rational coefficients*

$$\partial_a \xi_b = \frac{h\,\nu}{2} \sum_{\alpha\in\Delta_+} \frac{\alpha_a\alpha_b\alpha_c}{(\alpha, x)} G^{cd}\xi_d,\ \ \xi_a = \frac{\partial}{\partial x^a}\tilde{p}(t(x);\nu). \tag{5.22}$$

Example 2. To construct the almost Frobenius structure for $W = W(A_n)$ it is convenient to start with the standard action of $W = S_{n+1}$ on $\mathbb{R}^{n+1}$ by permutations of the coordinates $x_0, x_1, \ldots, x_n$. Introduce the function

$$F_*(x) = \frac{n+1}{8} \sum_{i<j} (x_i - x_j)^2 \log(x_i - x_j)^2. \tag{5.23}$$

Together with the Euclidean metric

$$(dx_i, dx_j) = \delta_{ij} \tag{5.24}$$

the third derivatives of $F_*(x)$ give the following multiplication law of tangent vectors $\partial_i := \partial/\partial x_i$

$$\partial_i * \partial_j = -\frac{n+1}{2}\frac{\partial_i - \partial_j}{x_i - x_j}, \quad i \neq j \tag{5.25}$$

$$\partial_i * \partial_i = -\sum_{j\neq i} \partial_i * \partial_j.$$

The vector field

$$\sum_i \partial_i$$

has zero products with everything. Factorizing over this direction one obtains an almost Frobenius structure on

$$M_* = \{x_0 + x_1 + \cdots + x_n = 0\} \setminus \cup_{i<j}\{x_i = x_j\}.$$

The unity = Euler vector field equals

$$E = \frac{1}{n+1}\sum_i x_i\partial_i$$

the vector field e of the axiom **AFM3** reads

$$e = -\sum_i \frac{1}{f'(x_i)}\partial_i, \qquad f(x) := \prod_{i=0}^n (x - x_i).$$

The equation (3.25) for the twisted periods coincides with the classical Euler - Poisson - Darboux equations

$$\partial_i \partial_j \tilde{p} = -\frac{\nu}{x_i - x_j}(\partial_i \tilde{p} - \partial_j \tilde{p}), \quad i \neq j. \tag{5.26}$$

Recently these equations proved to play an important role in the theory of dispersive waves [45, 22].

5.3. **Almost duality and Shephard groups.** Shephard groups are the symmetry groups of regular complex polytopes. It is a subclass of finite *unitary reflection groups.* By definition a unitary reflection is a linear transformation

$$g : \mathbb{C}^n \to \mathbb{C}^n$$

having a hyperplane of fixed points such that the only nontrivial eigenvalue of g is a root of unity. A finite unitary reflection group G by definition is a finite subgroup in $GL(n, \mathbb{C})$ generated by unitary reflections. The book [37] is an excellent introduction into the theory of unitary reflection groups.

Of course, any finite Coxeter group is a Shephard group. Besides this, there are two infinite series and fifteen exceptional cases of irreducible Shephard groups (see Table 1 below). It was discovered in [35] that, for any Shephard group G there is an accompanying finite Coxeter group W. The group G is uniquely determined by the pair (W, κ) where the number κ was defined in (5.28). Let us represent this correspondence describing the Shephard group associated with (W, κ) in terms of the monodromy of twisted periods on the Frobenius manifold M_W.

According to Chevalley theorem, the quotient

$$M_G = \mathbb{C}^n / G \tag{5.27}$$

carries a natural structure of graded affine variety. One can use a basis homogeneous G-invariant polynomials $f^1(z), \ldots, f^n(z)$, $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$ as the graded affine coordinates on M_G. Let us order them such that

$$\deg f^1(z) = \max = h_G > \deg f^2(z) \geq \cdots > \deg f^n(z) = \min =: 2\kappa. \tag{5.28}$$

The class of Shephard groups is selected by the following remarkable property [35], see also [37].

Theorem 5.4. *For a Shephard group the Hessian of the basic invariant polynomial of the lowest degree is a nondegenerate matrix for generic z. Conversely, nondegeneracy of the Hessian completely characterizes irreducible Shephard groups among all unitary reflection groups.*

We give here a nice differential-geometric interpretation of the Hessian reinterpreting the results of Orlik and Solomon [35, 36].

Let us denote

$$h_{ij}(z) = \frac{\partial^2 f^n(z)}{\partial z^i \partial z^j}, \quad \left(h^{ij}(z)\right) = (h_{ij}(z))^{-1}. \tag{5.29}$$

The G-bilinear form

$$(df^i, df^j) := \frac{\partial f^i}{\partial z^k}\frac{\partial f^j}{\partial z^k} h^{kl}(z) \tag{5.30}$$

is defined on the cotangent planes T^*M_G everywhere due to the following statement [35].

Theorem 5.5. *The functions* (df^i, df^j) *are polynomials. The determinant* $\det(df^i, df^j)$ *vanishes exactly on the collection of mirrors of the group* G.

Because of G-invariance one can represent (df^i, df^j) as polynomials in $f^1(z)$, …, $f^n(z)$

$$(df^i, df^j) =: g^{ij}(f), \quad f = (f^1, \dots, f^n). \tag{5.31}$$

We obtain a polynomial flat metric on T^*M_G. Define an analogue of K.Saito metric by

$$\eta^{ij}(f) := \frac{\partial g^{ij}(f)}{\partial f^1}. \tag{5.32}$$

From [36] it follows

Theorem 5.6. *The metric* (5.32) *on* T^*M_G *does not degenerate anywhere.*

The main step in establishing a connection between Shephard groups and Coxeter groups is

Theorem 5.7. *The polynomial metrics (5.31), (5.32) together with the unity vector field*

$$e = \frac{\partial}{\partial f^1}$$

and the Euler vector field

$$E = h_G^{-1} \sum_{i=1}^{n} (\deg f^i) f^i \frac{\partial}{\partial f^i}$$

form a flat pencil.

The flat coordinates for the metric (5.32) are given by a distinguished system of homogeneous *flat generators* in the ring of G-invariant polynomials. Flat generators exist also for other unitary reflection groups [37], but they are not flat coordinates of a natural metric on the orbit space if G is not a Shephard group.

The following two statements immediately follow from the Theorem 5.7.

Corollary 5.8. *The orbit space* M_G *of a Shephard group* G *carries a natural polynomial Frobenius structure.*

Due to the Hertling's theorem [23] the Frobenius manifold M_G must be isomorphic to the orbit space of a finite irreducible Coxeter group W. This is just the accompanying Coxeter group for G!

Corollary 5.9. *The Hessian quadratic form*

$$ds^2 := \frac{\partial^2 f^n(z)}{\partial z^i \partial z^j} dz^i dz^j \tag{5.33}$$

is a flat metric on

$$M_G^* := \mathbb{C}^n \setminus \{\text{mirrors of } G\}. \tag{5.34}$$

If G is itself a Coxeter group then the metric (5.33) coincides, up to a constant factor, with the Killing form.

Let us make a digression about the flat metrics representable in the Hessian form. Remarkably, our old friend associativity equations arises also in this setting!

Proposition 5.10. [27]. *Let $f(z)$ be a smooth function of $z = (z^1, \dots, z^n)$ such that the Hessian does not degenerate in some domain $\Omega \subset \mathbb{R}^n$*

$$\det \left(\frac{\partial^2 f(z)}{\partial z^i \partial z^j} \right) \neq 0.$$

Denote

$$\left(h^{ij}(z)\right) := \left(\frac{\partial^2 f(z)}{\partial z^i \partial z^j} \right)^{-1} \tag{5.35}$$

the inverse matrix. Then the metric

$$ds^2 := \frac{\partial^2 f(z)}{\partial z^i \partial z^j} dz^i dz^j$$

on Ω has a zero curvature iff *the function f satisfies the following system of associativity equations*

$$\frac{\partial^3 f(z)}{\partial z^i \partial z^j \partial z^s} h^{st}(z) \frac{\partial^3 f(z)}{\partial z^t \partial z^k \partial z^l} = \frac{\partial^3 f(z)}{\partial z^l \partial z^j \partial z^s} h^{st}(z) \frac{\partial^3 f(z)}{\partial z^t \partial z^k \partial z^i} \tag{5.36}$$

for all i, j, k, l.

Proof Let us denote by superscripts the partial derivatives of f w.r.t. z^i, z^j, etc. An easy calculation gives the Christoffel coefficients for the Levi-Civita connection for the metric ds^2:

$$\Gamma_{ij}^k = \frac{1}{2} h^{ks} f_{sij}.$$

From this one readily derives the following formula for the Riemann curvature tensor of the metric

$$R_{ijl}^k = \frac{1}{4} h^{kp} f_{pqj} h^{qs} f_{sil} - \frac{1}{4} h^{kp} f_{pqi} h^{qs} f_{slj}.$$

Vanishing of this expression is equivalent to (5.36). □

We will now describe a natural class of flat Hessian metrics associated with a Frobenius manifold.

Proposition 5.11. *Let M be an arbitrary Frobenius manifold with the charge $d \neq 1$. Let*

$$z^i := \tilde{p}^i(t;\nu), \ \ i = 1, \dots, n \tag{5.37}$$

be a system of independent twisted periods on $M \setminus \Sigma$. Here ν is a fixed complex number satisfying

$$\nu \neq \frac{1-d}{2}. \tag{5.38}$$

Denote

$$g_{ij}(z) := \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) \tag{5.39}$$

the Gram matrix of the intersection form (2.48) on TM written in the coordinates z^1, ..., z^n. Put

$$f(z) = \left[\frac{1-d}{2} - \nu \right] t_1 \tag{5.40}$$

where $t_1 = \eta_{1\alpha} t^\alpha$ must be represented as a function of the coordinates z. Then this flat metric can be written in the Hessian form

$$g_{ij}(z) = \frac{\partial^2 f(z)}{\partial z^i \partial z^j}. \tag{5.41}$$

Observe that the function $f(z)$ is a homogeneous function of $z = (z^1, \dots, z^n)$ of the degree

$$\deg f(z) = \frac{1-d}{\frac{1-d}{2} + \nu}. \tag{5.42}$$

This easily follows from

$$Lie_E t_1 = (1-d) t_1$$

and

$$Lie_E z^i = \frac{1-d}{2} + \nu$$

(see (3.39)).

Proof Let us compute the Hessian of the function (5.40). We have, due to (2.69)

$$\frac{2}{1-d} \frac{\partial^2 f}{\partial z^i \partial z^j} = G_{ab} \frac{\partial p^a}{\partial z^i} \frac{\partial p^b}{\partial z^j} + G_{ab} p^a \frac{\partial^2 p^b}{\partial z^i \partial z^j}. \tag{5.43}$$

The first term coincides with the metric (5.39). Let us compute the second one.

Rewriting

$$\frac{\partial^2 p^b}{\partial z^i \partial z^j} = \frac{\partial}{\partial p^c} \left(\frac{\partial p^b}{\partial z^i} \right) \frac{\partial p^c}{\partial z^j}$$

and using

$$\frac{\partial p^b}{\partial z^s} \frac{\partial z^s}{\partial p^d} = \delta^b_d$$

yields

$$\frac{\partial^2 p^b}{\partial z^i \partial z^j} = -\frac{\partial p^b}{\partial z^s} \frac{\partial^2 z^s}{\partial p^c \partial p^d} \frac{\partial p^d}{\partial z^i} \frac{\partial p^c}{\partial z^j}.$$

We now use the equation (3.25) for twisted periods to recast the second term in (5.43) as follows

$$G_{ab} p^a \frac{\partial^2 p^b}{\partial z^i \partial z^j} = -\nu\, p^a \overset{*}{c}_{acd} \frac{\partial p^c}{\partial z^i} \frac{\partial p^d}{\partial z^j}.$$

The last step in the proof is to use is that, the vector field

$$E = \frac{1-d}{2} p^a \frac{\partial}{\partial p^a}$$

is the unity on the dual almost Frobenius manifold M_*. Therefore

$$p^a \overset{*}{c}_{acd} = \frac{2}{1-d} G_{cd}.$$

This proves that the second term in (5.43) is proportional to the first one:

$$G_{ab} p^a \frac{\partial^2 p^b}{\partial z^i \partial z^j} = -\frac{2\nu}{1-d} G_{cd} \frac{\partial p^c}{\partial z^i} \frac{\partial p^d}{\partial z^j}.$$

This proves the Proposition. □

Summarizing the above results we arrive at the main statement of this Section.

Theorem 5.12. *Let W be a finite Coxeter group acting in the n-dimensional space. Denote h the Coxeter number of W. Let $e_1, \ldots, e_n$ be a basis of simple roots normalized by $(e_i, e_i) = 2$. Introduce an upper triangular matrix S such that*

$$S_{ii} = 1, \quad S_{ij} = (e_i, e_j) \quad \text{for } i < j.$$

Let ν be a rational number such that

$$\det(qS + S^T) \neq 0, \quad q := e^{2\pi i}$$

and the monodromy matrices $M_1(q), \ldots, M_n(q)$ of the form (4.33) generate a finite irreducible subgroup G in $GL(n, \mathbb{C})$. Then G is a Shephard group with the accompanying Coxeter group W and

$$\kappa = \frac{1}{1 - h\,\nu} \tag{5.44}$$

where h is the Coxeter number of W. Conversely, if (W, κ) is as in Table 1 then the monodromy matrices $M_1(q), \ldots, M_n(q)$ of twisted periods $\tilde{p}^a(t; \nu)$ with

$$\nu = \frac{1}{h}\left(1 - \frac{1}{\kappa}\right) \tag{5.45}$$

generate the Shephard group associated with (W, κ).

W	κ	G
A_1	$\frac{r}{2}$	$C(r)$
B_n	$\frac{r}{2}$	$G(r,1,n)$
A_2	2	G_4
A_2	4	G_8
A_2	10	G_{16}
B_2	3	G_5
B_2	6	G_{10}
B_2	15	G_{18}
$I_2(5)$	6	G_{20}
G_2	2	G_6
G_2	4	G_9
G_2	10	G_{17}
$I_2(8)$	3	G_{14}
$I_2(10)$	6	G_{21}
A_3	3	G_{25}
B_3	3	G_{26}
A_4	6	G_{32}

Table 1

Example 3. Let us consider the particular case of the group $G_{25} \subset GL(3)$. This is the group of symmetries of the celebrated Hessian configuration consisting of 9 inflections of a generic plane cubic. The ring of the G_{25}-invariant polynomials is generated by the classical Maschke polynomials

$$\begin{aligned}
&C_6(z) = z_1^6 + z_2^6 + z_3^6 - 10(z_1^3 z_2^3 + z_2^3 z_3^3 + z_3^3 z_1^3)\\
&C_9(z) = (z_1^3 - z_2^3)(z_2^3 - z_3^3)(z_3^3 - z_1^3)\\
&C_{12}(z) = (z_1^3 + z_2^3 + z_3^3)\left[(z_1^3 + z_2^3 + z_3^3)^3 + 216 z_1^3 z_2^3 z_3^3\right].
\end{aligned} \tag{5.46}$$

The Hessian flat metric reads

$$\begin{aligned}
\frac{1}{30}ds^2 = &\, 3(z_1^4 dz_1^2 + z_2^4 dz_2^2 + z_3^4 dz_3^2) - 2(z_1^3 + z_2^3 + z_3^3)(z_1 dz_1^2 + z_2 dz_2^2 + z_3 dz_3^2)\\
&-6(z_1^2 z_2^2 dz_1 dz_2 + z_2^2 z_3^2 dz_2 dz_3 + z_3^2 z_1^2 dz_3 dz_1).
\end{aligned} \tag{5.47}$$

The flat coordinates x_1, x_2, x_3 of this metric are algebraic functions of z_1, z_2, z_3 to be determined from the system

$$
\begin{aligned}
C_6(z) &= x_1^2 + x_2^2 + x_3^2 \\
C_9(z) &= \frac{1}{4} x_1 x_2 x_3 \\
C_{12}(z) &= (x_1^2 + x_2^2 + x_3^2)^2 - 3(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2),
\end{aligned} \tag{5.48}
$$

$$\frac{\partial z^i}{\partial x^k} \frac{\partial^2 C_6(z)}{\partial z^i \partial z^j} \frac{\partial z^j}{\partial x^l} = \frac{10}{3} \delta_{kl}.$$

The inverse functions $z_i = z_i(x_1, x_2, x_3)$ are twisted periods on the Frobenius manifold M_{A_3} with $\nu = \frac{1}{6}$. Comparing with the integral representation (5.11) of the twisted periods we obtain an interesting realization of the group G_{25} by the monodromy of Abelian integrals of the form

$$z = \oint (x^4 + a x^2 + b x + c)^{\frac{1}{6}} dx.$$

The Theorem 4.3 gives the following matrix realization of the generators of G_{25}

$$M_1 = \begin{pmatrix} -q & q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -q & q \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -q \end{pmatrix}$$

where

$$q = e^{\frac{\pi i}{3}}.$$

5.4. **Twisted periods for $QH^*(CP^4)$ and the mirror of the quintic.** Let $M = QH^*(CP^d)$ be the Frobenius manifold corresponding to the quantum cohomology of the d-dimensional complex projective space. It is a semisimple Frobenius manifold of the dimension $n = d + 1$. The flat coordinates t_1, t_2, $\dots$, t_{d+1} on it are in one-to-one correspondence with the standard basis 1, ω, $\dots$, ω^d in $H^*(CP^d)$ (we now write all lower indices for the sake of graphical simplicity). Here the two-form $\omega \in H^2(CP^d)$ is normalized by the condition

$$\int_{CP^d} \omega^d = 1.$$

The algebra structure on the tangent planes $T_t M$ becomes particularly simple at the points of the *small quantum cohomology locus*

$$t = (t_1, t_2, 0, \dots, 0). \tag{5.49}$$

At this points we have

$$T_t M \simeq \mathbb{C}[\omega]/\omega^{d+1} = e^{t_2}, \; \omega \leftrightarrow \partial_2. \tag{5.50}$$

At the point of classical limit $t_2 \to -\infty$ the algebra (5.50) goes to the classical cohomology algebra of the projective space. Let us describe the twisted periods at the locus (5.49). It will be convenient to consider them as a function of t_2 and λ introducing λ as in the beginning of Section 3.

Proposition 5.13. *The twisted periods $\tilde{p} = \tilde{p}(t;\nu)$ at the points (5.49) are determined from the following hypergeometric equation*

$$\partial_2^{d+1}\tilde{p} = e^{t_2} t_1^{-(d+1)} \prod_{m=0}^{d} \left[-(d+1)\partial_2 + \frac{1-d}{2} + \nu - m \right] \tilde{p} \tag{5.51}$$

and from the quasihomogeneity condition (3.39)

$$t_1\partial_1\tilde{p} + (d+1)\partial_2\tilde{p} = \left(\frac{1-d}{2} + \nu \right) \tilde{p}. \tag{5.52}$$

Proof The simplest way to derive (5.51) is to represent $\tilde{p}$ as the Laplace integral

$$\tilde{p} = \frac{i}{\sqrt{2\pi}} \int_0^\infty \tilde{t}(t;z) e^{-\lambda z} \frac{dz}{z^{\nu+\frac{1}{2}}}$$

(cf. (4.13)) and then to use the differential equation for the deformed flat coordinates $\tilde{t}$ on the original Frobenius manifold. On the locus (5.49) the latter reads

$$\partial_2^{d+1}\tilde{t} = z^{d+1} e^{t_2} \tilde{t} \tag{5.53}$$

$$\partial_1 \tilde{t} = z\tilde{t}$$

$$z\partial_z \tilde{t} = t_1\partial_1\tilde{t} + (d+1)\partial_2\tilde{t} + \frac{d-2}{2}\tilde{t}. \tag{5.54}$$

Substituting

$$\tilde{t}(t;z) = z^{\nu+\frac{1}{2}} \oint e^{\lambda z} \tilde{p}(t;\lambda)$$

into the equation (5.54) and integrating by parts we obtain

$$\left\{ \frac{1}{(d+1)^{d+1}} \left[-\lambda\partial_\lambda + \frac{1-d}{2} + \nu \right]^{d+1} - e^{t_2} \left(-\partial_\lambda \right)^{d+1} \right\} \tilde{p} = 0.$$

We obtain a similar equation for the dependence of $\tilde{p}$ on t_1 since $\partial_1 = -\partial_\lambda$. Setting then λ to zero and using the quasihomogeneity condition (5.52)), we obtain (5.51). Proposition is proved. □

Corollary 5.14. *Odd periods on the Frobenius manifold $M = QH^*(CP^4)$ at the points (5.49) with $t_1 = -1$ are given by the periods of the holomorphic three-form on the Calabi - Yau three-fold dual to the quintic in CP^4.*

Proof The equation (5.51) for the odd periods can be integrated once in t_2 to produce, at $t_1 = -1$, the Picard - Fuchs equation [5] for the periods of the Calabi - Yau three-fold

$$u_0 + \cdots + u_4 = 1, \; u_0 \ldots u_4 = e^{t_2}$$

dual to the quintic in CP^4:

$$\partial_2^4 \tilde{p} = 5\,(5\partial_2 + 1)(5\partial_2 + 2)(5\partial_2 + 3)(5\partial_2 + 4)\tilde{p}, \tag{5.55}$$

$$\tilde{p}(t_2) = \oint \frac{du_0 \wedge \cdots \wedge du_4}{d(u_0 + \cdots + u_4) \wedge d(u_0 \ldots u_4)}.$$

Corollary is proved. □

5.5. **From a Frobenius manifold to the Seiberg - Witten prepotential.** Let M be an arbitrary n-dimensional Frobenius manifold with eigenvalues of $\mathcal{V}$ distinct from 1/2. We will study the properties of the odd periods on the $2n$-dimensional manifold

$$M \otimes QH^*(CP^1). \tag{5.56}$$

The construction of tensor product of Frobenius manifolds, introduced by R.Kaufmann, M.Kontsevich, and Yu.I.Manin, generalizes the procedure of computation of Gromov - Witten invariants of Cartesian product of two smooth projective manifolds. Let us denote $t^1, \ldots, t^n$ the flat coordinates on M. For simplicity we will assume that $\mathcal{V} = \hat{\mu}$ is a diagonalizable matrix and we choose the flat coordinates in such a way that

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1},\ \mathcal{V} = \mathrm{diag}\,(\mu_1, \ldots, \mu_n),\ \mu_\alpha + \mu_{n-\alpha+1} = 0.$$

The flat coordinates on $QH^*(CP^1)$ we redenote (t^1, s), so the potential and the Euler vector field read

$$F_{CP^1} = \frac{1}{2}(t^1)^2 s + e^s,\ E = t^1\partial_1 + 2\partial_s. \tag{5.57}$$

The tangent spaces to (5.56) has the structure of the tensor product

$$T\left(M \otimes QH^*(CP^1)\right) = TM \otimes H^*(CP^1). \tag{5.58}$$

So the flat coordinates on the tensor product can be naturally labelled by pairs (α', α'') with $\alpha' = 1, \ldots, n$, $\alpha'' = 1, 2$. We identify $t^{\alpha'\,1''}$ with t^α and $t^{1'\,2''}$ with s, and we consider the "coordinate cross"

$$t^{\alpha'\,2''} = 0,\ \alpha' = 2, \ldots, n. \tag{5.59}$$

The points of the coordinate cross will be coordinatized by pairs (t, s), $t = (t^1, \ldots, t^n) \in M$.

The Frobenius structure on (5.56) is uniquely determined, according to [26], in a neiborhood of the coordinate cross by the initial condition that, on the coordinate cross,

$$T_{(t,s)}M \otimes QH^*(CP^1) = T_tM \otimes T_sQH^*(CP^1) \tag{5.60}$$

is an isomorphism of Frobenius algebras. The Euler vector field of (5.56) on the coordinate cross reads

$$E_{M\otimes QH^*(CP^1)} = E + 2\partial_s. \tag{5.61}$$

The operator μ of (5.56) thus acts as follows

$$\begin{aligned} \mu(e_\alpha \otimes e_1) &= (\mu_\alpha - \frac{1}{2})e_\alpha \otimes e_1 \\ \mu(e_\alpha \otimes e_2) &= (\mu_\alpha + \frac{1}{2})e_\alpha \otimes e_2. \end{aligned} \tag{5.62}$$

Hence the Poisson bracket (3.12) on the tensor product has the form

$$\begin{aligned} \left\{t^{\alpha'1''}, t^{\beta'2''}\right\} &= \eta^{\alpha\beta}(\mu_\beta + \frac{1}{2}) \\ \left\{t^{\alpha'2''}, t^{\beta'1''}\right\} &= \eta^{\alpha\beta}(\mu_\beta - \frac{1}{2}), \end{aligned} \tag{5.63}$$

other brackets vanish. This Poisson bracket induces a symplectic structure on the tensor product (5.56)

$$\Omega = \frac{2}{1-d} ds \wedge dt^n + \sum_{\alpha=2}^{n} (\mu_\alpha + \frac{1}{2})^{-1} dt^{\alpha' 2''} \wedge dt^{(n-\alpha+1)' 1''}. \tag{5.64}$$

Particularly, the n-dimensional planes

$$L_s := \left\{ s = \text{fixed},\ t^{\alpha' 2''} = 0, \alpha = 2, \dots, n,\ t^{\beta' 1''} = \text{arbitrary},\ \beta = 1, \dots, n \right\} \tag{5.65}$$

are Lagrangian submanifolds in $M \otimes QH^*(CP^1)$.

We will now construct another system $(X^1, \dots, X^n, Y_1, \dots, Y_n)$ of canonical coordinates using odd periods on $M \otimes QH^*(CP^1)$. We will choose an appropriate polarization in the space of odd periods on (5.56) and we will compute the generating function $S = S(X, s)$ of the family of Lagrangian submanifolds

$$L_s = \left\{ Y_a = \frac{\partial S(X, s)}{\partial X^a},\ a = 1, \dots, n \right\}. \tag{5.66}$$

The generating function will be found as an expansion near the point of "classical limit" $s = -\infty$. For the particular cases where M is one of the polynomial Frobenius manifolds of the ADE type (see Section 2 above) we will identify the generating function with the Seiberg - Witten prepotential of the four-dimensional supersymmetric Yang - Mills with one of the ADE gauge groups resp.

Let us spell out the differential equations for the components of odd period mapping on the tensor product (5.56). For any odd period ϖ we introduce row vectors $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$

$$p_\alpha = \frac{\partial \varpi}{\partial t^{\alpha' 1''}} \equiv \frac{\partial \varpi}{\partial t^\alpha},\ q_\alpha = \frac{\partial \varpi}{\partial t^{\alpha' 2''}},\ q_1 \equiv \frac{\partial \varpi}{\partial s}.$$

Lemma 5.15. *At the points of the coordinate cross (5.59) on $M \otimes QH^*(CP^1)$ the differential equations (4.1) with $\nu = 1/2$ read*

$$\begin{aligned} \partial_\alpha p \cdot \mathcal{U} + 2 \partial_\alpha q &= p(\mu - \frac{1}{2}) C_\alpha \\ \partial_\alpha q \cdot \mathcal{U} + 2Q \partial_\alpha p &= q(\mu + \frac{1}{2}) C_\alpha \end{aligned} \tag{5.67}$$

$$Q \partial_Q (p, q) \begin{pmatrix} \mathcal{U} & 2Q \\ 2 & \mathcal{U} \end{pmatrix} = (p, q) \begin{pmatrix} 0 & Q(\mu - \frac{1}{2}) \\ \mu + \frac{1}{2} & 0 \end{pmatrix} \tag{5.68}$$

where $Q = e^s$.

Proof is given by straightforward computation using (4.1), (5.60), (5.62), and (5.57).

Particularly, near $Q = 0$ one can rewrite the equation (5.68), for $t \in M \setminus \Sigma$, in the form

$$\partial_Q (p, q) = \frac{1}{Q} (p, q) A_0 + O(1) \tag{5.69}$$

where

$$A_0 = \begin{pmatrix} 0 & 0 \\ (\mu + \frac{1}{2})\mathcal{U}^{-1} & 0 \end{pmatrix}.$$

Therefore $Q = 0$ is a regular singularity of (5.68). The Jordan normal form of the matrix A_0 consists of n 2×2 nilpotent Jordan blocks. So the system (5.68) admits n independent solutions analytic at $Q = 0$. They can be completed to produce a basis by adding n solutions behaving like $\log Q$ at $Q \to 0$. We will now explain how to choose this basis in order to obtain solutions to the full system (5.67) - (5.68).

Theorem 5.16. *Let $(x^1(t), \ldots, x^n(t))$ be a system of independent flat coordinates of the intersection form on M defined locally on $M \setminus \Sigma$. Denote*

$$G^{ab} = (dx^a, dx^b),\ (G_{ab}) = (G^{ab})^{-1}.$$

Then there exists a basis of odd periods on $M \otimes QH^*(CP^1)$ can be represented in the form

$$(X^1(t,Q), \ldots, X^n(t,Q), Y_1(t,Q), \ldots, Y_n(t,Q))$$

where, at the points of the coordinate cross (5.59),

$$X^a = x^a(t) + O(Q),\ Q \to 0,\ a = 1, \ldots, n \tag{5.70}$$

$$Y_a = x_a(t) \log Q - 2\frac{\partial F_*(x)}{\partial x^a} + o(1),\ x_a(t) = G_{ab}x^b(t),\ a = 1, \ldots, n. \tag{5.71}$$

Here $F_*(x)$ is the potential of the dual Frobenius manifold M_*. The functions X^a, Y_b are canonically conjugated w.r.t. the symplectic structure (5.64):

$$\{X^a, Y_b\} = \delta^a_b,\ \{X^a, X^b\} = \{Y_a, Y_b\} = 0. \tag{5.72}$$

The coordinates X^a are determined uniquely, the coordinates Y_a are determined with the ambiguity that can be absorbed by a redefinition of the dual potential

$$F_*(x) \mapsto F_*(x) + \text{quadratic}.$$

Proof Left eigenvectors of the matrix A_0 are of the form

$$q = 0,\ p = \text{arbitrary}.$$

Hence for an arbitrary $p^0 = (p^0_1, \ldots, p^0_n)$ the system (5.68) admits a solution of the form

$$(p, q) = (p^0, 0) + O(Q),\ Q \to 0.$$

The dependence of this solution on t is to be determined from (5.67). Particularly, for p^0 one obtains the equations coinciding with the system of differential equations (2.60) for the gradients of the flat coordinates of the intersection form on M. This gives the solutions (5.70). The coefficients of the Q-expansion of the solutions are uniquely determined from the system (5.67), (5.68):

$$p^{(a)}_\alpha := \partial_\alpha X^a = \partial_\alpha x^a(t) + \partial_\alpha \sum_{k=1}^\infty \frac{Q^k}{(k!)^2} \partial_1^{2\,k} x^a(t)$$

$$q^{(a)}_\alpha := \partial_{\alpha' 2''} X^a = \partial_\alpha \sum_{k=1}^\infty \frac{k\, Q^k}{(k!)^2} \partial_1^{2k-1} x^a(t). \tag{5.73}$$

Let us find the second half of the solutions to (5.67), (5.68). These must have the form

$$(p, q) = \left(p^0 \log Q + r^0, q^0\right) + o(1), \; Q \to 0. \tag{5.74}$$

Substituting in (5.68) we find

$$q^0 = p^0 \mathcal{U}(\mu + \frac{1}{2})^{-1}. \tag{5.75}$$

Now we plug the expansion (5.74), (5.75) into differential equations (5.67). We find, as before, that $p^0 = (p_1^0, \dots, p_n^0)$ depends on t as

$$p_\alpha^0 = \partial_\alpha x(t) \tag{5.76}$$

for some flat function $x(t)$ of the intersection form on M. For the dependence of r^0 on t we obtain a system

$$\partial_\alpha r^0 \cdot \mathcal{U}(t) = r^0(\mu - \frac{1}{2})C_\alpha - 2p^0(t)C_\alpha(t).$$

Let us look for a solution of the linear inhomogeneous system in the form

$$r^0 = A_i(t)\xi^i(t), \; \xi^i(t) = (\partial_1 x^i(t), \dots, \partial_n x^i(t))$$

where the coefficients $A_i(t)$ are to be determined. This gives

$$\partial_\alpha A_i \, \xi^i \, \mathcal{U} = -2p^0 C_\alpha.$$

Multiplying both sides by $\eta^{-1}(\xi^c)^T$ we obtain

$$\partial_\alpha A_i G^{ic} = -2p_\beta^0 c_\alpha^{\beta\gamma} \frac{\partial x^c}{\partial t^\gamma}.$$

Applying chain rule we rewrite the last equation in the form

$$\frac{\partial A_i}{\partial x^b} G^{ic} = -2p_\beta^0 \frac{\partial t^\alpha}{\partial x^b} c_\alpha^{\beta\gamma} \frac{\partial x^c}{\partial t^\gamma}.$$

Choosing in (5.76) $x = x_a(t)$ we obtain

$$\begin{aligned} \frac{\partial A_i}{\partial x^b} &= -2G_{ic}G_{am} \frac{\partial x^m}{\partial t^\beta} \frac{\partial t^\alpha}{\partial x^b} c_\alpha^{\beta\gamma} \frac{\partial x^c}{\partial t^\gamma} \\ &= -2\frac{\partial^3 F_*(x)}{\partial x^i \partial x^a \partial x^b}. \end{aligned} \tag{5.77}$$

So

$$A_i = -2\frac{\partial^2 F_*(x)}{\partial x^i \partial x^a}$$

and

$$\begin{aligned} r_\alpha^0 &= -2\frac{\partial^2 F_*(x)}{\partial x^a \partial x^i} \frac{\partial x^i}{\partial t^\alpha} \\ &= -2\frac{\partial}{\partial t^\alpha} \frac{\partial F_*(x)}{\partial x^a}. \end{aligned} \tag{5.78}$$

This gives the solution Y_a of the form (5.71)). This solution is determined uniquely up to adding of a linear combination of the solutions analytic at $Q = 0$.

Let us compute the Poisson brackets of the functions X, Y. Due to (5.63) we have, for two arbitary functions ϖ_1, ϖ_2 on $M \otimes QH^*(CP^1)$

$$\{\varpi_1, \varpi_2\} = < p_1, q_2(\mu + \frac{1}{2}) > + < q_1, p_2(\mu - \frac{1}{2}) > .$$

Here

$$(p_i)_\alpha = \frac{\partial \varpi_i}{\partial t^{\alpha' 1''}}, \ (q_i)_\alpha = \frac{\partial \varpi_i}{\partial t^{\alpha' 2''}}, \ \alpha = 1, \dots, n, \ i = 1, 2.$$

Due to independence of the brackets of the functions X^a and Y_b on the point of the Frobenius manifold (5.56), we obtain

$$\{X^a, X^b\}_{Q=0} = 0, \ \{X^a, Y_b\} = < \xi^a, G_{bc}\xi^c \mathcal{U} > = \delta^a_b.$$

To prove that $\{Y_a, Y_b\} = 0$ we will first verify that $\{\tilde{Y}_a, \tilde{Y}_b\} = 0$ where

$$\tilde{Y}_a = Y_a + 2\frac{\partial F_*(x)}{\partial x^a} = x_a \log Q + o(1).$$

We have

$$\{\tilde{Y}_a, \tilde{Y}_b\} = \log Q \, G_{ac} G_{bd} \left[< \xi^c, \xi^d \mathcal{U} > + < \xi^c \mathcal{U}(\mu + \frac{1}{2})^{-1}, \xi^d(\mu - \frac{1}{2}) > \right] + o(1).$$

Using symmetry of $\mathcal{U}$ and skew-symmetry of μ we obtain that the coefficient of $\log Q$ vanishes. Finally we observe that the gradient shift

$$Y_a = \tilde{Y}_a - 2\frac{\partial F_*}{\partial x^a}$$

preserves the Poisson brackets. The Theorem is proved. □

We will now describe the generating function (5.66) of the family of Lagrangian manifolds L_s w.r.t. the canonical coordinates X^a, Y_b.

Theorem 5.17. *The generating function $S = S(X, Q)$, $Q = e^s$, of the family of Lagrangian manifolds L_s admits an expansion*

$$S = \frac{1}{2} G_{ab} X^a X^b \log Q - 2\, F_*(X) + \sum_{k \geq 1} S_k(X) Q^k, \ Q \to 0. \tag{5.79}$$

The coefficients $S_k(X)$ can be uniquely determined from the Hamilton - Jacobi equation

$$\frac{\partial S}{\partial s} = H \tag{5.80}$$

with the Hamiltonian

$$H = \frac{2}{1-d} t_1 = \frac{1}{2} G_{ab} x^a x^b \tag{5.81}$$

and from the expansions

$$X^a = x^a + \sum_{k=1}^{\infty} \frac{Q^k}{(k!)^2} \partial_1^{2k} x^a, \ a = 1, \dots, n. \tag{5.82}$$

Proof The shifts along s are given by a Hamiltonian flow with the linear Hamiltonian coinciding with the flat coordinate on (5.56) conjugate to s. From (5.64) we find (5.81) (the expression of the Hamiltonian via the flat coordinates of the intersection form follows from Lemma 2.6). Substituting the functions $x^a = x^a(X, Q)$ inverse to (5.82) one obtains an expansion

$$H = \frac{1}{2} G_{ab} X^a X^b + \sum_{k\geq 1} H_k(X) Q^k$$

of the Hamiltonian evaluated on the Lagrangian manifold L_s.

The generating function of a family of Lagrangian manifolds L_s moving along trajectories of a Hamiltonian flow is known to satisfy the Hamilton - Jacobi equation (5.80) where in the r.h.s. stands the Hamiltonian of the flow evaluated on the Lagrangian manifold (see, e.g., [15]). This gives

$$S = \frac{1}{2} G_{ab} X^a X^b \log Q + \sum_{k\geq 1} H_k(X) \frac{Q^k}{k} + S_0(X).$$

The integration constant $S_0(X)$ can be determined from the expansion (5.71) of $Y_a = \partial S/\partial X^a$. The Theorem is proved. □

We give now an integral representation of the odd periods $\varpi(t, Q)$ on $M \otimes QH^*(CP^1)$ via the even periods on M.

Proposition 5.18. *Let $x(t)$ be a period on $M \setminus \Sigma$. Then*

$$\varpi(t, Q) = \frac{1}{2\pi i} \oint \frac{x(t^1 - \lambda, t^2, \dots, t^n)}{\sqrt{\lambda^2 - Q}} d\lambda \tag{5.83}$$

is an odd period on $M \otimes QH^*(CP^1)$ evaluated on the coordinate cross (5.59).

Proof Using (4.1), (4.2) and integration by parts we can easily verify that the functions

$$p_\alpha = \frac{1}{2\pi i} \oint \frac{\partial_\alpha x(t^1 - \lambda, t^2, \dots, t^n)}{\sqrt{\lambda^2 - Q}} d\lambda, \;\; q_\alpha = \frac{1}{2\pi i} \oint \frac{\lambda \partial_\alpha x(t^1 - \lambda, t^2, \dots, t^n)}{\sqrt{\lambda^2 - Q}} d\lambda$$

satisfy the system (5.67), (5.68). The Proposition is proved. □

Example 1. In the simple case of one-dimensional Frobenius manifold M one obtains odd period mapping $(t, s) \mapsto (X, Y)$ on $QH^*(CP^1)$. The expansion (5.79) reads

$$X = x - \sum_{k\geq 1} \frac{Q^k}{x^{4k-1}} \frac{2^{2k}(4k-3)!!}{(k!)^2}.$$

The potential F_* is equal to $\frac{1}{2} x^2 \log x^2$. We obtain therefore the expansion of the generating function

$$S = \frac{1}{2} x^2 \log Q - x^2 \log x^2 + ...$$

The integral (5.83) expresses the periods X, Y via complete elliptic integrals

$$X = \frac{2}{\pi i}\int_{-\sqrt{Q}}^{\sqrt{Q}} \frac{\sqrt{t-\lambda}}{\sqrt{\lambda^2 - Q}} d\lambda,\ Y = \frac{2}{\pi i}\int_{-\sqrt{Q}}^{t} \frac{\sqrt{t-\lambda}}{\sqrt{\lambda^2 - Q}} d\lambda.$$

Example 2. For the polynomial Frobenius manifold $M = \mathbb{C}^n/W$ on the orbit space of a finite Coxeter group (see the end of Section 1) the derivatives $\partial_1 x^1, \ldots, \partial_1 x^n$ as functions of $x^1, \ldots, x^n$ can be found in a pure algebraic way from the linear system

$$\sum_{a=1}^{n} \partial_1 x^a \frac{\partial y^i}{\partial x^a} = \delta_1^i,\ i = 1, \ldots, n.$$

Iterating this procedure we obtain derivatives $\partial_1^{2k} x^a$ for any $k \geq 1$. This gives an algebraic algorithm of computing of the terms of the expansion (5.79)

$$S(X) = \frac{1}{2} G_{ab} X^a X^b \log Q - \frac{1}{2}\sum_{\alpha\in\Delta_+} (\alpha, X)^2 \log(\alpha, X)^2 + \sum_{k\geq 1} S_k(X) Q^k. \tag{5.84}$$

Our observation is that, for the case of W = one of the Weyl groups of the ADE type, the expansion (5.84) coincides, after redefining

$$Q = \Lambda^{\frac{4}{1-d}} = \Lambda^{2h}$$

h = Coxeter number of W, and changing the notations

$$X \to a,\ Y \to a_D,\ S \to \mathbf{F}$$

with the instanton expansion of the Seiberg - Witten prepotential of the 4-dimensional $N = 2$ supersymmetric Yang - Mills (see in [43, 44, 24, 30]). The shortest way to see this is to eliminate q from the systems (5.67), (5.68). We will do it assuming semisimplicity of the Frobenius manifold under an additional assumption $d \neq 3$.

Proposition 5.19. *The odd periods $\varpi(t, Q)$ evaluated on the coordinate cross (5.59) satisfy the following system of differential equations*

$$Lie_E \varpi + 2\partial_s \varpi = \frac{1-d}{2}\varpi \tag{5.85}$$

$$\partial_s^2 \varpi = Q \partial_1^2 \varpi \tag{5.86}$$

$$\partial_\alpha \partial_\beta \varpi = c_{\alpha\beta}^\gamma(t) \partial_1 \partial_\gamma \varpi. \tag{5.87}$$

Proof The equation (5.85) is just a manifestation of the general quasihomogeneity of the odd periods (see (3.39) for $\nu = \frac{1}{2}$). To derive (5.86) we consider the first component of the second equation in the system (5.68). It can be rewritten, due to $\mathcal{U}_1^\alpha = E^\alpha$ in the form

$$\partial_\alpha g = \frac{3-d}{2} q_\alpha$$

where we denoted

$$g := E^\epsilon q_\epsilon + 2Q\, p_1.$$

Substituting this expression into the first equation we obtain, for $p_\epsilon = \partial_\epsilon \varpi$

$$\partial_\alpha \partial_\epsilon \varpi\, \mathcal{U}_\beta^\epsilon - p_\sigma(\mu - \frac{1}{2})_\epsilon^\sigma c_{\alpha\beta}^\epsilon = -\frac{4}{3-d}\partial_\alpha \partial_\beta g.$$

Hence the operator $\mathcal{U}$ of multiplication by E is symmetric w.r.t. the bilinear form with the matrix

$$(\partial_\alpha \partial_\beta \varpi). \tag{5.88}$$

Any power of $\mathcal{U}$ will still be symmetric w.r.t. the same bilinear form. On a semisimple Frobenius manifold powers of E span the whole algebra on $T\,M$. So, the bilinear form (5.88) must be an invariant form on the Frobenius algebra. Thus it must have a representation

$$\partial_\alpha \partial_\beta \varpi = c_{\alpha\beta}^{\gamma} f_\epsilon$$

for some covector f_ϵ. Obviously

$$f_\alpha = \partial_\alpha \partial_1 \varpi.$$

This gives equation (5.86).

To derive (5.87) we rewrite the first component of the equation

$$2Q\partial_s p + \partial_s q\,\mathcal{U} = Q\,p\,(\mu - \frac{1}{2})$$

(see (5.68)) in the form

$$2Q\partial_s p_1 + \partial_s (E^\epsilon q_\epsilon) = -Q\frac{1+d}{2} p_1.$$

From the definition of the function g it follows that $E^\epsilon q_\epsilon = g - 2Q\,p_1$. So

$$\partial_s g = \frac{3-d}{2} Q\,p_1.$$

Differentiating this equation along t^1 and using that

$$\partial_1 g = \frac{3-d}{2} \partial_s \varpi$$

we obtain (5.87). Lemma is proved.

For the case of polynomial Frobenius manifolds on the orbit spaces of the ADE Weyl groups the structure constants $c_{\alpha\beta}^{\gamma}(t)$ are the same as in (5.6) for the corresponding ADE singularity. As it was shown in [25] the latter equations coincide with the Picard - Fuchs equations for the periods of Seiberg - Witten differential on the spectral curve (the latter does not appear in our formalism). Observe that our Hamilton - Jacobi equation (5.80) coincides with the renormalization group equation introduced in [32, 24].

Taking into account that Frobenius manifolds M arise in the setting of 2D topological field theory, it would be interesting to figure out if there is a physical motivation for the tensor product $M \otimes QH^*(CP^1)$ to carry all the structures of the quantum moduli space (perhaps, all but positivity of the kinetic energy $\partial^2 S / \partial X^i \partial X^j$) of a $\mathcal{N} = 2$ supersymmetric 4D theory.

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