Virasoro Symmetries of the Extended Toda Hierarchy

Boris Dubrovin[1], **Youjin Zhang**[2]

[1] SISSA, Via Beirut 2–4, 34014 Trieste, Italy
[2] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China

Abstract: We prove that the extended Toda hierarchy of [1] admits a nonabelian Lie algebra of infinitesimal symmetries isomorphic to half of the Virasoro algebra. The generators L_m, $m \geq -1$ of the Lie algebra act by linear differential operators onto the tau function of the hierarchy. We also prove that the tau function of a generic solution to the extended Toda hierarchy is annihilated by a combination of the Virasoro operators and the flows of the hierarchy. As an application we show that the validity of the Virasoro constraints for the CP^1 Gromov-Witten invariants and their descendents implies that their generating function is the logarithm of a particular tau function of the extended Toda hierarchy.

1. Introduction

The extended Toda hierarchy was introduced in [30, 16, 1] in an attempt to encode the recursion relations among the CP^1 Gromov-Witten invariants into that of a hierarchy of integrable systems. As it was shown in [1], this hierarchy can be represented in a Lax pair formalism through the Lax operator

$$L = \Lambda + v(x) + e^{u(x)}\Lambda^{-1}. \tag{1.1}$$

The functions v, u serve as the dependent variables for the hierarchy with spatial variable x and $\Lambda = e^{\epsilon \partial_x}$ is the shift operator, ϵ is a small parameter. We will also introduce a two-component vector

$$w = (w^1, w^2), \quad w^1 := v, \ w^2 := u$$

to use in the formulae where many summations enter.

The flows of the hierarchy are defined via Lax representation

$$\epsilon \frac{\partial L}{\partial t^{\beta,q}} = [A_{\beta,q}, L] := A_{\beta,q}L - LA_{\beta,q}, \quad \beta = 1, 2; \ q \geq 0. \tag{1.2}$$

Here the operators $A_{\beta,q}$ have the expression

$$A_{1,q} = \frac{2}{q!} \left[L^q (\log L - c_q) \right]_+ , \quad A_{2,q} = \frac{1}{(q+1)!} \left[L^{q+1} \right]_+ , \tag{1.3}$$

$$c_0 = 0, \ c_q = 1 + \frac{1}{2} + \cdots + \frac{1}{q} \tag{1.4}$$

with the positive part B_+ of the difference operator $B = \sum B_k \Lambda^k$ given by $B_+ = \sum_{k \geq 0} B_k \Lambda^k$. The logarithm of the operator L is defined as follows [1]. Let us first introduce the dressing operators P and Q of the form

$$P = \sum_{k \geq 0} p_k \Lambda^{-k}, \quad Q = \sum_{k \geq 0} q_k \Lambda^k, \quad p_0 = 1 \tag{1.5}$$

such that

$$L = P \Lambda P^{-1} = Q \Lambda^{-1} Q^{-1}. \tag{1.6}$$

Then we define

$$\log L := \frac{1}{2} \left(P \epsilon \partial_x P^{-1} - Q \epsilon \partial_x Q^{-1} \right). \tag{1.7}$$

Equations (1.2) for $\beta = 2$ coincide with the standard flows of the Toda hierarchy. In particular for $q = 0$ one obtains the equations of Toda lattice

$$\frac{\partial v}{\partial t^{2,0}} = \frac{1}{\epsilon} \left(e^{u(x+\epsilon)} - e^{u(x)} \right) = \sum_{k \geq 0} \frac{\epsilon^k}{(k+1)!} \partial_x^{k+1} e^u,$$

$$\frac{\partial u}{\partial t^{2,0}} = \frac{1}{\epsilon} \left(v(x) - v(x - \epsilon) \right) = \sum_{k \geq 0} (-1)^k \frac{\epsilon^k}{(k+1)!} \partial_x^{k+1} v, \tag{1.8}$$

written in the interpolated form. To return to the original discrete setup of [26] one introduces the lattice variables

$$u_n := u(n\epsilon), \ v_n := v(n\epsilon).$$

The parameter ϵ plays the role of the mesh of the lattice. Another part, for $\beta = 1$ is a new one. For $q = 0$ one obtains just the spatial translations

$$\frac{\partial v}{\partial t^{1,0}} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial t^{1,0}} = \frac{\partial u}{\partial x}. \tag{1.9}$$

The flow for $\beta = 1$ and $q = 1$ is less trivial:

$$\frac{\partial v}{\partial t^{1,1}} = v v_x + \frac{1}{\epsilon} \left[e^{u(x+\epsilon)} \left(\mathcal{B}_- u(x + \epsilon) - 2 \right) - e^{u(x)} \left(\mathcal{B}_- u(x - \epsilon) - 2 \right) \right],$$

$$\frac{\partial u}{\partial t^{1,1}} = \frac{1}{\epsilon} \left[v(x) \left(\mathcal{B}_- u(x) - 2 \right) - v(x - \epsilon) \left(\mathcal{B}_- u(x - \epsilon) - 2 \right) \right.$$

$$\left. + \mathcal{B}_+ v(x + \epsilon) - \mathcal{B}_+ v(x - \epsilon) \right], \tag{1.10}$$

where the operators $\mathcal{B}_\pm$ are defined by

$$\mathcal{B}_+ := (\Lambda - 1)^{-1}\epsilon\partial_x = \sum_{k\geq 0} \frac{B_k}{k!}(\epsilon\partial_x)^k,$$

$$\mathcal{B}_- := (1 - \Lambda^{-1})^{-1}\epsilon\partial_x = \sum_{k\geq 0} \frac{B_k}{k!}(-\epsilon\partial_x)^k. \tag{1.11}$$

Here B_k are the Bernoulli numbers.

The flows of the extended Toda hierarchy can be represented as Hamiltonian systems

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \{w^\alpha(x), H_{\beta,q}\}_1 \equiv U_1^{\alpha\gamma}\frac{\delta H_{\beta,q}}{\delta w^\gamma(x)} \tag{1.12}$$

with the Hamiltonian operators

$$U_1^{11} = U_1^{22} = 0, \quad U_1^{12} = \frac{1}{\epsilon}(\Lambda - 1), \quad U_1^{21} = \frac{1}{\epsilon}(1 - \Lambda^{-1}), \tag{1.13}$$

and the Hamiltonians

$$H_{\beta,q} = \int h_{\beta,q}dx, \quad \beta = 1, 2; \ q \geq -1,$$

with the densities $h_{\beta,q}$ defined by

$$h_{1,q} = \frac{2}{(q + 1)!} \operatorname{res}\left[L^{q+1}(\log L - c_{q+1})\right] = \operatorname{res} A_{1,q+1},$$

$$h_{2,q} = \frac{1}{(q + 2)!}\operatorname{res} L^{q+2} = \operatorname{res} A_{2,q+1}. \tag{1.14}$$

By definition the residue of a difference operator

$$A = \sum_{k\in\mathbb{Z}} a_k\Lambda^k$$

is given by

$$\operatorname{res} A := a_0.$$

Note that

$$h_{1,-1} = \mathcal{B}_-u(x), \quad h_{2,-1} = v(x) \tag{1.15}$$

are densities of Casimirs of the Poisson bracket, i.e.

$$\{., H_{1,-1}\}_1 = \{., H_{2,-1}\}_1 \equiv 0.$$

Denote $\mathcal{A}$ the graded ring of formal power series of the form $\sum_{k\geq 0} \epsilon^k f_k(w, w_x, \dots)$, where f_k are polynomials of $v, u, e^{\pm u}, v^{(m)}, u^{(m)}, m \geq 1$. The gradation is defined by

$$\deg v^{(m)} = 1 - m, \ \deg u^{(m)} = -m, \ \text{for } m \geq 0, \ \deg e^u = 2, \ \deg \epsilon = 1. \tag{1.16}$$

As it was shown in [1], all the Hamiltonian densities (1.14) as well as the right hand sides of the flows of the extended Toda hierarchy are homogeneous elements of the ring $\mathcal{A}$.

Introduce functions $\Omega_{\alpha, p; \beta, q} \in \mathcal{A}$ by

$$
\frac{1}{\epsilon} (\Lambda - 1) \Omega_{\alpha, p; \beta, q} := \frac{\partial h_{\alpha, p-1}}{\partial t^{\beta, q}} = \begin{cases} \frac{1}{\epsilon} \frac{2}{p!} \operatorname{res} \left([A_{\beta, q}, L^p (\log L - c_p)] \right), & \alpha = 1, \\ \frac{1}{\epsilon} \frac{1}{(p+1)!} \operatorname{res} \left[A_{\beta, q}, L^{p+1} \right], & \alpha = 2. \end{cases}
\tag{1.17}
$$

Existence of such functions and an important property of τ-*symmetry*

$$
\Omega_{\beta, q; \alpha, p} = \Omega_{\alpha, p; \beta, q}
$$

was established in [1]. These elements of the ring $\mathcal{A}$ are uniquely determined by the above formulae and by the homogeneity condition

$$
\deg \Omega_{\alpha, p; \beta, q} = p + q + 1 + \mu_\alpha + \mu_\beta.
$$

Here

$$
\mu_1 = -\frac{1}{2}, \quad \mu_2 = \frac{1}{2}.
$$

In this paper we will consider the solutions to the extended Toda hierarchy in the class of formal series in ϵ

$$
w^\alpha(x, \mathbf{t}; \epsilon) = \sum_{k \geq 0} \epsilon^k w_k^\alpha(x, \mathbf{t}), \quad \alpha = 1, 2,
$$
$$
\mathbf{t} = (t^{1,0}, t^{2,0}, t^{1,1}, t^{2,1}, \dots).
\tag{1.18}
$$

As it follows from the definition (1.3),

$$
A_{1,0} = \epsilon \partial_x
$$

modulo terms commuting with L. So

$$
\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x},
$$

i.e., the solution depends on x, $t^{1,0}$ only via the combination $x + t^{1,0}$. We will therefore often suppress the explicit dependence on x in the formulae.

As it follows from Proposition 4.1 of [30], after the transformation of dependent variables $u \mapsto \hat{u}$, $v \mapsto \hat{v}$,

$$
\hat{v}(x) = \mathcal{B}_+ v(x),
$$
$$
\hat{u}(x) = \mathcal{B}_+ \mathcal{B}_- u(x),
\tag{1.19}
$$

only *even* powers of ϵ remain in the ϵ-expansions of the right hand sides of equations of the hierarchy (1.2).

Definition ([1]). *For any solution $v(x, \mathbf{t}; \epsilon)$, $u(x, \mathbf{t}; \epsilon)$ of the extended Toda hierarchy there exists a function*

$$\tau = \tau(x, \mathbf{t}; \epsilon) = e^{\sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{t})}$$

such that the functions $\Omega_{\alpha, p; \beta, q}$ evaluated on this solution can be represented in the form

$$\Omega_{\alpha, p; \beta, q} = \epsilon^2 \frac{\partial^2 \log \tau}{\partial t^{\alpha, p} \partial t^{\beta, q}} \tag{1.20}$$

*for any $\alpha, \beta = 1, 2$, $p, q \geq 0$. It is called the **tau function** of the solution (1.18) of the extended Toda hierarchy.*

In particular, we have

$$h_{\alpha, p} = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial t^{\alpha, p+1}}, \quad \alpha = 1, 2, \ p \geq -1, \tag{1.21}$$

$$v = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial t^{2,0}} = \epsilon \frac{\partial}{\partial t^{2,0}} \log \frac{\tau(x + \epsilon, \mathbf{t}; \epsilon)}{\tau(x, \mathbf{t}; \epsilon)},$$

$$u = (\Lambda - 1)(1 - \Lambda^{-1}) \log \tau = \log \frac{\tau(x + \epsilon, \mathbf{t}; \epsilon)\tau(x - \epsilon, \mathbf{t}; \epsilon)}{\tau^2(x, \mathbf{t}; \epsilon)}. \tag{1.22}$$

Another important property of this hierarchy is that, apart from its Hamiltonian structure described above, it also possesses a second Hamiltonian structure which is compatible with the first one (see (4.15) below). The bihamilonian structure and the tau symmetry property of the extended Toda hierarchy imply, due to a general theorem of [9], *quasi-triviality* of the extended Toda hierarchy. The precise formulation of the quasi-triviality property in the case of interest is given by the following theorem (cf. Corollary 3.10.23 of [9]):

Theorem 1.1. *Any solution v, u of the extended Toda hierarchy is obtained from a solution v_0, u_0 of the dispersionless extended Toda hierarchy through the quasi-Miura transformation of the form*

$$v = v_0 + \sum_{g \geq 1} \epsilon^{2g-1}(\Lambda - 1) \frac{\partial F_g(w_0, \ldots, w_0^{(3g-2)})}{\partial t^{2,0}},$$

$$u = u_0 + \sum_{g \geq 1} \epsilon^{2g-2}(\Lambda - 1)(1 - \Lambda^{-1}) F_g(w_0, \ldots, w_0^{(3g-2)}), \tag{1.23}$$

*(cf. (1.19)) and the corresponding tau function of the solution admits the following **genus expansion***

$$\log \tau = \epsilon^{-2} \log \tau^{[0]} + \sum_{g \geq 1} \epsilon^{2g-2} F_g(w_0, \ldots, w_0^{(3g-2)}). \tag{1.24}$$

In these formulae the functions $F_g(w, w_x, \ldots, w^{(3g-2)})$ do not depend on the choice of the solution.

Here $\tau^{[0]} = \tau^{[0]}(x, \mathbf{t})$ is the tau function for the solution v_0, u_0 of the dispersionless extended Toda hierarchy, i.e., it is related to the leading term of the solution (1.18)

$$w_0 = (v_0, u_0)$$

via

$$v_0(x, \mathbf{t}) = \frac{\partial^2 \log \tau^{[0]}(x, \mathbf{t})}{\partial x \partial t^{2,0}},$$

$$u_0(x, \mathbf{t}) = \frac{\partial^2 \log \tau^{[0]}(x, \mathbf{t})}{\partial x^2}. \tag{1.25}$$

Recall that the dispersionless extended Toda hierarchy is obtained from the extended Toda hierarchy by setting $\epsilon = 0$. All the flows of the dispersionless extended Toda hierarchy are systems of hydrodynamic type, i.e. systems of two first order quasilinear PDEs. For $\beta = 1$, $q = 0$ the dispersionless flow still coincides with the spatial translations (1.9). For $\beta = 2$, $q = 0$ one obtains

$$\frac{\partial v}{\partial t^{2,0}} = \frac{\partial}{\partial x} e^u,$$

$$\frac{\partial u}{\partial t^{2,0}} = \frac{\partial v}{\partial x}.$$

Eliminating v yields the so-called long wave limit of the Toda lattice equations

$$u_{tt} = \left(e^u\right)_{xx},$$

where $t = t^{2,0}$. The dispersionless limit of (1.10) reads

$$\frac{\partial v}{\partial t^{1,1}} = \frac{\partial}{\partial x} \left[\frac{1}{2} v^2 + (u - 1)e^u \right],$$

$$\frac{\partial u}{\partial t^{1,1}} = \frac{\partial}{\partial x} (u \, v).$$

Changing the sign of time $t = -t^{1,1}$ one identifies these with the equations of motion of the one-dimensional polytropic gas with the speed v and density u and the equation of state of the form $p = (u^2 - 2u + 2)e^u - 2$.

It is time to remind the reader that the theory of dispersionless (extended) Toda hierarchy can be nicely encoded [5, 4] in terms of a particular two-dimensional Frobenius manifold M_{Toda}. The latter can be identified with the quantum cohomology of complex projective line

$$M_{\text{Toda}} = QH^*(CP^1).$$

Alternatively, the Frobenius manifold in question is isomorphic to the orbit space of the simplest extended affine Weyl group [6]

$$M_{\text{Toda}} = \mathbb{C}^2 / \tilde{W}(A_1).$$

Denote v, u the coordinates on M_{Toda}. The potential of the Frobenius manifold reads

$$F = \frac{1}{2} v^2 u + e^u. \tag{1.26}$$

The third derivatives of the potential define the multiplication law of tangent vectors at each point of M such that $\partial / \partial v$ is the unity and

$$\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} = e^u \frac{\partial}{\partial v}.$$

The flat metric on M reads

$$< \ , \ \ >= 2du \, dv. \tag{1.27}$$

The Euler vector field is

$$E = v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial u}.$$

We will not recall here the universal construction, due to [3] of a "dispersionless" integrable hierarchy valid for an arbitrary Frobenius manifold M. The hierarchy can be considered as an infinite family of pairwise commuting flows on the loop space $\mathcal{L}(M)$. All these flows are represented by first order quasilinear PDEs; for this reason this hierarchy is called dispersionless. The word "hierarchy" means that the systems of integrable PDEs are organized by means of the action of a bihamiltonian recursion operator.

In [9] we addressed the problem of extending the correspondence

$$\text{Frobenius manifolds} \ \rightarrow \ \text{hierarchies of integrable PDEs}$$

to an arbitrary Frobenius manifold. We proved that, indeed such a *universal correspondence* exists for an arbitrary semisimple M provided a suitable completion of the loop space $\mathcal{L}(M)$ is made allowing to work with infinite order PDEs. By definition semisimplicity of a Frobenius manifold means that, for a generic point $w \in M$ the algebra on $T_w M$ is semisimple. We leave as an exercise to the reader to verify that M_{Toda} is a semisimple Frobenius manifold. So, according to the main result of [9], there exists an integrable hierarchy associated with the Frobenius manifold M_{Toda}. The main aim of the present paper is to identify this integrable hierarchy with the extended Toda hierarchy (1.2).

The crucial role in such identification, apart from the Lax representation and tau structure obtained in [1], play *Virasoro symmetries*. According to our paper [8] for an arbitrary Frobenius manifold there exists a universal construction of second order linear differential operators

$$L_m = L_m(\epsilon^{-1}\mathbf{t}, \epsilon\frac{\partial}{\partial \mathbf{t}}), \ m \geq -1$$

satisfying the Virasoro commutation relations

$$[L_i, L_j] = (i - j)L_{i+j}, \quad i, \, j \geq -1. \tag{1.28}$$

For M_{Toda} these Virasoro operators are given by the following explicit expressions (cf. [12]):

$$L_{-1} = \sum_{k \geq 1} t^{\alpha,k} \frac{\partial}{\partial t^{\alpha,k-1}} + \frac{1}{\epsilon^2} t^{1,0} t^{2,0},$$

$$L_0 = \sum_{k \geq 1} k \left(t^{1,k} \frac{\partial}{\partial t^{1,k}} + t^{2,k-1} \frac{\partial}{\partial t^{2,k-1}} \right) + 2 \sum_{k \geq 1} t^{1,k} \frac{\partial}{\partial t^{2,k-1}} + \frac{1}{\epsilon^2} \left(t^{1,0} \right)^2,$$

$$L_m = \epsilon^2 \sum_{k=1}^{m-1} k! \, (m-k)! \frac{\partial^2}{\partial t^{2,k-1} \partial t^{2,m-k-1}} + \sum_{k \geq 1} \frac{(m+k)!}{(k-1)!} \left(t^{1,k} \frac{\partial}{\partial t^{1,m+k}} \right.$$

$$\left. + t^{2,k-1} \frac{\partial}{\partial t^{2,m+k-1}} \right) + 2 \sum_{k \geq 0} \alpha_m(k) t^{1,k} \frac{\partial}{\partial t^{2,m+k-1}}, \quad m \geq 1, \tag{1.29}$$

where

$$\alpha_m(0) = m!, \quad \alpha_m(k) = \frac{(m+k)!}{(k-1)!} \sum_{j=k}^{m+k} \frac{1}{j}, \ k > 0.$$

Again, we will not reproduce here the *universal construction* of the Virasoro operators; however, we will give below the free field realization of these operator for the M_{Toda} case obtained in Sect. 3.10.2 of our paper [9].

It is the *Virasoro invariance* property of a hierarchy of integrable PDEs that allows to reconstruct it uniquely starting from a given semisimple Frobenius manifold, along with bihamiltonian structure and existence of a tau function. That means that the *linear* action of the Virasoro operators onto tau functions defines symmetries of the hierarchy[1].

The central result of this paper is the following

Main Theorem. *1. The transformations*

$$\tau \mapsto \tau + \delta\, L_m \tau, \quad \delta \to 0 \tag{1.30}$$

for any $m \geq -1$ are **infinitesimal symmetries** *of the extended Toda hierarchy, i.e., given a tau function $\tau = \tau(\mathbf{t}; \epsilon)$ of a solution $v(\mathbf{t}; \epsilon)$, $u(\mathbf{t}; \epsilon)$ of the form (1.18), the functions*

$$\tilde{v}(\mathbf{t}; \epsilon) = v(\mathbf{t}; \epsilon) + \epsilon\delta(\Lambda - 1)\frac{\partial}{\partial t^{2,0}}\frac{L_m \tau}{\tau} + O(\delta^2),$$

$$\tilde{u}(\mathbf{t}; \epsilon) = u(\mathbf{t}; \epsilon) + \delta(\Lambda - 1)(1 - \Lambda^{-1})\frac{L_m \tau}{\tau} + O(\delta^2)$$

satisfy equations of the extended Toda hierarchy modulo terms of order $O(\delta^2)$.
2. For a generic solution (1.18) of the extended Toda hierarchy, the corresponding tau function satisfies the Virasoro constraints

$$L_m\left(\epsilon^{-1}(\mathbf{t} - \mathbf{c}(\epsilon)), \epsilon\frac{\partial}{\partial \mathbf{t}}\right)\tau = 0, \quad m \geq -1 \tag{1.31}$$

for some $\mathbf{c}(\epsilon)$. Here $\mathbf{c}(\epsilon) = (c^{\alpha, p}(\epsilon))$ is a collection of formal power series in ϵ.

We will describe the class of generic solutions in Sect. 4. Note that in the above formulae we omit writing explicitly the x-dependence since x enters only through the combination $x + t^{1,0}$.

The above two theorems and the uniqueness result of [9] (see Theorem 3.10.31) imply

Corollary 1.2. *The hierarchy of PDEs associated, according to [9], with the Frobenius manifold M_{Toda} coincides with (1.2).*

According to [9], the functions $F_g = F_g(w_0; \ldots, w_0^{(3g-2)})$ in the genus expansion (1.24) are uniquely determined by the loop equation that was introduced in Sect. 3.10.7

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[1] In Sect. 3.10.4 of [9] we called this property *linearization of the Virasoro symmetries*. The reason for this name was the following. The dispersionless hierarchy on $\mathcal{L}(M)$ constructed in [3] is always invariant with respect to an action of half of the Virasoro algebra [8]. However, the generators of the Virasoro action do not act linearly onto the dispersionless tau function $\tau^{[0]}(\mathbf{t})$. The full hierarchy constructed in [9] is a deformation of the dispersionless one that transforms the nonlinear action of the Virasoro symmetries to the linear one.

of [9] for any semisimple Frobenius manifold. For M_{Toda} the loop equation will be discussed in Sect. 5. It will also be explained how one can compute Gromov - Witten invariants of CP^1 and their descendents of any genus using the loop equation.

Due to the uniqueness of solution of the loop equation and the validity of the Virasoro constraints for CP^1 Gromov-Witten invariants [20], we have the following

Corollary 1.3. *The generating function of the CP^1 Gromov-Witten invariants and their descendents is uniquely determined by the system of Virasoro constraints. It coincides with the logarithm of the tau function of a particular solution of the extended Toda hierarchy.*

This particular solution for the CP^1 Gromov-Witten invariants will be described in Sect. 5.

Summarizing, we can say that, in the theory of Gromov - Witten invariants of CP^1 and their descendents the extended Toda hierarchy (1.2) plays the role similar to one played by the KdV hierarchy in the Kontsevich - Witten formulation [28, 21, 29] of the intersection theory of the tautological classes on the Deligne - Mumford moduli spaces of stable punctured curves.

2. Some Formulae for the Functions $\Omega_{\alpha, p; \beta, q}$

We present in this section some important identities for the functions $\Omega_{\alpha, p; \beta, q}$ defined in (1.17) and for the Hamiltonians $H_{\alpha, p}$.

Let us first recall the definitions of [1] of the logarithmic x-derivatives of the dressing operators P and Q. Let us look for these logarithmic derivatives in the form

$$\epsilon P_x P^{-1} = \sum_{j \geq 1} b_j \Lambda^{-j},$$

$$\epsilon Q_x Q^{-1} = \sum_{j \geq 0} c_j \Lambda^j. \tag{2.1}$$

Lemma 2.1. *There exist unique elements b_j, $c_j \in \mathcal{A}$ homogeneous of the degrees*

$$\deg b_j = j, \quad \deg c_j = -j$$

such that the operators (2.1) satisfy the following system of equations:

$$\operatorname{res}\left([\epsilon P_x P^{-1}, L^m] - \epsilon \partial_x L^m \right) = \operatorname{res}\left([\epsilon Q_x Q^{-1}, L^m] - \epsilon \partial_x L^m \right) = 0, \quad m \geq 1.$$

Proof. See [1]. For example,

$$b_1 = -\mathcal{B}_+ v, \quad c_0 = \mathcal{B}_- u, \quad c_1 = e^{-u(x+\epsilon)} \mathcal{B}_- v.$$

□

Lemma 2.2. *The following identities hold true*

$$\frac{\partial \Omega_{\alpha, p; \beta, q}}{\partial v} = \Omega_{\alpha, p-1; \beta, q} + \Omega_{\alpha, p; \beta, q-1} + (\delta_{\alpha,1}\delta_{\beta,2} + \delta_{\alpha,2}\delta_{\beta,1})\delta_{p,0}\delta_{q,0}; \tag{2.2}$$

$$\left(\sum_{m \geq 0} v^{(m)} \frac{\partial}{\partial v^{(m)}} + 2 \frac{\partial}{\partial u} \right) \Omega_{\alpha, p; \beta, q}$$

$$= (p + q + 1 + \mu_\alpha + \mu_\beta)\Omega_{\alpha, p; \beta, q} + R_\alpha^\gamma \Omega_{\gamma, p-1; \beta, q} + R_\beta^\gamma \Omega_{\alpha, p; \gamma, q-1}$$

$$+ 2\, \delta_{\alpha,1}\delta_{\beta,1}\delta_{p,0}\delta_{q,0}. \tag{2.3}$$

Here the numbers μ_α and a 2×2 matrix $R = \left(R_\beta^\gamma \right)$ are defined by

$$\mu_1 = -\mu_2 = -\frac{1}{2}, \quad R_\beta^\gamma = 2\delta_2^\gamma \delta_{\beta,1}, \tag{2.4}$$

and we assume that $\Omega_{\alpha,p-1;\beta,q} = \Omega_{\beta,q;\alpha,p-1} = 0$ when $p = 0$.

Proof. Let us consider the difference operators

$$B := \frac{\partial}{\partial v} P \epsilon \partial_x P^{-1} = -\sum_{j \geq 1} \frac{\partial b_j}{\partial v} \Lambda^{-j},$$

$$C := \frac{\partial}{\partial v} Q \epsilon \partial_x Q^{-1} = -\sum_{j \geq 0} \frac{\partial c_j}{\partial v} \Lambda^j.$$

The coefficients $b_j \in \mathcal{A}$, $c_j \in \mathcal{A}$ were defined in (2.1). We want to show that

$$LB = 1, \quad LC = -1. \tag{2.5}$$

Indeed, differentiating the identity

$$[P \epsilon \partial_x P^{-1}, L] = 0$$

with respect to v we derive that the operators B and L commute. Therefore

$$[B - P\Lambda^{-1} P^{-1}, L^m] = 0, \quad m \geq 1.$$

Since $b_1 = -\mathcal{B}_+ v$,

$$\frac{\partial b_1}{\partial v} = -1.$$

So the expansion of the difference operator $B - P\Lambda^{-1} P^{-1}$ begins with Λ^{-2}. From the equations

$$\text{res}[B - P\Lambda^{-1} P^{-1}, L^m] = 0 \quad \text{for any } m \geq 1,$$

we prove that the coefficient of Λ^{-2} of the operator $B - P\Lambda^{-1} P^{-1}$ is a constant. Since

$$\deg \frac{\partial b_j}{\partial v} = j - 1,$$

the degree of this coefficient, as an element of $\mathcal{A}$, is equal to 1. So, the coefficient must be equal to zero. Continuing this process we prove that

$$B = P\Lambda^{-1} P^{-1} = \Lambda^{-1} + O(\Lambda^{-2}).$$

In a similar way we can prove that

$$C = -Q\Lambda Q^{-1} = -e^{-u(x+\epsilon)}\Lambda + O(\Lambda^2).$$

Now, using the definition (1.7) of $\log L$ we obtain

$$\frac{\partial}{\partial v} \log L = \frac{1}{2}(B - C) = \frac{1}{2}(P\Lambda^{-1} P^{-1} + Q\Lambda Q^{-1}).$$

Therefore

$$L^q \frac{\partial \log L}{\partial v} = L^{q-1}. \tag{2.6}$$

From the last equation it readily follows that

$$\frac{\partial}{\partial v} \left[\frac{2}{p!} L^p (\log L - c_p) \right] = \frac{2}{(p-1)!} L^{p-1} (\log L - c_{p-1}), \quad p \geq 1,$$

$$\frac{\partial}{\partial v} \left[\frac{1}{(p+1)!} L^{p+1} \right] = \frac{1}{p!} L^p, \quad p \geq 0.$$

These two equations yield, together with

$$h_{1,-1} = (1 - \Lambda^{-1})^{-1} \epsilon \partial_x u, \quad h_{2,-1} = v$$

and the definition (1.17), the formula (2.2).

To prove the second formula of the lemma, let us introduce the operators $\mathcal{H}$ and $\mathcal{E}$ that act on the space of difference operators of the form $\sum b_k \Lambda^k$ with coefficients in $\mathcal{A}$ as follows:

$$\mathcal{H} \sum b_k \Lambda^k = \sum k b_k \Lambda^k, \tag{2.7}$$

$$\mathcal{E} \sum b_k \Lambda^k = \sum \left(\sum_{m \geq 0} v^{(m)} \frac{\partial b_k}{\partial v^{(m)}} + 2 \frac{\partial b_k}{\partial u} \right) \Lambda^k. \tag{2.8}$$

It is easy to see that these operators satisfy the Leibnitz rule

$$\mathcal{P} \left[\left(\sum a_k \Lambda^k \right) \left(\sum b_l \Lambda^l \right) \right] = \left(\mathcal{P} \sum a_k \Lambda^k \right) \sum b_l \Lambda^l + \sum a_k \Lambda^k \left(\mathcal{P} \sum b_l \Lambda^l \right). \tag{2.9}$$

Here $\mathcal{P} = \mathcal{H}$ or $\mathcal{P} = \mathcal{E}$. By our definition it follows that

$$(\mathcal{H} + \mathcal{E}) L = L. \tag{2.10}$$

Due to Lemma 2.1

$$(\mathcal{H} + \mathcal{E}) \log L = 1. \tag{2.11}$$

Now by using (2.10) and (2.11) we obtain

$$(\Lambda - 1)\mathcal{E}\Omega_{2,p;1,q} = \frac{1}{(p+1)!} \mathcal{E} \operatorname{res} \left[A_{1,q}, L^{p+1} \right]$$

$$= \frac{1}{(p+1)!} (\mathcal{H} + \mathcal{E}) \operatorname{res} \left[A_{1,q}, L^{p+1} \right]$$

$$= \frac{1}{(p+1)!} \operatorname{res} \left[q A_{1,q} + 2 A_{2,q-1}, L^{p+1} \right] + \frac{1}{p!} \operatorname{res} \left[A_{1,q}, L^{p+1} \right]$$

$$= (p + q + 1)(\Lambda - 1)\Omega_{2,p;1,q} + 2(\Lambda - 1)\Omega_{2,p;2,q-1}. \tag{2.12}$$

So, from the homogeneity condition for $\Omega_{\alpha,p;\beta,q}$ we arrive at

$$\mathcal{E}\Omega_{2,p;1,q} = (p + q + 1)\Omega_{2,p;1,q} + 2\,\Omega_{2,p;2,q-1}.$$

Other cases of the formula (2.3) can be proved in a similar way. The lemma is proved.
□

Lemma 2.3. *The variational derivatives of the Hamiltonians $H_{\beta,q}$ are given by the following formulae*

$$\frac{\delta H_{\beta,q}}{\delta v(x)} = h_{\beta,q-1} = \epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{\beta,q}}, \tag{2.13}$$

$$\frac{\delta H_{\beta,q}}{\delta u(x)} = \Omega_{2,0;\beta,q} = \epsilon^2 \frac{\partial^2 \log \tau}{\partial t^{2,0}\partial t^{\beta,q}}. \tag{2.14}$$

Proof. From the Hamiltonian representation (1.12) of the extended Toda hierarchy we have

$$\frac{\partial v}{\partial t^{\beta,q}} = \frac{1}{\epsilon}(\Lambda - 1)\frac{\delta H_{\beta,q}}{\delta u}, \qquad \frac{\partial u}{\partial t^{\beta,q}} = \frac{1}{\epsilon}\left(1 - \Lambda^{-1}\right)\frac{\delta H_{\beta,q}}{\delta v}. \tag{2.15}$$

On the other hand, the formulae (1.21) and (1.22) imply that

$$\frac{\partial v}{\partial t^{\beta,q}} = \frac{1}{\epsilon}(\Lambda - 1)\,\Omega_{2,0;\beta,q}, \qquad \frac{\partial u}{\partial t^{\beta,q}} = \frac{1}{\epsilon}\left(1 - \Lambda^{-1}\right)h_{\beta,q-1}. \tag{2.16}$$

So, due to the homogeneity property of the densities of the Hamiltonians we arrive at the expressions (2.13) and (2.14). The lemma is proved. $\square$

3. Virasoro Operators for the Frobenius Manifold M_{Toda}

Recall that, according to the general algorithm of [9] the Virasoro operators associated with a given Frobenius manifold are obtained by the following free field realization[2] that we now present for the example of M_{Toda}. Let $a_{1,p}$, $a_{2,p}$, $p \in \mathbb{Z}$ be free bosonic operators satisfying the following commutation relation[3]

$$[a_{1,p}, a_{2,q}] = (-1)^p \delta_{p+q+1,0}. \tag{3.1}$$

Introduce vectors of operators

$$\mathbf{a}_p := (a_{1,p}, a_{2,p}), \quad p \in \mathbb{Z}. \tag{3.2}$$

Consider the generating function

$$\mathbf{f}(z) := \sum_{p\in\mathbb{Z}} \mathbf{a}_p z^{p+\mu} z^R = \sum_{p\in\mathbb{Z}} \mathbf{a}_p \begin{pmatrix} z^{p-\frac{1}{2}} & 0 \\ 2z^{p+\frac{1}{2}}\log z & z^{p+\frac{1}{2}} \end{pmatrix}. \tag{3.3}$$

Here the diagonal matrix μ and nilpotent matrix R correspond to the *spectrum at the origin* of the Frobenius manifold M_{Toda},

$$\mu = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{3.4}$$

---

[2] In [8] we used a different free field realization of the Virasoro operators inspired by [10].
[3] Note change of notations: in [9] we used half integer labels.

The current $\phi^{(\nu)}(\lambda)$ for any non-integer ν is defined by a (suitably defined) Laplace-type transform

$$\phi^{(\nu)}(\lambda) = \int_0^\infty \frac{dz}{z^{\frac{1}{2}+\nu}} e^{-\lambda z} \mathbf{f}(z)$$

$$= \sum_{p \in \mathbb{Z}} \frac{\mathbf{a}_p}{\lambda^{p-\nu}} \begin{pmatrix} \Gamma(p-\nu) & 0 \\ \frac{2}{\lambda} \left[\Gamma'(p-\nu+1) - \log\lambda\, \Gamma(p-\nu+1) \right] & \frac{1}{\lambda}\Gamma(p-\nu+1) \end{pmatrix}.$$

(3.5)

The generating function of the regularized Virasoro operators $L_m^{(\nu)}$ is defined by the following quadratic combination of the derivatives of the currents

$$T^{(\nu)}(\lambda)$$

$$= \sum_{m \in \mathbb{Z}} \frac{L_m^{(\nu)}}{\lambda^{m+2}}$$

$$= \frac{1}{2} : \partial_\lambda \phi^{(-\nu)} G(\nu) \left[\partial_\lambda \phi^{(\nu)} \right]^T : + \frac{1}{4\lambda^2} \operatorname{tr}\left(\frac{1}{4} - \mu^2 \right)$$

$$= \sum_{p,q \in \mathbb{Z}} \frac{1}{\lambda^{p+q+3}} : \mathbf{a}_p M_{pq}(\nu, \lambda)\, \mathbf{a}_q^T :, \quad \text{where} \tag{3.6}$$

$$M_{pq}(\nu, \lambda)$$

$$= \begin{pmatrix} 0 & \frac{\sin\pi\nu}{2\pi}\Gamma(p+\nu+1)\Gamma(q-\nu+2) \\ -\frac{\sin\pi\nu}{2\pi}\Gamma(q-\nu+1)\Gamma(p+\nu+2) & \frac{1}{\lambda}\partial_\nu\left[\frac{\sin\pi\nu}{\pi}\Gamma(p+\nu+2)\Gamma(q-\nu+2)\right] \end{pmatrix}.$$

In this formula the Gram matrix $G(\nu)$ reads

$$G(\nu) = \frac{1}{2\pi}\left[e^{\pi i R} e^{\pi i(\mu+\nu)} + e^{-\pi i R} e^{-\pi i(\mu+\nu)} \right] \eta^{-1}$$

$$= \frac{1}{\pi}\begin{pmatrix} 0 & \sin\pi\nu \\ -\sin\pi\nu & 2\pi\cos\pi\nu \end{pmatrix}, \tag{3.7}$$

η is the Gram matrix of the metric (1.27), i.e., in our case

$$\eta = \eta^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

the term with the trace in (3.6) does not contribute since $\mu^2 = \frac{1}{4}$. The normal ordering in (3.6) is defined in such a way that all the operators $a_{1,p}$ and $a_{2,p}$ with nonnegative p are to be written on the right. So

$$L_m^{(\nu)} = \sum_{p+q=m-2} : a_{2,p} a_{2,q} : \partial_\nu \left[\frac{\sin\pi\nu}{\pi}\Gamma(p+\nu+2)\Gamma(q-\nu+2) \right]$$

$$+ \sum_{p+q=m-1} : a_{1,p} a_{2,q} : \frac{\sin\pi\nu}{2\pi}\left[\Gamma(p+\nu+1)\Gamma(q-\nu+2) \right.$$

$$\left. -\Gamma(p-\nu+1)\Gamma(q+\nu+2) \right].$$

Using the standard identities of the theory of gamma-functions

$$\Gamma(x+1) = x\,\Gamma(x), \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

we finally obtain the following expression for the regularized Virasoro operators

$$L_m^{(\nu)} = \frac{1}{2}\sum_p (-1)^{p+1}\left[\frac{\Gamma(\nu - p + m + 1)}{\Gamma(\nu - p)} + \frac{\Gamma(-\nu - p + m + 1)}{\Gamma(-\nu - p)}\right] : a_{1,p}a_{2,m-p-1} :$$

$$+ \sum_p (-1)^p \partial_\nu \left[\frac{\Gamma(m - \nu - p)}{\Gamma(-\nu - p - 1)}\right] : a_{2,p}a_{2,m-p-2} : . \tag{3.8}$$

The operators

$$L_{-1}^{(\nu)} = \sum_p (-1)^{p+1} : a_{1,p}a_{2,-p-2} :$$

and

$$L_0^{(\nu)} = \sum_p (-1)^p p : a_{1,p}a_{2,-p-1} : + \sum_p (-1)^{p+1} : a_{2,p}a_{2,-p-2} :$$

do not depend on ν. The operators (3.8) with $m > 0$ depend polynomially on ν; those with $m < -1$ depend rationally on ν. Therefore there exist limits

$$L_m := \lim_{\nu \to 0} L_m^{(\nu)}, \quad m \geq -1.$$

To arrive at the Virasoro operators given above in the Introduction one use the following realization of the bosonic operators $a_{\alpha,p}$:

$$a_{\alpha,p} = \begin{cases} \epsilon \frac{\partial}{\partial t^{\alpha,p}}, & p \geq 0 \\ \epsilon^{-1}(-1)^{p+1} t_{\alpha,-p-1}, & p < 0 \end{cases}. \tag{3.9}$$

Here we use the matrix η for lowering the indices

$$t_{\alpha,p} := \eta_{\alpha\beta} t^{\beta,p}.$$

In the next section we will prove that the linear action of the Virasoro operators L_m with $m \geq -1$ defines infinitesimal symmetries of the extended Toda hierarchy.

4. Proof of the Main Theorem

We first consider the following system of Euler-Lagrange equations:

$$\sum_{p \geq 0} \tilde{t}^{\alpha,p} \frac{\delta H_{\alpha,p-1}}{\delta v(x)} = 0,$$

$$\sum_{p \geq 0} \tilde{t}^{\alpha,p} \frac{\delta H_{\alpha,p-1}}{\delta u(x)} = 0 \tag{4.1}$$

with

$$\tilde{t}^{\alpha,p} = t^{\alpha,p} - c^{\alpha,p}(\epsilon) + \delta_1^\alpha \delta_{p,0} x \tag{4.2}$$

for some formal power series $c^{\alpha,p}(\epsilon)$. Here and henceforth summations with respect to the repeated Greek indices are assumed. We assume that only finitely many of them are nonzero. The series must satisfy the condition of *genericity* that we shall now formulate.

Let us expand the Hamiltonian densities (1.14) in powers of ϵ,

$$h_{\alpha,p} = \theta_{\alpha,p+1}(v,u) + O(\epsilon). \tag{4.3}$$

The explicit formulae for the functions $\theta_{\alpha,p}(v,u)$ can be found in [9] (see also the formulae (A.12), (A.13) below). Let us impose the following assumptions for the leading terms of the series $c^{\alpha,p}(\epsilon)$.

1. There exist values $\bar{v}, \bar{u}$ such that

$$\sum_{p\geq 0} c^{\alpha,p}(0) \left.\frac{\partial\theta_{\alpha,p}(v,u)}{\partial v}\right|_{v=\bar{v},u=\bar{u}} = 0,$$

$$\sum_{p\geq 0} c^{\alpha,p}(0) \left.\frac{\partial\theta_{\alpha,p}(v,u)}{\partial u}\right|_{v=\bar{v},u=\bar{u}} = 0,$$

$$\tag{4.4}$$

and

2. The operator of multiplication by the vector

$$\nabla \sum_{p\geq 1} c^{\alpha,p}(0)\theta_{\alpha,p-1}(\bar{v},\bar{u}) \tag{4.5}$$

is invertible element of the Frobenius algebra $T_{\bar{v},\bar{u}} M_{\text{Toda}}$.

Under these assumptions the following lemma holds true (cf. Lemma 3.8.2 of [9]).

Lemma 4.1. *There exists a unique solution to the Euler - Lagrange equations (4.1) in the class of formal series*

$$v = v(x,\mathbf{t},\epsilon) = a_0(\epsilon) + \sum_{k>0} a_{\alpha_1,p_1;\dots;\alpha_k,p_k}(\epsilon)t^{\alpha_1,p_1}\dots t^{\alpha_k,p_k}|_{t^{1,0}\mapsto t^{1,0}+x},$$

$$u = u(x,\mathbf{t},\epsilon) = b_0(\epsilon) + \sum_{k>0} b_{\alpha_1,p_1;\dots;\alpha_k,p_k}(\epsilon)t^{\alpha_1,p_1}\dots t^{\alpha_k,p_k}|_{t^{1,0}\mapsto t^{1,0}+x}, \tag{4.6}$$

where

$$a_0(0) = \bar{v}, \quad b_0(0) = \bar{u}. \tag{4.7}$$

Proof. In the leading order in ϵ the Euler - Lagrange equations (4.1) become just equations for the critical points of the function

$$\sum_{p\geq 0}(t^{\alpha,p} - c^{\alpha,p}(0) + x\delta_1^\alpha \delta_0^p)\theta_{\alpha,p}(v,u).$$

The above two assumptions imply that there exists a unique critical point

$$v_0 = v_0(x,\mathbf{t},\epsilon), \quad u_0 = u_0(x,\mathbf{t},\epsilon)$$

of the function

$$\sum_{p\geq 0}(t^{\alpha,p} - c^{\alpha,p}(\epsilon) + x\delta_1^\alpha\delta_0^p)\theta_{\alpha,p}(v,u)$$

in the class of formal power series of the structure similar to (4.6) with

$$v_0(0, \mathbf{0}, 0) = \bar{v}, \quad u_0(0, \mathbf{0}, 0) = \bar{u}.$$

It is easy to see that these functions can be uniquely extended to a solution to the full Euler - Lagrange equations (4.1). The lemma is proved. □

Lemma 4.2. *The space of solutions of the Euler-Lagrange equation (4.1) is invariant with respect to the flows of the extended Toda hierarchy.*

Proof. Let us represent the difference operators $A_{\beta,q}$ defined in (1.3) by

$$A_{\beta,q} = \sum_{k\geq 0} a_{\beta,q;k}\Lambda^k, \quad \beta = 1, 2, \; q \geq 0. \tag{4.8}$$

Then by using Lemma 2.3 we obtain

$$\frac{\delta H_{\alpha,p}}{\delta v(x)} = a_{\alpha,p;0}(x), \quad \frac{\delta H_{\alpha,p}}{\delta u(x)} = a_{\alpha,p;1}(x - \epsilon)\, e^{u(x)}. \tag{4.9}$$

The Lax pair representation (1.2) of the extended Toda hierarchy yields

$$\frac{\partial A_{\alpha,p}}{\partial t^{\beta,q}} - \frac{\partial A_{\beta,q}}{\partial t^{\alpha,p}} = [A_{\beta,q}, A_{\alpha,p}],$$

$$\frac{\partial e^u}{\partial t^{\beta,q}} = \left[a_{\beta,q;0}(x) - a_{\beta,q;0}(x - \epsilon)\right] e^{u(x)}.$$

These equations together with (4.8) imply

$$\frac{\partial}{\partial t^{\beta,q}}\left(\frac{\delta H_{\alpha,p}}{\delta w^\gamma(x)}\right) = \frac{\partial}{\partial t^{\alpha,p}}\left(\frac{\delta H_{\beta,q}}{\delta w^\gamma(x)}\right), \quad \alpha, \beta, \gamma = 1, 2; \; p, q \geq 0. \tag{4.10}$$

So, under the flows of the extended Toda hierarchy we have

$$\frac{\partial}{\partial t^{\beta,q}}\left(\sum_{p\geq 0}\tilde{t}^{\alpha,p}\frac{\delta H_{\alpha,p-1}}{\delta w^\gamma(x)}\right) = \frac{\delta H_{\beta,q-1}}{\delta w^\gamma(x)} + \sum_{m\geq 0}\frac{\partial}{\partial w^{\xi,m}}\left(\frac{\delta H_{\beta,q}}{\delta w^\gamma(x)}\right)\partial_x^m\left(\sum_{p\geq 1}\tilde{t}^{\alpha,p}\frac{\partial w^\xi}{\partial t^{\alpha,p-1}}\right). \tag{4.11}$$

Here and below we use the following notations for the x-derivatives of the functions u and v

$$w^{\xi,m} := \frac{\partial^m w^\xi}{\partial x^m}, \quad \xi = 1, 2, \quad m \geq 0.$$

So

$$w^{1,m} = v^{(m)}, \quad w^{2,m} = u^{(m)}.$$

By using (4.12) that we will prove in Lemma 4.4 below we know that the r.h.s. of (4.11) can be rewritten as

$$\frac{\delta H_{\beta,q-1}}{\delta w^\gamma(x)} - \frac{\partial}{\partial v}\left(\frac{\delta H_{\beta,q}}{\delta w^\gamma(x)}\right)$$

which equals zero due to (4.9), (2.6) and the identity $\frac{\partial L}{\partial v} = 1$. The lemma is proved. □

Due to the uniqueness of solutions of the initial value problem for the Euler-Lagrange equation (4.1) and the above theorem, we have

Theorem 4.3. *Any solution of Eq. (4.1) gives a solution to the extended Toda hierarchy.*

Using quasitriviality it can be shown that the class of solutions of the extended Toda hierarchy that is given by the above theorem forms a dense subset of the class of its analytic solutions $w^\alpha(x, \mathbf{t}, \epsilon)$, $\alpha = 1, 2$ (see Theorem 3.6.15 and 3.10.31 in [9]). We call this class of solutions the generic class of solutions of the extended Toda hierarchy, and we will restrict ourselves to it henceforth.

Lemma 4.4. *Any solution (v, u) of the Euler-Lagrange equation (4.1) satisfies the equations*

$$\sum_{p \geq 1} \tilde{t}^{\alpha, p} \frac{\partial v}{\partial t^{\alpha, p-1}} + 1 = 0,$$

$$\sum_{p \geq 1} \tilde{t}^{\alpha, p} \frac{\partial u}{\partial t^{\alpha, p-1}} = 0, \tag{4.12}$$

$$\sum_{q \geq 1} \left[q \left(\tilde{t}^{1,q} \frac{\partial v}{\partial t^{1,q}} + \tilde{t}^{2,q-1} \frac{\partial v}{\partial t^{2,q-1}} \right) + 2\tilde{t}^{1,q} \frac{\partial v}{\partial t^{2,q-1}} \right] + v = 0,$$

$$\sum_{q \geq 1} \left[q \left(\tilde{t}^{1,q} \frac{\partial u}{\partial t^{1,q}} + \tilde{t}^{2,q-1} \frac{\partial u}{\partial t^{2,q-1}} \right) + 2\tilde{t}^{1,q} \frac{\partial u}{\partial t^{2,q-1}} \right] + 2 = 0. \tag{4.13}$$

Proof. Equation (4.12) is the result of the application of the operators $\frac{\partial}{\partial t^{2,0}}$ and $\frac{1}{\epsilon}(\Lambda - 1)$ on the Euler-Lagrange equation (4.1). To prove Equation (4.13), we need to use the following bihamiltonian recursion relation of the extended Toda hierarchy [1]:

$$U_2^{\alpha\gamma} \frac{\delta H_{\beta, q-1}}{\delta w^\gamma} = (q + \mu_\beta + \frac{1}{2}) U_1^{\alpha\gamma} \frac{\delta H_{\beta, q}}{\delta w^\gamma} + R_\beta^\gamma U_1^{\alpha\xi} \frac{\delta H_{\gamma, q-1}}{\delta w^\xi}. \tag{4.14}$$

Here the first Hamiltonian structure $U_1^{\alpha\beta}$ is defined in (1.13) and the second one is given by

$$U_2^{11} = \frac{1}{\epsilon} \left(e^{\epsilon \partial_x} e^{u(x)} - e^{u(x)} e^{-\epsilon \partial_x} \right), \quad U_2^{12} = \frac{1}{\epsilon} v(x) \left(e^{\epsilon \partial_x} - 1 \right),$$

$$U_2^{21} = \frac{1}{\epsilon} \left(1 - e^{-\epsilon \partial_x} \right) v(x), \quad U_2^{22} = \frac{1}{\epsilon} \left(e^{\epsilon \partial_x} - e^{-\epsilon \partial_x} \right). \tag{4.15}$$

The matrices R and μ are defined by (2.4) (see also (3.4)). Then Eq. (4.13) is obtained by applying the operator $U_2^{\alpha\gamma}$ to both sides of the Euler-Lagrange equation (4.1) and by using the bihamiltonian recursion relation (4.14). The lemma is proved. □

Let v, u be any solution of the Euler-Lagrange equation (4.1) specified by a choice of the series $c^{\alpha, p}(\epsilon)$ and of the leading term $\bar{v}, \bar{u}$ in (4.4). Due to Theorem 1.1 and Theorem 4.3 this solution can be obtained from a solution v_0, u_0 of the dispersionless Toda hierarchy. Denote by $\tau^{[0]}$ and τ the corresponding tau functions with the relation

$$\log \tau = \epsilon^{-2} \log \tau^{[0]} + \sum_{g \geq 1} \epsilon^{2g-2} F_g(w_0, \dots, w_0^{(3g-2)}). \tag{4.16}$$

Note that the genus zero tau function $\tau^{[0]}$ is defined up to multiplication by a function of the form $e^{\sum c^{[0]}_{\alpha,p} t^{\alpha,p}}$ with constants $c^{[0]}_{\alpha,p}$. We now fix this ambiguity by taking

$$\log \tau^{[0]} = \frac{1}{2} \sum \Omega^{[0]}_{\alpha,p;\beta,q}(v_0, u_0) \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q}, \tag{4.17}$$

where

$$\Omega^{[0]}_{\alpha,p;\beta,q} = \Omega_{\alpha,p;\beta,q}\big|_{\epsilon=0}. \tag{4.18}$$

The validity of this definition for the tau function of the solution v_0, u_0 of the dispersionless extended Toda hierarchy is based on the fact that v_0, u_0 satisfy the genus zero Euler-Lagrange equation

$$\sum \tilde{t}^{\alpha,p} \frac{\partial h^{[0]}_{\alpha,p-1}(v_0, u_0)}{\partial v_0} = 0, \quad \sum \tilde{t}^{\alpha,p} \frac{\partial h^{[0]}_{\alpha,p-1}(v_0, u_0)}{\partial u_0} = 0 \tag{4.19}$$

with $h^{[0]}_{\alpha,p-1} = h_{\alpha,p-1}\big|_{\epsilon=0}$. From this equation it readily follows that

$$\Omega^{[0]}_{\alpha,p;\beta,q} = \frac{\partial^2 \log \tau^{[0]}}{\partial t^{\alpha,p} \partial t^{\beta,q}}, \tag{4.20}$$

and that the genus zero tau function satisfies the string equation

$$\sum_{p \geq 1} \tilde{t}^{\alpha,p} \frac{\partial \log \tau^{[0]}}{\partial t^{\alpha,p-1}} + \tilde{t}^{1,0} \tilde{t}^{2,0} = 0. \tag{4.21}$$

The proof of the above statement can be found in [5]. It was proved in [8] that such a tau function also satisfies the genus zero Virasoro constraints given by the Virasoro operators (1.29). The action of these operators on tau functions of the form (4.16) can be expressed as

$$L_m(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon \frac{\partial}{\partial \mathbf{t}}) \tau = \left(\sum_{g \geq 0} \epsilon^{2g-2} Z_g \right) \tau, \quad m \geq -1. \tag{4.22}$$

The genus zero Virasoro constraints are given by $Z_0 = 0$. We are to prove below that the tau function of a generic solution to the extended Toda hierarchy satisfies the full genera Virasoro constraints $Z_g = 0$, $g \geq 0$. Let us begin with the L_{-1} and L_0 constraints.

Lemma 4.5. *The tau function (4.16) satisfies the constraints*

$$L_{-1}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon \frac{\partial}{\partial \mathbf{t}}) \tau = 0. \quad L_0(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon \frac{\partial}{\partial \mathbf{t}}) \tau = c_0(\epsilon) = \sum_{g \geq 1} \epsilon^{2g-2} c^{[g]}_0 \tag{4.23}$$

with a certain constant $c_0(\epsilon)$.

Proof. Let us apply the operator $\frac{\partial^2}{\partial t^{\sigma,k}\partial t^{\rho,l}}$ to the l.h.s. of the first equation of (4.23). Using the definition (1.20) for the tau function and Eq. (4.12) we get

$$
\epsilon^2 \frac{\partial^2}{\partial t^{\sigma,k}\partial t^{\rho,l}} \left(\sum_{p\geq 1} \tilde{t}^{\alpha,p} \frac{\partial \log \tau}{\partial t^{\alpha,p-1}} + \frac{1}{\epsilon^2} \tilde{t}^{1,0}\tilde{t}^{2,0} \right)
$$

$$
= \Omega_{\sigma,k-1;\rho,l} + \Omega_{\sigma,k;\rho,l-1} + \sum_{p\geq 1} \tilde{t}^{\alpha,p} \frac{\partial \Omega_{\sigma,k;\rho,l}}{\partial t^{\alpha,p-1}} + \left(\delta_{\sigma,1}\delta_{\rho,2} + \delta_{\sigma,2}\delta_{\rho,1} \right) \delta_{k,0}\delta_{l,0}
$$

$$
= \Omega_{\sigma,k-1;\rho,l} + \Omega_{\sigma,k;\rho,l-1} + \sum_{p\geq 1} \tilde{t}^{\alpha,p} \frac{\partial \Omega_{\sigma,k;\rho,l}}{\partial w^{\xi,m}} \partial_x^m \left(\frac{\partial w^\xi}{\partial t^{\alpha,p-1}} \right)
$$

$$
+ \left(\delta_{\sigma,1}\delta_{\rho,2} + \delta_{\sigma,2}\delta_{\rho,1} \right) \delta_{k,0}\delta_{l,0}
$$

$$
= \Omega_{\sigma,k-1;\rho,l} + \Omega_{\sigma,k;\rho,l-1} - \frac{\partial \Omega_{\sigma,k;\rho,l}}{\partial v} + \left(\delta_{\sigma,1}\delta_{\rho,2} + \delta_{\sigma,2}\delta_{\rho,1} \right) \delta_{k,0}\delta_{l,0}
$$

$$
= 0. \tag{4.24}
$$

Here the last equality is due to (2.2). On the other hand, the Euler-Lagrange equation (4.1) implies that the l.h.s. of the first formula of (4.23) does not depend on $\tilde{t}^{1,0}$ and $\tilde{t}^{2,0}$, so there exist constants $c_{-1}^{[g]}$, $c_{\alpha,p}^{[g]}$, $\alpha = 1, 2$, $p \geq 0$, $g \geq 1$ such that

$$
\sum_{p\geq 1} \tilde{t}^{\alpha,p} \frac{\partial \log \tau}{\partial t^{\alpha,p-1}} + \frac{1}{\epsilon^2} \tilde{t}^{1,0}\tilde{t}^{2,0} = \sum_{p\geq 1, g\geq 1} \epsilon^{2g-2} c_{\alpha,p-1}^{[g]} \tilde{t}^{\alpha,p} + \sum_{g\geq 1} \epsilon^{2g-2} c_{-1}^{[g]}. \tag{4.25}
$$

Here the vanishing of the ϵ^{-2} term in the r.h.s. of the above identity is due to (4.21). Thus if we modify the tau function by

$$
\tau \mapsto \tilde{\tau} = \tau\, e^{-\sum_{p\geq 0, g\geq 1} \epsilon^{2g-2} c_{\alpha,p}^{[g]} \tilde{t}^{\alpha,p}}, \tag{4.26}
$$

then we obtain

$$
L_{-1}(\epsilon^{-1}\mathbf{t}, \epsilon \frac{\partial}{\partial \mathbf{t}})\, \tilde{\tau} = c_{-1}(\epsilon)\, \tilde{\tau} = \left(\sum_{g\geq 1} \epsilon^{2g-2} c_{-1}^{[g]} \right) \tilde{\tau}. \tag{4.27}
$$

We will prove the vanishing of the constants $c_{\alpha,p}^{[g]}$, $c_{-1}(\epsilon)$ in a moment.

By using the formula (2.3) and a similar argument as that given in the proof of (4.24), we can prove the validity of the following identity:

$$
\frac{\partial^2}{\partial t^{\sigma,k}\partial t^{\rho,l}} \left(\frac{L_0 \tilde{\tau}}{\tilde{\tau}} \right) = 0. \tag{4.28}
$$

Here

$$
L_0 = L_0(\epsilon^{-1}\mathbf{t}, \epsilon \frac{\partial}{\partial \mathbf{t}}).
$$

So there exist constants $c_0(\epsilon)$ and $b_{\alpha,p}(\epsilon)$ such that

$$
\frac{L_0 \tilde{\tau}}{\tilde{\tau}} = \sum_{\alpha,p} b_{\alpha,p}(\epsilon) \tilde{t}^{\alpha,p} + c_0(\epsilon). \tag{4.29}
$$

By using the commutation relation $[L_{-1}, L_0] = -L_{-1}$ we obtain

$$L_{-1}\left[\left(\sum b_{\alpha,p}(\epsilon)\tilde{t}^{\alpha,p} + c_0(\epsilon)\right)\tilde{\tau}\right] - L_0\left(c_{-1}(\epsilon)\tilde{\tau}\right) = -c_{-1}(\epsilon)\tilde{\tau}. \qquad (4.30)$$

The l.h.s. of the above equality reads

$$\left(\sum_{p\geq 1} b_{\alpha,p-1}\tilde{t}^{\alpha,p}\right)\tilde{\tau} + \left(\sum b_{\alpha,p}\tilde{t}^{\alpha,p} + c_0(\epsilon)\right)L_{-1}\tilde{\tau} - c_{-1}(\epsilon)L_0\tilde{\tau}$$

$$= \left(\sum_{p\geq 1} b_{\alpha,p-1}\tilde{t}^{\alpha,p}\right)\tilde{\tau}. \qquad (4.31)$$

So from (4.30) it follows that

$$\left(\sum_{p\geq 1} b_{\alpha,p-1}\tilde{t}^{\alpha,p}\right)\tilde{\tau} = -c_{-1}(\epsilon)\tilde{\tau}, \qquad (4.32)$$

from which we obtain $c_{-1}(\epsilon) = b_{\alpha,p}(\epsilon) = 0$.

Now we proceed to proving the vanishing of the constants $c_{\alpha,p}^{[g]}$. From the above argument we already have the identity

$$L_{-1}(\epsilon^{-1}\tilde{t}, \epsilon\frac{\partial}{\partial\mathbf{t}})\left(\tau\, e^{-\sum_{g\geq 1}\epsilon^{2g-2}c_{\alpha,p}^{[g]}\tilde{t}^{\alpha,p}}\right) = 0.$$

At the genus one level we have

$$\sum_{p\geq 1}\tilde{t}^{\alpha,p}\frac{\partial F_1(w, w_x)}{\partial t^{\alpha,p-1}} - \sum_{p\geq 1}\tilde{t}^{\alpha,p}c_{\alpha,p-1}^{[1]} = 0. \qquad (4.33)$$

Starting from this formula till the end of the proof of the lemma we will redenote for the sake of brevity the arguments $w_0 = (v_0, u_0)$ and $w_{0x} = (v_{0x}, u_{0x})$ of the function $F_1(w_0, w_{0x})$ by $w = (v, u)$ and $w_x = (v_x, u_x)$.

Since $\tau^{[0]}$ satisfies the genus zero Virasoro constraints, we can use the vanishing of the genus zero Virasoro symmetries to obtain, as we did in [8, 9], the following formula:

$$\sum_{p\geq 1}\tilde{t}^{\alpha,p}\frac{\partial F_1(w, w_x)}{\partial t^{\alpha,p-1}} = -\frac{\partial F_1}{\partial v}. \qquad (4.34)$$

Thus the identity (4.33) can be rewritten as

$$-\sum_{p\geq 1}\tilde{t}^{\alpha,p}c_{\alpha,p-1}^{[1]} = \frac{\partial F_1}{\partial v}. \qquad (4.35)$$

By applying the operator $\sum_{p\geq 0} z^p\frac{\partial}{\partial t^{\alpha,p}}$ to the above identity we get

$$-\sum_{p\geq 1} c^{[1]}_{\alpha,p-1} z^p$$

$$= \sum_{p\geq 0} \left[\frac{\partial^2 F_1}{\partial v \partial w^\gamma} \partial_x \left(\frac{\partial \theta_{\alpha,p+1}}{\partial w_\gamma} \right) + \frac{\partial^2 F_1}{\partial v \partial w^\gamma_x} \partial_x^2 \left(\frac{\partial \theta_{\alpha,p+1}}{\partial w_\gamma} \right) \right] z^p$$

$$= \left[\frac{\partial^2 F_1}{\partial v \partial w^\gamma} c^{\gamma\sigma}_\rho w^\rho_x + \frac{\partial^2 F_1}{\partial v \partial w^\gamma_x} \partial_x (c^{\gamma\sigma}_\rho w^\rho_x) + z \frac{\partial^2 F_1}{\partial v \partial w^\gamma_x} c^{\gamma\sigma_1}_{\rho_1} w^{\rho_1}_x c^\sigma_{\sigma_1\rho_2} w^{\rho_2}_x \right] \frac{\partial \theta_\alpha(z)}{\partial w^\sigma}.$$
$$(4.36)$$

Here the functions $\theta_{\alpha,p} = \theta_{\alpha,p}(w)$, $c_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}(w)$ are given by

$$\theta_{\alpha,p} = h_{\alpha,p-1}|_{\epsilon=0}, \qquad \theta_\alpha(z) = \sum_{p\geq 0} \theta_{\alpha,p} z^p,$$

$$c_{\alpha\beta\gamma} = \frac{\partial^3}{\partial w^\alpha \partial w^\beta \partial w^\gamma} \left(\frac{1}{2} v^2 u + e^u \right),$$
$$(4.37)$$

and the raising of indices in $c_{\alpha\beta\gamma}$ is done by the metric (1.27), i.e. $\eta^{11} = \eta^{22} = 0$, $\eta^{12} = \eta^{21} = 1$. In the above computation we used the horizontality of the differentials of the functions $\theta_\alpha(w; z)$ w.r.t. the deformed flat connection on M_{Toda}, i.e. the equations

$$\frac{\partial^2 \theta_\alpha(z)}{\partial w^\beta \partial w^\gamma} = z c^\xi_{\beta\gamma} \frac{\partial \theta_\alpha(z)}{\partial w^\xi}.$$
$$(4.38)$$

So from (4.36) we get

$$\frac{\partial^2 F_1}{\partial v \partial w^\gamma} c^{\gamma\sigma}_\rho w^\rho_x + \frac{\partial^2 F_1}{\partial v \partial w^\gamma_x} \partial_x (c^{\gamma\sigma}_\rho w^\rho_x) = 0,$$

$$\left(\frac{\partial^2 F_1}{\partial v \partial w^\gamma_x} c^{\gamma\sigma_1}_{\rho_1} w^{\rho_1}_x c^\sigma_{\sigma_1\rho_2} w^{\rho_2}_x \right) \eta_{\sigma\alpha} = -c^{[1]}_{\alpha,0},$$
$$(4.39)$$

and together with (4.36) these formulae in turn yield

$$\sum_{p\geq 0} c^{[1]}_{\alpha,p} z^p = c^{[1]}_{\gamma,0} \partial^\gamma \theta_\alpha(z).$$
$$(4.40)$$

By differentiating both sides of the above equation w.r.t. x we get

$$0 = c^{[1]}_{\gamma,0} c^\gamma_{\xi\sigma} w^\sigma_x \partial^\xi \theta_\alpha(z)$$
$$(4.41)$$

which implies

$$c^{[1]}_{\gamma,0} = 0.$$

So from (4.40) we obtain

$$c^{[1]}_{\gamma,p} = 0, \quad p \geq 1.$$

In a completely similar way we can prove that

$$c^{[g]}_{\gamma,p} = 0, \quad p \geq 0, \; g \geq 2.$$

The lemma is proved. $\square$

The following lemma represents the bihamiltonian recursion relation for the extended Toda hierarchy in terms of its tau functions:

Lemma 4.6. *The following recursion relations hold true for any $q \geq 1$ for the tau functions of generic solutions to the extended Toda hierarchy:*

$$q \, (\Lambda - 1) \, \frac{\partial \log \tau}{\partial t^{1,q}} = \mathcal{R} \frac{\partial \log \tau}{\partial t^{1,q-1}} - 2 \, (\Lambda - 1) \, \frac{\partial \log \tau}{\partial t^{2,q-1}}, \tag{4.42}$$

$$(q + 1) \, (\Lambda - 1) \, \frac{\partial \log \tau}{\partial t^{2,q}} = \mathcal{R} \frac{\partial \log \tau}{\partial t^{2,q-1}}, \tag{4.43}$$

where the operator $\mathcal{R}$ is defined by

$$\mathcal{R} = v(x)(\Lambda - 1) + \epsilon(\Lambda + 1) \frac{\partial}{\partial t^{2,0}}. \tag{4.44}$$

Proof. Denote

$$W_{\beta,q} := \mathcal{R} \frac{\partial \log \tau}{\partial t^{\beta,q-1}} - (q + \mu_\beta + \frac{1}{2})(\Lambda - 1) \frac{\partial \log \tau}{\partial t^{\beta,q}} - (\Lambda - 1) R_\beta^\gamma \frac{\partial \log \tau}{\partial t^{\gamma,q-1}}. \tag{4.45}$$

We are to prove that $W_{\beta,q} = 0$ for $\beta = 1, 2, q \geq 1$. From Lemma 2.3 and from the bihamiltonian recursion relation (4.14) with $\alpha = 2$ we obtain by a direct calculation that

$$(\Lambda - 1) W_{\beta,q} = 0. \tag{4.46}$$

We note that $\epsilon \, W_{\beta,q}$ can be expressed as homogeneous differential polynomials in $w^{i,m}$, $e^{\pm u}$, $i = 1, 2, m \geq 0$ of degree $q + \frac{1}{2} + \mu_\beta$. Recall that the degree of such differential polynomials is defined in (1.16). So the lemma follows from the above equation (4.46). The lemma is proved. ☐

Proof of the Main Theorem. Let us first prove that the following recursion relation holds true:

$$\mathcal{R} \frac{L_m \tau}{\tau} = (\Lambda - 1) \frac{L_{m+1} \tau}{\tau}. \tag{4.47}$$

Here and below

$$L_m = L_m(\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \frac{\partial}{\partial \mathbf{t}}).$$

From the definition of the operator $\mathcal{R}$ we have

$$\mathcal{R} \sum_{k=1}^{m-1} k!(m - k)! \left(\frac{1}{\tau} \frac{\partial^2 \tau}{\partial t^{2,k-1} \partial t^{2,m-1-k}} \right)$$

$$= \sum_{k=1}^{m-1} k!(m - k)! \left[\frac{\partial}{\partial t^{2,k-1}} \mathcal{R} \frac{\partial \log \tau}{\partial t^{2,m-1-k}} + \left(\mathcal{R} \frac{\partial \log \tau}{\partial t^{2,k-1}} \right) \Lambda \frac{\partial \log \tau}{\partial t^{2,m-1-k}} \right.$$

$$\left. + \frac{\partial \log \tau}{\partial t^{2,k-1}} \mathcal{R} \frac{\partial \log \tau}{\partial t^{2,m-1-k}} \right], \quad m \geq 1.$$

So by using the recursion relations (4.42), (4.43) we can deduce the relation (4.47) for $m \geq 1$ as follows:

$$\mathcal{R}\frac{L_m \tau}{\tau} = \epsilon^2(\Lambda - 1)\sum_{k=1}^{m} k!(m+1-k)! \frac{1}{\tau} \frac{\partial^2 \tau}{\partial t^{2,k-1}\partial t^{2,m-k}}$$

$$-\epsilon^2 m!(\Lambda - 1)\frac{\partial^2 \log \tau}{\partial t^{2,m-1}\partial t^{2,0}} - m!\left[\epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{2,0}}\right]\epsilon(\Lambda + 1)\frac{\partial \log \tau}{\partial t^{2,m-1}}$$

$$+\sum_{k\geq 1}\frac{(m+k)!}{(k-1)!}(\Lambda - 1)\left[(m+k+1)\left(\tilde{\imath}^{1,k}\frac{\partial \log \tau}{\partial t^{1,m+k+1}} + \tilde{\imath}^{2,k-1}\frac{\partial \log \tau}{\partial t^{2,m+k}}\right)\right.$$

$$\left.+2\tilde{\imath}^{1,k}\frac{\partial \log \tau}{\partial t^{2,m+k}}\right] + \epsilon(m+1)!(\Lambda + 1)\frac{\partial \log \tau}{\partial t^{2,m}}$$

$$+2\sum_{k\geq 0}(m+k+1)\alpha_m(k)\tilde{\imath}^{1,k}(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{2,m+k}}$$

$$+2\epsilon\alpha_m(0)v\Lambda\frac{\partial \log \tau}{\partial t^{2,m-1}} + 2\epsilon^2\alpha_m(0)\Lambda\frac{\partial^2 \log \tau}{\partial t^{2,m-1}\partial t^{2,0}}$$

$$= (\Lambda - 1)\frac{L_{m+1}\tau}{\tau} - \epsilon^2 m!(\Lambda - 1)\frac{\partial^2 \log \tau}{\partial t^{2,m-1}\partial t^{2,0}}$$

$$-m!\left[\epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{2,0}}\right]\epsilon(\Lambda + 1)\frac{\partial \log \tau}{\partial t^{2,m-1}}$$

$$+\epsilon(m+1)!(\Lambda + 1)\frac{\partial \log \tau}{\partial t^{2,m}} - 2\epsilon(m+1)!\Lambda\frac{\partial \log \tau}{\partial t^{2,m}}$$

$$+2\epsilon\alpha_m(0)v\Lambda\frac{\partial \log \tau}{\partial t^{2,m-1}} + 2\epsilon\alpha_m(0)\Lambda\frac{\partial^2 \log \tau}{\partial t^{2,m-1}\partial t^{2,0}}$$

$$= (\Lambda - 1)\frac{L_{m+1}\tau}{\tau} + \epsilon m!\,\mathcal{R}\frac{\partial \log \tau}{\partial t^{2,m-1}} - \epsilon(m+1)!(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{2,m}}$$

$$= (\Lambda - 1)\frac{L_{m+1}\tau}{\tau}. \tag{4.48}$$

It can be easily checked that the recursion relation (4.47) is also true for $m = -1, 0$.

From (4.23) and the recursion relation (4.47) we know that

$$(\Lambda - 1)\left(\frac{L_1 \tau}{\tau}\right) = 0. \tag{4.49}$$

Since the function $\tau^{[0]}$ satisfies the genus zero Virasoro constraints, it follows from (4.16) that

$$\frac{L_1 \tau}{\tau} = \sum_{g\geq 1}\epsilon^{2g-2}W_g(w_0, w_0', \ldots, w_0^{(m_g)}). \tag{4.50}$$

Thus from (4.49) we arrive at the formula

$$\frac{L_1 \tau}{\tau} = c_1(\epsilon) = \sum_{g\geq 1}\epsilon^{2g-2}c_1^{[g]} \tag{4.51}$$

for some constants $c_1^{[g]}$.

On the other hand, by using the the commutation relation (1.28) and Eqs. (4.23) we obtain

$$L_{-1}(L_m\tau) = -(m+1)L_{m-1}\tau, \quad L_0(L_m\tau) = (c_0(\epsilon) - m)L_m\tau. \quad (4.52)$$

These formulae can be rewritten as

$$\hat{L}_{-1}\left(\frac{L_m\tau}{\tau}\right) = -(m+1)\left(\frac{L_{m-1}\tau}{\tau}\right), \quad \hat{L}_0\left(\frac{L_m\tau}{\tau}\right) = -m\left(\frac{L_m\tau}{\tau}\right). \quad (4.53)$$

Here $\hat{L}_{-1} = L_{-1} - \frac{1}{\epsilon^2}\tilde{t}^{1,0}\,\tilde{t}^{2,0}$ and $\hat{L}_0 = L_0 - \frac{1}{\epsilon^2}\left(\tilde{t}^{1,0}\right)^2$. By putting $m = 1$ into the above two relations we get

$$\hat{L}_{-1}\left(\frac{L_1\tau}{\tau}\right) = -2\,c_0(\epsilon), \quad \hat{L}_0\left(\frac{L_1\tau}{\tau}\right) = -\left(\frac{L_1\tau}{\tau}\right). \quad (4.54)$$

So from (4.51) we have $c_0(\epsilon) = c_1(\epsilon) = 0$, and we proved the vanishing of $L_0\tau$ and $L_1\tau$.

By using the recursion relation (4.47) with $m = 1$ we obtain

$$(\Lambda - 1)\left(\frac{L_2\tau}{\tau}\right) = 0. \quad (4.55)$$

Due to the same reason as the one we used to derive (4.51) we have

$$\frac{L_2\tau}{\tau} = c_2(\epsilon) = \sum_{g\geq 1}\epsilon^{2g-2}c_2^{[g]} \quad (4.56)$$

for some constants $c_2^{[g]}$. So by using the second formula in (4.53) we get $L_2\tau = 0$. Now, the Virasoro commutation relation (1.28) implies the validity of all the Virasoro constraints

$$L_m\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\frac{\partial}{\partial\mathbf{t}}\right)\tau = 0, \ m \geq -1.$$

It remains to prove that linear action of the Virasoro operators (1.29) defines infinitesimal symmetries of the extended Toda hierarchy. To this end we observe that

$$L_m\left(\epsilon^{-1}(\mathbf{t} - \mathbf{c}(\epsilon)), \epsilon\frac{\partial}{\partial\mathbf{t}}\right) = L_m\left(\epsilon^{-1}\mathbf{t}, \epsilon\frac{\partial}{\partial\mathbf{t}}\right) - a^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p}} - b_{\alpha,p}t^{\alpha,p} - c,$$

where $a^{\alpha,p}$, $b_{\alpha,p}$ and c are some series in ϵ that may also depend on m. Note that the a series contains only nonnegative powers of ϵ. From the already proven Virasoro constraints it follows that, for any generic solution to the extended Toda hierarchy the action of the Virasoro operators on the tau function can be recast as

$$L_m\left(\epsilon^{-1}\mathbf{t}, \epsilon\frac{\partial}{\partial\mathbf{t}}\right)\tau = \left(a^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p}} + b_{\alpha,p}t^{\alpha,p} + c\right)\tau.$$

So, for a small parameter δ

$$\tau + \delta\,L_m\left(\epsilon^{-1}\mathbf{t}, \epsilon\frac{\partial}{\partial\mathbf{t}}\right)\tau = e^{\delta(b_{\alpha,p}t^{\alpha,p}+c)}e^{\delta\,a^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p}}}\tau + O(\delta^2). \quad (4.57)$$

The operator

$$e^{\delta a^{\alpha, p} \frac{\partial}{\partial t^{\alpha, p}}}$$

is nothing but the shift along trajectories of the hierarchy. Note that such a shift leaves invariant the class of generic solutions. Multiplication by the exponential of a linear function in the times for obvious reasons maps a tau function to another one for the same solution to the hierarchy. This proves that (4.57) is again a tau function of some solution of the extended Toda hierarchy. The theorem is proved. □

5. The Topological Solution of the Extended Toda Hierarchy

Let us briefly recall the definition of Gromov - Witten invariants and their descendents and the construction of Witten's generating function (physicists call it the *free energy* of the two-dimensional CP^1 topological sigma model). Denote $\phi_1 = 1 \in H^0(CP^1)$, $\phi_2 = \omega \in H^2(CP^1)$ the basis in the cohomology space $H^*(CP^1)$. The 2-form ω is assumed to be normalized by the condition

$$\int_{CP^1} \omega = 1.$$

The free energy of the CP^1 topological sigma-model is a function of infinite number of *coupling parameters*

$$\mathbf{t} = (t^{1,0}, t^{2,0}, t^{1,1}, t^{2,1}, \dots)$$

and of ϵ defined by the following genus expansion form:

$$\mathcal{F}(\mathbf{t}; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}). \tag{5.1}$$

The parameter ϵ is called here the string coupling constant, and the function $\mathcal{F}_g = \mathcal{F}_g(\mathbf{t})$ is called the genus g free energy which is given by

$$\mathcal{F}_g = \sum \frac{1}{m!} t^{\alpha_1, p_1} \dots t^{\alpha_m, p_m} \langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g, \tag{5.2}$$

where $\tau_p(\phi_\alpha)$ are the gravitational descendents of the primary fields ϕ_α, $t^{\alpha, p}$ is the corresponding coupling constants, and the rational numbers $\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g$ are given by the following intersection numbers on the moduli spaces of CP^1-valued stable curves of genus g:

$$\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g$$
$$= \sum_\beta q^\beta \int_{[\bar{M}_{g,m}(CP^1, \beta)]^{\text{virt}}} \text{ev}_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \dots \wedge \text{ev}_m^* \phi_{\alpha_m} \wedge \psi_m^{p_m}. \tag{5.3}$$

Here $\bar{M}_{g,m}(CP^1, \beta)$ is the moduli space of stable curves of genus g with m markings of the given degree $\beta \in H_2(CP^1; \mathbb{Z})$, ev_i is the evaluation map

$$\text{ev}_i : \bar{M}_{g,m}(CP^1, \beta) \to CP^1$$

corresponding to the ith marking, ψ_i is the first Chern class of the tautological line bundle over the moduli space corresponding to the ith marking. According to the divisor axiom

[22] the dependence of the Gromov - Witten potential on the indeterminate q appears only through the combination $q\, e^{t^{2,0}}$. We will therefore omit the dependence on q in the formulae.

Let us clarify the relationship between our theory of Virasoro symmetries of the extended Toda hierarchy and the Virasoro conjecture of T.Eguchi, K.Hori, and C.-S.Xiong [12, 13] extended by S.Katz. Denote

$$Z_{CP^1}(\mathbf{t}; \epsilon) := e^{\mathcal{F}(\mathbf{t};\epsilon)}$$

the partition function of the CP^1 topological sigma-model. Here $\mathcal{F}(\mathbf{t}; \epsilon)$ is the generating function of the CP^1 Gromov - Witten invariants and their descendents defined above. According to the results of A.Givental [19, 20] this partition function satisfies the following infinite sequence of linear Virasoro constraints:

$$L_{-1}Z_{CP^1} = \frac{\partial}{\partial t^{1,0}}Z_{CP^1},$$

$$L_m Z_{CP^1} = (m+1)!\left[\frac{\partial}{\partial t^{1,m+1}} + 2\kappa_m \frac{\partial}{\partial t^{2,m}}\right]Z_{CP^1}, \quad m \geq 0, \qquad (5.4)$$

where

$$\kappa_m = \sum_{j=1}^{m+1}\frac{1}{j}.$$

Here L_m are just the Virasoro operators defined in (1.29). For the particular case of CP^1 (5.4) coincide with the Virasoro constraints conjectured in [12]. However, in their papers Eguchi, Hori and Xiong formulated a somewhat bolder conjecture that says that all $g \geq 1$ Gromov - Witten invariants and their descendents of a smooth projective variety can be *uniquely determined* by solving recursively the linear system of Virasoro constraints. Although this conjecture seems to be too nice to be true in general (Calabi - Yau manifolds give counterexamples to uniqueness, see [2]), in certain cases it can be justified.

Let us give our version of the Eguchi - Hori - Xiong Virasoro constraints programme adapted to computing Gromov - Witten invariants of CP^1.

Step 1. Computation of the genus zero Gromov - Witten potential $\mathcal{F}_0(\mathbf{t})$. This can be done in terms of the Frobenius manifold M_{Toda} as in [3, 4]. For the reader's convenience we recall the algorithm of computation of $\mathcal{F}_0(\mathbf{t})$ in the Appendix below. Introduce functions

$$v_0 = v_0(\mathbf{t}) := \frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0}\partial t^{2,0}},$$

$$u_0 = u_0(\mathbf{t}) := \frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0}\partial t^{1,0}}.$$

We will denote u_0', v_0', u_0'', v_0'' etc. the derivatives of these functions along $t^{1,0}$.

Step 2. Eguchi - Hori - Xiong $(3g - 2)$-ansatz for the higher genus corrections. Look for the $g \geq 1$ terms in the genus expansion (5.1) in the form

$$\mathcal{F}_g(\mathbf{t}) = F_g(v_0(\mathbf{t}), u_0(\mathbf{t}), v_0'(\mathbf{t}), u_0'(\mathbf{t}), \ldots, v_0^{(3g-2)}(\mathbf{t}), u_0^{(3g-2)}(\mathbf{t})), \quad g \geq 1. \quad (5.5)$$

The ansatz (5.5) was proved by E. Getzler in [18]. In the setup of our theory [9] of integrable systems the $(3g-2)$-ansatz is a consequence of a deep result about quasitriviality of tau-symmetric deformations of Poisson pencils (see Theorem 3.9.5 in [9]).

Step 3. Virasoro Conjecture.

Part 1. The series (5.1), where $\mathcal{F}_0(\mathbf{t})$ is the genus zero Gromov - Witten potential and the terms of positive genera have the form (5.5), satisfies the Virasoro constraints (5.4). (Clearly the $(3g-2)$-ansatz is of no importance so far.)
Part 2. The degree $2g-2$ homogeneous functions F_g on the $(3g-2)$ jet space of M_{Toda} for all $g \geq 1$ are uniquely determined from the Virasoro constraints (5.4) by solving recursively systems of linear equations.

Part 1 of the Virasoro Conjecture was proved by A. Givental [19, 20]. Part 2 was proved in the much more general framework of an arbitrary semisimple Frobenius manifold in Theorem 3.10.20 of [9]. Combining these results we arrive at

Theorem 5.1. *1. The partition function $Z_{CP^1}(\mathbf{t};\epsilon)$ of the CP^1 topological sigma-model is uniquely determined by the Virasoro Conjecture equations.*
2. It coincides with the tau function τ_{CP^1} of a particular solution to the extended Toda hierarchy (1.2)

$$Z_{CP^1}(\mathbf{t};\epsilon) = \tau_{CP^1}(\mathbf{t};\epsilon) \tag{5.6}$$

specified by the following choice of the shift parameters $c^{\alpha,p}(\epsilon)$ and the initial point $\bar{v}, \bar{u}$:

$$c^{\alpha,p}(\epsilon) = \delta_1^\alpha \delta_1^p, \quad \bar{v} = \bar{u} = 0. \tag{5.7}$$

The choice (5.7) selects the solution satisfying the string equation

$$\sum_{p\geq 1} t^{\alpha,p} \frac{\partial \mathcal{F}}{\partial t^{\alpha,p-1}} + \frac{1}{\epsilon^2} t^{1,0} t^{2,0} = \frac{\partial \mathcal{F}}{\partial t^{1,0}}. \tag{5.8}$$

Proof. As it was shown in Sect. 3.10.7 of [9], from validity of the Virasoro constraints for the sum $\Delta\mathcal{F}$ of all $g \geq 1$ corrections to the Gromov - Witten potential represented via the $(3g-2)$-ansatz,

$$\Delta\mathcal{F} := \sum_{g\geq 1} \epsilon^{2g} F_g(v, u; v_x, u_x, \ldots, v^{(3g-2)}, u^{(3g-2)}),$$

the *loop equation* below (cf. Example 3.10.27 in [9]) follows:

$$\sum_{r\geq 0} \left(\frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \partial_x^r \frac{v-\lambda}{D} - 2\frac{\partial \Delta\mathcal{F}}{\partial u^{(r)}} \partial_x^r \frac{1}{D} \right)$$

$$+ \sum_{r\geq 1} \sum_{k=1}^r \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{D}} \left(\frac{\partial \Delta\mathcal{F}}{\partial v^{(r)}} \partial_x^{r-k+1} \frac{v-\lambda}{\sqrt{D}} - 2\frac{\partial \Delta\mathcal{F}}{\partial u^{(r)}} \partial_x^{r-k+1} \frac{1}{\sqrt{D}} \right)$$

$$= D^{-3} e^u \left(4 e^u + (v-\lambda)^2 \right)$$

$$+ \sum_{k,l} \frac{\epsilon^2}{4} \left[-\left(\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial v^{(l)}} \right) \partial_x^{k+1} \frac{v-\lambda}{\sqrt{D}} \partial_x^{l+1} \frac{v-\lambda}{\sqrt{D}} \right.$$

$$+ 4 \left(\frac{\partial^2 \Delta\mathcal{F}}{\partial v^{(k)} \partial u^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial u^{(l)}} \right) \partial_x^{k+1} \frac{v-\lambda}{\sqrt{D}} \partial_x^{l+1} \frac{1}{\sqrt{D}}$$

$$\left. - 4 \left(\frac{\partial^2 \Delta\mathcal{F}}{\partial u^{(k)} \partial u^{(l)}} + \frac{\partial \Delta\mathcal{F}}{\partial u^{(k)}} \frac{\partial \Delta\mathcal{F}}{\partial u^{(l)}} \right) \partial_x^{k+1} \frac{1}{\sqrt{D}} \partial_x^{l+1} \frac{1}{\sqrt{D}} \right]$$

$$- \frac{\epsilon^2}{2} \sum_k \left\{ \frac{\partial \Delta\mathcal{F}}{\partial v^{(k)}} \partial_x^{k+1} e^u \frac{4 e^u (v-\lambda) u' - [(v-\lambda)^2 + 4 e^u] v'}{D^3} \right.$$

$$\left. + \frac{\partial \Delta\mathcal{F}}{\partial u^{(k)}} \partial_x^{k+1} e^u \frac{4 (v-\lambda) v' - [(v-\lambda)^2 + 4 e^u] u'}{D^3} \right\}, \tag{5.9}$$

where

$$D = (v-\lambda)^2 - 4 e^u.$$

Here λ is an arbitrary complex parameter. Expanding the loop equation near $\lambda = \infty$ reproduces the Virasoro constraints for $\Delta\mathcal{F}$. The proof of existence and uniqueness of the solution to this equation is based on expanding the loop equation near zeroes

$$u_\pm = v \pm 2 e^{u/2}$$

of D (these are the *canonical coordinates* on the Frobenius manifold M_{Toda}). The uniqueness of the solution to the loop equation proves the first part of the theorem.

To prove the second part we use the following arguments. From [4, 15, 11, 14] we already know that

$$\tau^{[0]}(\mathbf{t}) := \mathcal{F}_0(\mathbf{t})$$

is the tau function of the dispersionless extended Toda hierarchy. This solution is specified by the shift parameters and the leading term (5.7).

The transformation

$$\log \tau^{[0]} \mapsto \log \tau^{[0]} + \Delta\mathcal{F} =: \epsilon^2 \log \tau \tag{5.10}$$

maps dispersionless tau functions to tau functions of the *full hierarchy* associated with the semisimple Frobenius manifold M_{Toda}. The full hierarchy is uniquely determined for the given semisimple Frobenius manifold by the following properties:

– bihamiltonian structure satisfying certain nondegeneracy conditions;
– tau symmetry that provides existence of a tau function for a generic solution;
– invariance with respect to the linear action of the Virasoro operators L_m, $m \geq -1$ onto the tau functions.

As we explained in the Introduction, the first two properties are met by the extended Toda hierarchy due to results of [1]. The last property of Virasoro invariance is established in the present paper. This implies that the full hierarchy associated with the Frobenius manifold M_{Toda} coincides with the extended Toda hierarchy. Therefore the transformation (5.10) maps the tau function $\tau^{[0]}$ of an arbitrary solution of the dispersionless hierarchy to the tau function τ of a solution of the full extended Toda hierarchy. Taking $\tau^{[0]} = \mathcal{F}_0$ one obtains $\tau = Z_{CP^1}$. The theorem is proved. □

Clearly the theorem covers Corollaries 1.2 and 1.3 formulated in the Introduction.

To illustrate the algorithm of computation of the genus expansion (5.1) for CP^1 let us write down the first two terms of the expansion. The formulae become simpler when written in the canonical coordinates

$$u_\pm = v_0 \pm 2e^{\frac{u_0}{2}}.$$

Genus 1:

$$F_1 = \frac{1}{24} \log u'_+ u'_- - \frac{1}{12} \log \frac{u_+ - u_-}{4}.$$

Genus 2:

$$24^2 F_2 = \frac{4 u''_+{}^3 (u_+ - u_-)}{5 u'_+{}^4} - \frac{4 u''_-{}^3 (u_+ - u_-)}{5 u'_-{}^4} - \frac{u''_+ u''_-}{4 u'_+ u'_-}$$

$$+ \frac{3 u''_+}{4 u'_+{}^3} \left[\frac{1}{2} u''_+ u'_- - \frac{7}{5} u'''_+ (u_+ - u_-) \right]$$

$$+ \frac{3 u''_-}{4 u'_-{}^3} \left[\frac{1}{2} u''_- u'_+ + \frac{7}{5} u'''_- (u_+ - u_-) \right]$$

$$+ \frac{1}{4 u'_+{}^2} \left[\frac{33}{10} u''_+{}^2 - \frac{9}{10} u'''_+ u'_- + \frac{1}{10} u''_+ u''_- + u^{IV}_+ (u_+ - u_-) \right]$$

$$+ \frac{1}{4 u'_-{}^2} \left[\frac{33}{10} u''_-{}^2 - \frac{9}{10} u'''_- u'_+ + \frac{1}{10} u''_+ u''_- - u^{IV}_- (u_+ - u_-) \right]$$

$$- \frac{1}{4 u'_+} \left(\frac{17}{5} u'''_+ + \frac{1}{2} u'''_- \right) - \frac{1}{4 u'_-} \left(\frac{17}{5} u'''_- + \frac{1}{2} u'''_+ \right)$$

$$- \frac{1}{10 (u_+ - u_-)^2} \left(\frac{u'_+{}^3}{u'_-} + \frac{u'_-{}^3}{u'_+} \right) - \frac{1}{(u_+ - u_-)^2} \left(u'_+{}^2 - \frac{11}{5} u'_+ u'_- + u'_-{}^2 \right)$$

$$+ \frac{u''_+ - u''_-}{u_+ - u_-} \left(\frac{u'_-}{5 u'_+} + \frac{u'_+}{5 u'_-} + 1 \right). \tag{5.11}$$

Remark 1. In [16], Getzler proved that, under the assumption of the recursion relation (4.43), validity of the Virasoro constraints for τ_{CP^1} is equivalent to (4.42). In his proof a recursion relation of the form (4.47) was used. The recursion (4.43) for τ_{CP^1} was proved in [24] on the subspace $\{t^{1,k} = 0, \ k > 1\}$ of the large phase space of all couplings. Using this result Getzler also proved (4.42) and (4.43) under the assumption of the Virasoro constraints for τ_{CP^1}. He did not consider connections between recursion relations and Virasoro constraints for other solutions to the extended Toda hierarchy. Our Corollary 1.3 shows that the recursion relations (4.42), (4.43) for τ_{CP^1} follow directly from validity of the Virasoro constraints.

Remark 2. In [25] A. Okounkov and R. Pandharipande proved that the Gromov - Witten potential of the *equivariant* GW invariants of CP^1 and their descendents is the logarithm of the tau function of the 2D Toda hierarchy of K. Ueno and K. Takasaki [27]. The tau function of [25] depends on an additional small parameter t. The non-equivariant limit corresponds to $t \to 0$. It would be interesting to derive the Lax representation of the extended Toda lattice by applying a suitable limiting procedure to that of the 2D

Toda hierarchy of [27]. An interesting construction, due to Getzler [17] of a nontrivial reduction of the 2D Toda hierarchy depending on the parameter t (it was called the equivariant Toda lattice) could give a clue to such a limiting procedure. We plan to study the relationships between 2D and extended Toda hierarchies in a subsequent publication.

Appendix. Genus Zero Gromov-Witten Potential of CP^1

To compute the genus zero Gromov-Witten potential $\mathcal{F}_0(\mathbf{t})$ according to the general scheme of [3, 4] one is to perform the following computations (cf. Sects. 3.6.2, 3.6.10 of [9]).

1. Compute the functions $\theta_{\alpha,p}(v, u)$ as the coefficients of expansion of the following series:

$$
\begin{aligned}
\theta_1(v, u; z) &= \sum_{p \geq 0} \theta_{1,p}(v, u) z^p \\
&= -2 e^{zv} \left(K_0(2ze^{\frac{1}{2}u}) + (\log z + \gamma) I_0(2ze^{\frac{1}{2}u}) \right) \\
&= -2e^{zv} \sum_{m \geq 0} (\gamma - \frac{1}{2}u + \psi(m+1)) e^{mu} \frac{z^{2m}}{(m!)^2},
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
\theta_2(v, u; z) &= \sum_{p \geq 0} \theta_{2,p}(v, u) z^p \\
&= z^{-1} e^{zv} I_0(2ze^{\frac{1}{2}u}) - z^{-1} = z^{-1} \left(\sum_{m \geq 0} e^{mu+zv} \frac{z^{2m}}{(m!)^2} - 1 \right).
\end{aligned}
\tag{A.13}
$$

Here γ denotes Euler's constant, $\psi(z)$ stands for the digamma function, $K_0(x)$ and $I_0(x)$ are modified Bessel functions.

2. Compute the functions $\Omega^{[0]}_{\alpha,p;\beta,q}(v, u)$ as the coefficients of the following generating series

$$
\begin{aligned}
&\sum_{p,q \geq 0} \Omega^{[0]}_{\alpha,p;\beta,q}(v, u) z^p w^q \\
&= \frac{1}{z+w} \left[\frac{\partial \theta_\alpha(v, u; z)}{\partial v} \frac{\partial \theta_\beta(v, u; w)}{\partial u} + \frac{\partial \theta_\alpha(v, u; z)}{\partial u} \frac{\partial \theta_\beta(v, u; w)}{\partial v} - \eta_{\alpha\beta} \right].
\end{aligned}
\tag{A.14}
$$

3. Define the functions $v(\mathbf{t})$, $u(\mathbf{t})$ as the unique solution of the system

$$
\begin{aligned}
v &= \sum t^{\beta,q} \frac{\partial \theta_{\beta,q}}{\partial u}, \\
u &= \sum t^{\beta,q} \frac{\partial \theta_{\beta,q}}{\partial v}
\end{aligned}
\tag{A.15}
$$

having the expansion

$$
\begin{aligned}
v(\mathbf{t}) &= t^{1,0} + o(t), \\
u(\mathbf{t}) &= t^{2,0} + o(t).
\end{aligned}
$$

4. The genus zero Gromov - Witten potential of CP^1 is given by

$$\mathcal{F}_0(\mathbf{t}) = \frac{1}{2} \sum \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q} \Omega^{[0]}_{\alpha,p;\beta,q}(v(\mathbf{t}), u(\mathbf{t})). \tag{A.16}$$

Here

$$\tilde{t}^{\alpha,p} = \begin{cases} t^{1,1} - 1, & \alpha = 1, \ p = 1 \\ t^{\alpha,p}, & \text{otherwise.} \end{cases} \tag{A.17}$$

Let us write the first few terms of the expansion of the resulting genus zero Gromov - Witten potential. For simplicity we denote

$$t_p := t^{1,p}, \quad s_p := t^{2,p}, \quad p \geq 0.$$

The potential is expanded in powers of t_p, s_p and in e^{s_0}. The powers of the exponential separate the intersection numbers on the moduli spaces $\bar{M}_{0,m}(CP^1, \beta)$ for different β. Thus, the terms without exponential correspond to $\beta = 0$ etc. In our expansion we collect the intersection numbers up to degree $\beta = 3$ and up to $m \leq 4$ punctures. We also restrict the potential onto the subspace of couplings $t^{\alpha,p}$ with $p \leq 3$. The maximal number of descendents is restricted to three.

$$
\begin{aligned}
\mathcal{F}_0 = {} & \frac{t_0^2 s_0}{2!} + \frac{t_0^2 t_1 s_0}{2!} + \frac{t_0^3 s_1}{3!} + \frac{t_0^3 t_2 s_0}{3!} + \frac{t_0^4 s_2}{4!} + \frac{t_0^2 t_1^2 s_0}{2!} \\
& + 2\frac{t_0^3 t_1 s_1}{3!} + \frac{t_0^4 t_3 s_0}{4!} + \frac{t_0^5 s_3}{5!} + 3\frac{t_0^3 t_1 t_2 s_0}{3!} + 3\frac{t_0^4 t_2 s_1}{4!} + 3\frac{t_0^4 t_1 s_2}{4!} \\
& + e^{s_0}\Bigg[1 - 2t_1 + 2\frac{t_1^2}{2!} - 2t_0 t_2 + t_1 s_0 + t_0 s_1 - 2\frac{t_0^2 t_3}{2!} \\
& - 2\frac{t_1^2 s_0}{2!} + t_0 t_2 s_0 + \frac{t_0^2 s_2}{2!} - 2\frac{t_0^2 t_1 t_3}{2!} - t_0 t_1 t_2 s_0 + \frac{t_0^2 t_3 s_0}{2!} \\
& + 2\frac{t_1^2 s_0^2}{(2!)^2} - \frac{t_0^2 t_2 s_1}{2!} + t_0 t_1 s_0 s_1 + \frac{t_0^2 s_1^2}{2!} + \frac{t_0^2 t_1 s_2}{2!} + \frac{t_0^3 s_3}{3!} \Bigg] \\
& + e^{2s_0}\Bigg[-\frac{3}{4}t_3 + \frac{1}{4}s_2 + \frac{5}{4}\frac{t_2^2}{2!} + \frac{3}{4}t_1 t_3 + \frac{1}{4}t_3 s_0 - \frac{3}{4}t_2 s_1 + \frac{1}{2}\frac{s_1^2}{2!} \\
& - \frac{1}{4}t_1 s_2 + \frac{1}{4}t_0 s_3 + 2t_0 t_2 t_3 - \frac{3}{2}\frac{t_2^2 s_0}{2!} - \frac{3}{2}t_1 t_3 s_0 - 2t_0 t_3 s_1 \\
& + \frac{1}{2}t_2 s_0 s_1 - t_0 t_2 s_2 + \frac{1}{2}t_1 s_0 s_2 + t_0 s_1 s_2 + 2t_0 t_1 t_2 t_3 + 4\frac{t_0^2 t_3^2}{(2!)^2} \\
& + \frac{t_1 t_2^2 s_0}{2!} - 3t_0 t_2 t_3 s_0 + \frac{t_2^2 s_0^2}{(2!)^2} + \frac{t_1 t_3 s_0^2}{2!} + \frac{t_0 t_2^2 s_1}{2!} \\
& - 2t_0 t_1 t_3 s_1 - t_1 t_2 s_0 s_1 + t_0 t_3 s_0 s_1 - 2\frac{t_0 t_2 s_1^2}{2!} + \frac{t_1 s_0 s_1^2}{2!} \\
& + 3\frac{t_0 s_1^3}{3!} - t_0 t_1 t_2 s_2 - 3\frac{t_0^2 t_3 s_2}{2!} + t_0 t_2 s_0 s_2 + t_0 t_1 s_1 s_2 \\
& + 2\frac{t_0^2 s_2^2}{(2!)^2} - \frac{t_0^2 t_2 s_3}{2!} + \frac{t_0 t_1 s_0 s_3}{2} + \frac{3}{2}\frac{t_0^2 s_1 s_3}{2!} \Bigg]
\end{aligned}
$$

$$+e^{3s_0}\left[\frac{50}{27}\frac{t_3{}^2}{2!}-\frac{7}{9}\,t_3\,s_2+\frac{1}{3}\frac{s_2{}^2}{2!}-\frac{2}{9}\,t_2\,s_3+\frac{1}{6}s_1\,s_3-2\frac{t_2{}^2\,t_3}{2!}-\frac{14}{9}\frac{t_3{}^2\,s_0}{2!}\right.$$

$$+2\,t_2\,t_3\,s_1-2\frac{t_3\,s_1{}^2}{2!}+\frac{t_2{}^2\,s_2}{2!}+\frac{1}{3}t_3\,s_0\,s_2-t_2\,s_1\,s_2+\frac{s_1{}^2\,s_2}{2!}-t_0\,t_3\,s_3$$

$$+\frac{1}{6}t_2\,s_0\,s_3+\frac{1}{2}t_0\,s_2\,s_3-4\frac{t_0\,t_2\,t_3{}^2}{2!}+5\frac{t_2{}^2\,t_3\,s_0}{2!}+4\frac{t_1\,t_3{}^2\,s_0}{2!}+\frac{2}{3}\frac{t_3{}^2\,s_0{}^2}{(2!)^2}$$

$$+8\frac{t_0\,t_3{}^2\,s_1}{2!}-3\,t_2\,t_3\,s_0\,s_1+\frac{t_3\,s_0\,s_1{}^2}{2!}+3\,t_0\,t_2\,t_3\,s_2-2\frac{t_2{}^2\,s_0\,s_2}{2!}$$

$$-2\,t_1\,t_3\,s_0\,s_2-5\,t_0\,t_3\,s_1\,s_2+t_2\,s_0\,s_1\,s_2-2\frac{t_0\,t_2\,s_2{}^2}{2!}+\frac{t_1\,s_0\,s_2{}^2}{2!}$$

$$+3\frac{t_0\,s_1\,s_2{}^2}{2!}+\frac{t_0\,t_2{}^2\,s_3}{2!}-t_0\,t_1\,t_3\,s_3-\frac{1}{2}t_1\,t_2\,s_0\,s_3+\frac{1}{2}t_0\,t_3\,s_0\,s_3$$

$$-\frac{3}{2}t_0\,t_2\,s_1\,s_3+\frac{1}{2}t_1\,s_0\,s_1\,s_3+2\frac{t_0\,s_1{}^2\,s_3}{2!}+\frac{1}{2}t_0\,t_1\,s_2\,s_3+\frac{t_0{}^2\,s_3{}^2}{(2!)^2}\left.\right].$$

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