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Hamiltonian PDEs: deformations, integrability, solutions

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Abstract

We review recent classification results on the theory of systems of nonlinear Hamiltonian partial differential equations with one spatial dimension, including a perturbative approach to the integrability theory of such systems, and discuss universality conjectures describing critical behaviour of solutions to such systems.

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1. Introduction

The main subject of our research is the study of Hamiltonian perturbations of systems of quasilinear hyperbolic[1] PDEs:

$$u^i_t + A^i_j(u)u^j_x + \text{higher order derivatives} = 0, \qquad i = 1, \ldots, n.$$

(Here and below the summation over repeated indices will be assumed.) They can be obtained, in particular, by applying the procedure of *weak dispersion expansion*: starting from a system of PDEs

$$u^i_t + F^i(u, u_x, u_{xx}, \ldots) = 0, \qquad i = 1, \ldots, n.$$

With the analytic right-hand side let us introduce *slow variables*

$$x \mapsto \epsilon x, \qquad t \mapsto \epsilon t.$$

Expanding in ϵ one obtains, after dividing by ϵ, a system of the above form

$$u^i_t + \frac{1}{\epsilon}F^i(u, \epsilon u_x, \epsilon^2 u_{xx}, \ldots) = u^i_t + A^i_j(u)u^j_x + \epsilon\left(B^i_j(u)u^j_{xx} + \frac{1}{2}C^i_{jk}(u)u^j_x u^k_x\right) + \cdots$$

[1] Always the strong hyperbolicity will be assumed, i.e. the eigenvalues of the $n \times n$ matrix $(A^i_j(u))$ are all real and pairwise distinct for all $u = (u^1, \ldots, u^n)$ in the domain under consideration. Certain parts of the formalism developed in this paper will be applicable also to the quasilinear systems with complex distinct eigenvalues of the coefficient matrix.

assuming all the dependent variables are *slow*, i.e. the terms of the order $1/\epsilon$ vanish:

$$F^i(u, 0, 0, \dots) \equiv 0, \qquad i = 1, \dots, n.$$

The celebrated Korteweg–de Vries (KdV) equation

$$u_t + u\, u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \tag{1.1}$$

is one of the most well-known examples of such a weakly dispersive Hamiltonian PDE.

The systems of PDEs under investigation

$$u_t^i + A_j^i(u) u_x^j + \epsilon \left(B_j^i(u) u_{xx}^j + \tfrac{1}{2} C_{jk}^i(u) u_x^j u_x^k \right) + \cdots = 0, \qquad i = 1, \dots, n, \tag{1.2}$$

depending on a small parameter ϵ will be considered as Hamiltonian vector fields on the 'infinite-dimensional manifold'

$$\mathcal{L}(M^n) \otimes \mathbb{R}[[\epsilon]], \tag{1.3}$$

where M^n is an n-dimensional manifold (in all our examples it will have the topology of a ball) and

$$\mathcal{L}(M^n) = \{ S^1 \to M^n \}$$

is the space of loops on M^n. The dependent variables

$$u = (u^1, \dots, u^n) \in M^n$$

are local coordinates on M^n. In expansion (1.2) the terms of order ϵ^k are *polynomials* of degree $k+1$ in the derivatives $u_x, u_{xx}, \dots$, where

$$\deg u^{(m)} = m, \qquad m = 1, 2, \dots.$$

The coefficients of these polynomials are smooth functions defined in every coordinate chart on M^n.

The systems of the form (1.2) will be assumed to be Hamiltonian flows

$$u_t^i = \{u^i(x), H\} = \sum_{k \geqslant 0} \epsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij}(u; u_x, \dots, u^{(m)}) \partial_x^{k-m+1} \frac{\delta H}{\delta u^j(x)} \tag{1.4}$$

with respect to a *local Poisson bracket*

$$\begin{aligned} \{u^i(x), u^j(y)\} &= \sum_{k \geqslant 0} \epsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij}(u(x); u_x(x), \dots, u^{(m)}(x)) \delta^{(k-m+1)}(x-y) \\ \deg A_{k,m}^{ij}(u; u_x, \dots, u^{(m)}) &= m \end{aligned} \tag{1.5}$$

with *local Hamiltonians*

$$\begin{aligned} H &= \sum_{k \geqslant 0} \epsilon^k \int h_k(u; u_x, \dots, u^{(k)})\, \mathrm{d}x = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots \\ H_k &= \int h_k(u; u_x, \dots, u^{(k)})\, \mathrm{d}x, \qquad \deg h_k(u; u_x, \dots, u^{(k)}) = k. \end{aligned} \tag{1.6}$$

Here $\delta(x)$ is the Dirac delta-function and

$$\frac{\delta H}{\delta u^j(x)} = \frac{\partial h}{\partial u^j} - \partial_x \frac{\partial h}{\partial u_x^j} + \partial_x^2 \frac{\partial h}{\partial u_{xx}^j} - \cdots$$

is the Euler–Lagrange operator of a local Hamiltonian

$$H = \int h(u; u_x, u_{xx}, \dots)\, \mathrm{d}x.$$

The coefficients of the Poisson bracket and Hamiltonian densities will always be assumed to be differential polynomials in every order in ϵ. The antisymmetry and Jacobi identity for the Poisson bracket (1.5) are understood as identities for formal power series in ϵ. Note that for a differential polynomial $f(u; u_x, u_{xx}, \ldots, u^{(m)})$ the local functional $\int f(u; u_x, u_{xx}, \ldots, u^{(m)})\,\mathrm{d}x$ is defined as the equivalence class of the differential polynomial f modulo $\operatorname{Im}\partial_x$ where

$$\partial_x = \sum u^{i,k+1}\frac{\partial}{\partial u^{i,k}}, \qquad u^{i,k} := \frac{\partial^k u^i}{\partial x^k} \tag{1.7}$$

is the operator of the total x-derivative. The Poisson bracket of local functionals of the form

$$\begin{gathered} F = \sum_{k\geqslant 0}\epsilon^k \int f_k(u; u_x, \ldots, u^{(k)})\,\mathrm{d}x, \qquad G = \sum_{l\geqslant 0}\epsilon^l \int g_l(u; u_x, \ldots, u^{(l)})\,\mathrm{d}x \\ \deg f_k(u; u_x, \ldots, u^{(k)}) = k, \qquad \deg g_l(u; u_x, \ldots, u^{(l)}) = l \end{gathered} \tag{1.8}$$

reads

$$\begin{aligned} \{F, G\} &= \int \frac{\delta F}{\delta u^i(x)} A^{ij} \frac{\delta G}{\delta u^j(x)}\,\mathrm{d}x \\ A^{ij} &:= \sum_{k\geqslant 0}\epsilon^k \sum_{m=0}^{k+1} A^{ij}_{k,m}(u; u_x, \ldots, u^{(m)})\partial_x^{k-m+1}. \end{aligned} \tag{1.9}$$

Therefore, a local Poisson bracket (1.5) defines a structure of Lie algebra on the space $\mathcal{G}_{\mathrm{loc}}$ of local functionals.

Example 1.1. The KdV equation (1.1) admits a Hamiltonian representation

$$u_t + \{u(x), H\} \equiv u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0 \tag{1.10}$$

with the Poisson bracket

$$\{u(x), u(y)\} = \delta'(x-y) \tag{1.11}$$

and the Hamiltonian

$$H = \int \left(\frac{1}{6}u^3 - \frac{\epsilon^2}{24}u_x^2\right)\mathrm{d}x. \tag{1.12}$$

Let us now introduce a class of 'coordinate changes' on the infinite-dimensional manifold $\mathcal{L}(M^n)\otimes\mathbb{R}[[\epsilon]]$. Define a *generalized Miura transformation:*

$$\begin{aligned} &u^i \mapsto \tilde{u}^i = \sum_{k\geqslant 0}\epsilon^k F^i_k(u; u_x, \ldots, u^{(k)}) \\ &\deg F^i_k(u; u_x, \ldots, u^{(k)}) = k \\ &\det\left(\frac{\partial F^i_0(u)}{\partial u^j}\right) \neq 0. \end{aligned} \tag{1.13}$$

The coefficients $F^i_k(u; u_x, \ldots, u^{(k)})$ are differential polynomials. It is easy to see that the transformations of the form (1.13) form a group. The classes of evolution PDEs (1.2), local Poisson brackets (1.5) and local Hamiltonians (1.6) are invariant with respect to the action of the group of generalized Miura transformations. We say that two objects of our theory (i.e. two evolutionary vector fields of the form (1.2), two local Poisson brackets of the form (1.5), or two local Hamiltonians of the form (1.6)) are *equivalent* if they are related by a generalized Miura transformation.

Our main goals are as follows:

- classification of Hamiltonian PDEs,
- selection of integrable PDEs and
- study of the general properties of solutions to these PDEs.

2. Classification problems

Setting $\epsilon = 0$ in (1.4)–(1.6) one obtains the so-called *dispersionless limit* of the PDE

$$u^i_t = A^i_j(u)u^j_x. \tag{2.1}$$

The dispersionless limit is itself a Hamiltonian PDE

$$u^i_t = \{u^i(x), H_0\}_0 \tag{2.2}$$

with respect to the Poisson bracket

$$\{u^i(x), u^j(y)\}_0 = A^{ij}_{0,0}(u)\delta'(x-y) + A^{ij}_{0,1}(u; u_x)\delta(x-y) \tag{2.3}$$

with a very simple local Hamiltonian

$$H_0 = \int h_0(u)\,\mathrm{d}x. \tag{2.4}$$

The original system (1.4) with its Hamiltonian structure (1.5) and (1.6) can be considered as a *deformation* of (2.1)–(2.4).

The problem of classification of Hamiltonian systems of our class can therefore be decomposed in two steps:

- classification of dispersionless Hamiltonian systems;
- classification of their perturbations.

The dispersionless systems and their Hamiltonian structures will be classified with respect to the action of the group of (local) diffeomorphisms, while the perturbations have to be classified up to equivalences established by the action of the group of generalized Miura transformations.

The answer to the first question was obtained by Novikov and the author of [16] under the so-called *non-degeneracy assumption*[2]

$$\det\left(A^{ij}_{0,0}(u)\right) \neq 0. \tag{2.5}$$

The non-degeneracy condition is invariant with respect to the action of the group of (local) diffeomorphisms.

Theorem 2.1. *Poisson brackets of the form (2.3) satisfying the non-degeneracy condition (2.5) by a change of local coordinates $\tilde u^i = \tilde u^i(u^1, \dots, u^n)$, $\det(\partial\tilde u^i/\partial u^j) \neq 0$ can be reduced to the following standard form:*

$$\{\tilde u^i(x), \tilde u^j(y)\}_0 = \eta^{ij}\delta'(x-y), \tag{2.6}$$

where (η^{ij}) is a constant symmetric non-degenerate matrix.

The crucial step in the proof of this theorem is the observation that the Riemannian or pseudo-Riemannian metric

$$\mathrm{d}s^2 = g_{ij}(u)\,\mathrm{d}u^i\,\mathrm{d}u^j, \qquad (g_{ij}(u)) = \left(A^{ij}_{0,0}(u)\right)^{-1} \tag{2.7}$$

[2] To avoid confusion with non-degeneracy of the Poisson bracket. The Poisson bracket (2.3) satisfying (2.5) is always degenerate. The symplectic leaves of this bracket have a codimension n.

defined by the inverse to the matrix $\left(A^{ij}_{0,0}(u)\right)$ has a vanishing curvature. Moreover, the second coefficient of bracket (2.3) can be expressed via the Christoffel coefficients of the Levi-Civita connection for the metric

$$A^{ij}_{0,1}(u;u_x) = -A^{is}_{0,0}(u)\Gamma^j_{sk}(u)u^k_x. \tag{2.8}$$

Choosing locally a system of flat coordinates one reduces the metric to a constant form

$$\mathrm{d}s^2 = \eta_{ij}\,\mathrm{d}v^i\,\mathrm{d}v^j. \tag{2.9}$$

The Christoffel coefficients in these coordinates vanish.

The matrix (η^{ij}) in (2.6) is the inverse to (η_{ij}).

We will omit tildes in the notation for the flat coordinates. From the previous theorem one obtains

Corollary 2.2. *Any Hamiltonian system of dispersionless PDEs can be locally reduced to the form*

$$u^i_t = \eta^{ij}\partial_x\frac{\delta H_0}{\delta u^j(x)} = \eta^{ij}\frac{\partial^2 h(u)}{\partial u^j\partial u^k}u^k_x, \qquad i = 1,\ldots,n, \qquad H_0 = \int h(u)\,\mathrm{d}x. \tag{2.10}$$

Let us now proceed to the study of perturbations of Hamiltonian systems (2.10). *A priori*, there are two types of perturbations:

- perturbations of Hamiltonians (2.4)
- perturbation of Poisson brackets (2.3).

It turns out that one can discard the second type of perturbations to arrive at.

Theorem 2.3. *Any system of Hamiltonian PDEs of classes (1.2) and (1.4)–(1.6) satisfying the non-degeneracy assumption (2.5) can be reduced to the following standard form:*

$$u^i_t = \eta^{ij}\partial_x\frac{\delta H}{\delta u^j(x)}, \qquad i = 1,\ldots,n, \tag{2.11}$$

with the Hamiltonian of the form (1.6). Here (η^{ij}) is a constant symmetric non-degenerate matrix. Two such systems are equivalent iff the Hamiltonians are related by a canonical transformation

$$\begin{aligned} &H \mapsto H + \epsilon\{K,H\} + \frac{\epsilon^2}{2}\{K,\{K,H\}\} + \cdots,\\ &K = \sum_{k\geqslant 0}\epsilon^k\int f_k(u;u_x,\ldots,u^{(k)})\,\mathrm{d}x, \qquad \deg f_k(u;u_x,\ldots,u^{(k)}) = k. \end{aligned} \tag{2.12}$$

Observe that the Poisson bracket for the system reads

$$\{u^i(x),u^j(y)\} = \eta^{ij}\delta'(x-y), \qquad \eta^{ij} = \eta^{ji} = \text{const}, \qquad \det(\eta^{ij}) \neq 0. \tag{2.13}$$

The main idea in the proof of this theorem is in the study of suitably defined Poisson cohomology of bracket (2.13). Triviality of Poisson cohomology in positive degrees in ϵ was proved in [9, 18]; see also [17] for an extensive discussion of the deformation problem and its generalizations.

Example 2.4. The Hopf equation

$$v_t + vv_x = 0 \tag{2.14}$$

is a Hamiltonian system

$$v_t + \partial_x\frac{\delta H_0}{\delta v(x)} = 0$$

with the Hamiltonian

$$H_0 = \int \frac{v^3}{6}\, \mathrm{d}x \tag{2.15}$$

and the Poisson bracket of the form (2.13):

$$\{v(x), v(y)\} = \delta'(x-y). \tag{2.16}$$

Any Hamiltonian perturbation of this equation of order ϵ^4 can be reduced to the following normal form parametrized by two arbitrary functions of one variable $c = c(u)$, $p = p(u)$:

$$\begin{aligned} u_t + uu_x + \frac{\epsilon^2}{24}\left[2cu_{xxx} + 4c'u_xu_{xx} + c''u_x^3\right] + \epsilon^4\left[2pu_{xxxxx} + 2p'(5u_{xx}u_{xxx} + 3u_xu_{xxxx})\right.\\ \left. + p''\left(7u_xu_{xx}^2 + 6u_x^2u_{xxx}\right) + 2p'''u_x^3u_{xx}\right] = 0. \end{aligned} \tag{2.17}$$

The Hamiltonian has the form

$$H = \int \left[\frac{u^3}{6} - \epsilon^2\frac{c(u)}{24}u_x^2 + \epsilon^4 p(u)u_{xx}^2\right]\mathrm{d}x. \tag{2.18}$$

Two such perturbations are equivalent iff the associated functional parameters $c(u)$, $p(u)$ coincide [10].

3. Selecting integrable PDEs

The theory of integrability of Hamiltonian systems studies the families of commuting Hamiltonians. Observe that the commutativity $\{H, F\} = 0$ of local Hamiltonians with respect to the Poisson bracket (2.13) reduces to the following system of differential equations for their densities:

$$\frac{\delta}{\delta u^k(x)}\left(\frac{\delta H}{\delta u^i(x)}\eta^{ij}\partial_x\frac{\delta F}{\delta u^j(x)}\right) = 0, \qquad k = 1, \ldots, n. \tag{3.1}$$

Commutativity of formal series

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots, \qquad F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots$$

must hold true at every order in ϵ:

$$\begin{aligned} &\{H_0, F_0\} = 0\\ &\{H_0, F_1\} + \{H_1, F_0\} = 0\\ &\{H_0, F_2\} + \{H_1, F_1\} + \{H_2, F_0\} = 0, \quad \text{etc.} \end{aligned} \tag{3.2}$$

In particular the commutativity of the leading terms

$$H_0 = \int h(u)\,\mathrm{d}x, \qquad F_0 = \int f(u)\,\mathrm{d}x$$

reduces to the following bilinear system of differential equations:

$$\frac{\partial^2 f}{\partial v^i \partial v^l}\eta^{ij}\frac{\partial^2 h}{\partial v^j \partial v^k} = \frac{\partial^2 f}{\partial v^i \partial v^k}\eta^{ij}\frac{\partial^2 h}{\partial v^j \partial v^l}, \qquad k, l = 1, \ldots, n. \tag{3.3}$$

Replacing the commutativity with the condition

$$\{H, F\} = \mathcal{O}(\epsilon^{N+1}) \tag{3.4}$$

one obtains *N-commuting* Hamiltonians. Clearly the conditions of N-commutativity involve only first N terms of the series H and F.

After these preliminaries let us consider a system of Hamiltonian PDEs of the form

$$u_t^i = \eta^{ij}\partial_x \frac{\delta H}{\delta u^j(x)}, \qquad i = 1, \ldots, n, \tag{3.5}$$

$$\begin{aligned} H &= \sum_{k\geqslant 0} \epsilon^k \int h_k(u; u_x, \ldots, u^{(k)})\, \mathrm{d}x = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots \\ H_k &= \int h_k(u; u_x, \ldots, u^{(k)})\, \mathrm{d}x, \qquad \deg h_k(u; u_x, \ldots, u^{(k)}) = k. \end{aligned} \tag{3.6}$$

Here, as above (η^{ij}) is a constant symmetric non-degenerate matrix. The eigenvalues of the leading coefficient matrix

$$A_j^i(u) = \eta^{is}\frac{\partial^2 h(u)}{\partial u^s \partial u^j}, \qquad h(u) \equiv h_0(u) \tag{3.7}$$

are assumed to be pairwise distinct for all $u \in M^n$.

Definition 3.1. *The system (3.5) is called integrable if*

- *the leading-order hyperbolic system*

$$\begin{aligned} u_t^i &= \eta^{ij}\partial_x \frac{\delta H_0}{\delta u^j(x)} = A_j^i(u) u_x^j \\ H_0 &= \int h(u)\, \mathrm{d}x \end{aligned} \tag{3.8}$$

  is integrable;
- *for any first integral*

$$\begin{aligned} H_0^f &= \int f(u)\, \mathrm{d}x \\ \{H_0^f, H_0\} &= 0 \end{aligned} \tag{3.9}$$

  of the hyperbolic system (3.8) there exists a first integral of (3.5) with the prescribed leading term H_0^f:

$$\begin{aligned} &\{H^f, H\} = 0 \\ &H^f = \sum_{k\geqslant 0} \epsilon^k \int f_k(u; u_x, \ldots, u^{(k)})\, \mathrm{d}x = H_0^f + \epsilon H_1^f + \epsilon^2 H_2^f + \cdots \\ &H_k^f = \int f_k(u; u_x, \ldots, u^{(k)})\, \mathrm{d}x, \qquad \deg f_k(u; u_x, \ldots, u^{(k)}) = k; \end{aligned} \tag{3.10}$$

- *for any pair of first integrals H_0^f, H_0^g of the hyperbolic system (3.8) the Hamiltonians H^f and H^g commute*[3]

$$\{H^f, H^g\} = 0.$$

Replacing commutativity $\{H^f, H^g\} = 0$ with N-commutativity one obtains the definition of an *N-integrable* Hamiltonian PDE.

Remark 3.2. Under certain natural assumptions (see corollary 3.9 below) the coefficients of the differential polynomials $f_k(u; u_x, \ldots, u^{(k)})$ can be expressed via partial derivatives of the function $f(u)$.

[3] In the examples considered so far the last condition is redundant. It would be interesting to prove this in general.

The above definition relies on the knowledge of the theory of integrability [29] of hyperbolic systems (also called *systems of hydrodynamic type*). In a nutshell this theory states that a hyperbolic Hamiltonian system (3.8) is integrable if it can be reduced to a diagonal form. This means that locally a system of coordinates $(r^1(u), \ldots, r^n(u))$ exists such that the coefficient matrix of the system (3.8) becomes diagonal $A = \mathrm{diag}(\lambda^1(r), \ldots, \lambda^n(r))$ in the new coordinates

$$r^i_t + \lambda^i(r) r^i_x = 0, \qquad i = 1, \ldots, n. \tag{3.11}$$

Thus, the new coordinates are *Riemann invariants* of the hyperbolic system (3.8).

The integrability test of a Hamiltonian hyperbolic system can be formulated with the help of the classical Haantjes torsion [20] that in the present situation takes the following form [13].

Theorem 3.3. *The hyperbolic system (3.8) is integrable iff the following conditions hold true:*

$$\mathrm{H}_{ijk} := (h_{ipq} h_{jr} h_{ks} + h_{jpq} h_{kr} h_{is} + h_{kpq} h_{ir} h_{js}) h_{ab} \eta^{pa} \delta^{qbsr} = 0, \qquad 1 \leqslant i, j, \qquad k \leqslant n. \tag{3.12}$$

Here

$$h_{ij} := \frac{\partial^2 h}{\partial u^i \partial u^j}, \qquad h_{ijk} := \frac{\partial^3 h}{\partial u^i \partial u^j \partial u^k}$$
$$\delta^{ijkl} := \det \begin{pmatrix} \eta^{ik} & \eta^{il} \\ \eta^{jk} & \eta^{jl} \end{pmatrix}.$$

The torsion (3.12) is a rank 3 antisymmetric tensor. So the conditions of the theorem become nontrivial starting from $n \geqslant 3$.

Densities $f(u)$ of the first integrals $H_0^f = \int f(u)\, \mathrm{d}x$ of the hyperbolic system (3.8) have to be determined from the system of linear PDEs (3.3). Vanishing of the Haantjes torsion (3.12) provides the compatibility conditions for this overdetermined system. For a given Hamiltonian density $h(u)$ satisfying (3.12) the solutions to (3.3) form a linear space of first integrals of the system (3.8). They can be parametrized by n arbitrary functions of one variable.

The commutativity (3.3) admits the following simple interpretation. The Hamiltonian flow

$$u^i_s = B^i_j(u) u^j_x, \qquad B^i_j(u) = \eta^{is} \frac{\partial^2 f(u)}{\partial u^s \partial u^j} \tag{3.13}$$

generated by any solution to (3.2) is an infinitesimal symmetry of the hyperbolic system (3.8):

$$\frac{\partial}{\partial s} \frac{\partial u^i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial u^i}{\partial s}, \qquad i = 1, \ldots, n. \tag{3.14}$$

Equation (3.3) states that the coefficient matrices $A = \left(A^i_j(u)\right)$ and $B = \left(B^i_j(u)\right)$ commute at every point v. Since the eigenvalues of the matrix A are pairwise distinct, the centralizer of A is commutative. Hence

Theorem 3.4. *First integrals of the form (3.2) of a Hamiltonian hyperbolic system commute pairwise.*

The diagonal coordinates $(r^1, \ldots, r^n)$ are also Riemann invariants for any commuting flow (3.13) and (3.14). Moreover, metric (2.9) becomes diagonal in the new coordinates

$$\mathrm{d}s^2 = \sum_{i=1}^n \eta_i^2(r) (\mathrm{d}r^i)^2. \tag{3.15}$$

Thus, Riemann invariants $(r^1, \ldots, r^n)$ define an orthogonal curvilinear system of local coordinates in the pseudo-Euclidean space with metric (2.9). The linear space of commuting Hamiltonians for the hyperbolic system (3.11) coincides with the space of functions $f(r)$ satisfying the following property: the covariant Hessian of the function $f(r)$ is a diagonal matrix

$$\nabla_i \nabla_j f(r) = 0 \qquad \text{for} \quad i \neq j. \tag{3.16}$$

Here ∇ is the Levi-Civita connection for the metric $\mathrm{d}s^2$.

Denote

$$\mathcal{K}_0 \subset \mathcal{G}_{\mathrm{loc}} \tag{3.17}$$

the linear space of all first integrals of the form $H_0^f = \int f(u)\,\mathrm{d}x$ of an integrable system of hyperbolic PDEs. It is a commutative subalgebra in the Lie algebra of local Hamiltonians. Let us remind the following standard definition.

Definition 3.5. *A commutative subalgebra in a Lie algebra is called maximal if any element of the Lie algebra commuting with all elements of the subalgebra belongs to the subalgebra.*

Theorem 3.6. *The first integrals of an integrable system of hyperbolic PDEs form a maximal Abelian subalgebra in the Lie algebra of local Hamiltonians.*

The proof of this theorem[4] is based on the following.

Lemma 3.7. *Let a local Hamiltonian $F = \int f(u, u_x, \ldots, u^{(m)})\,\mathrm{d}x$ commute with all Hamiltonians in $\mathcal{K}_0$. Then the variational derivatives $\delta F/\delta u^j(x)$ do not depend on the jets*

$$\begin{aligned} &u^{i,k} = \frac{\partial^k u^i}{\partial x^k}, \\ &\frac{\partial}{\partial u^{i,k}} \frac{\delta F}{\delta u^j(x)} = 0, \qquad k = 1, \ldots, 2m, \qquad i, j = 1, \ldots, n. \end{aligned} \tag{3.18}$$

Denote by $\mathcal{K}$ the commutative Lie algebra of first integrals of the perturbed system (3.5). It is easy to see that this subalgebra is maximal as well. Consider the linear map (sometimes also called the *dispersionless limit*)

$$\begin{aligned} &\mathcal{K} \to \mathcal{K}_0 \\ &F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots \mapsto F_0 \in \mathcal{K}_0. \end{aligned} \tag{3.19}$$

Theorem 3.8. *The map (3.19) is an isomorphism.*

Corollary 3.9. *For any N-integrable Hamiltonian PDE (3.5) there exists a linear differential operator*

$$\begin{aligned} &D_N = D^{[0]} + \epsilon D^{[1]} + \epsilon^2 D^{[2]} + \cdots + \epsilon^N D^{[N]} \\ &D^{[0]} = \mathrm{id}, \qquad D^{[k]} = \sum b^{[k]}_{i_1 \ldots i_{m(k)}}(u; u_x, \ldots, u^{(k)}) \frac{\partial^{m(k)}}{\partial u^{i_1} \ldots \partial u^{i_{m(k)}}} \\ &\deg b^{[k]}_{i_1 \ldots i_{m(k)}}(u; u_x, \ldots, u^{(k)}) = k, \qquad k \geqslant 1, \end{aligned} \tag{3.20}$$

defined on the common kernel of linear operators

$$h_{jk} \eta^{ij} \frac{\partial^2}{\partial v^i \partial v^l} - h_{jl} \eta^{ij} \frac{\partial^2}{\partial v^i \partial v^k}, \qquad k, l = 1, \ldots, n, \tag{3.21}$$

[4] A somewhat different approach to completeness of the Lie algebra $\mathcal{K}_0$ is discussed in section 5 of [29].

such that for any two solutions $f(u)$, $g(u)$ to equations (3.3) the Hamiltonians

$$H_N^f := \int D_N f \, \mathrm{d}x, \qquad H_N^g := \int D_N g \, \mathrm{d}x \tag{3.22}$$

satisfy

$$\{H_N^f, H_N^g\} = O(\epsilon^{N+1}). \tag{3.23}$$

Here $m(k)$ is an integer. It is easy to see that

$$m(k) \leqslant \left[\frac{3k}{2}\right].$$

Operator (3.20) is called the *D-operator* of an integrable Hamiltonian PDE (3.5). It is defined up to Im ∂_x.

Example 3.10. The perturbed Hopf equation (2.17) is five-integrable for an arbitrary choice of the functional parameters $c(u)$, $p(u)$ [11]. Indeed, the first integrals of the unperturbed system have the form

$$F_0 = \int f(v) \, \mathrm{d}x$$

for an arbitrary function $f(v)$. Define deformed functionals by the formula

$$F = \int D_{c,p} f \, \mathrm{d}x,$$

where the D-operator $D_5 \equiv D_{c,p}$ (see [12] for details) transforming the first integrals of the unperturbed system to the first integrals (modulo $O(\epsilon^6)$) of the perturbed one:

$$\begin{aligned} D_{c,p} f = f - \frac{\epsilon^2}{24} c f''' u_x^2 + \epsilon^4 \Bigg[&\left(p f''' + \frac{c^2 f^{(4)}}{480} \right) u_{xx}^2 \\ &- \left(\frac{c c'' f^{(4)}}{1152} + \frac{c c' f^{(5)}}{1152} + \frac{c^2 f^{(6)}}{3456} + \frac{p' f^{(4)}}{6} + \frac{p f^{(5)}}{6} \right) u_x^4 \Bigg]. \end{aligned} \tag{3.24}$$

It is an interesting open problem to prove existence and uniqueness, for a given pair of the functional parameters $c(u) \neq 0$, $p(u)$, of an extension to all orders in ϵ of the perturbed system (2.17) in order to obtain an integrable PDE. So far the existence of such an extension is known only for some particular cases including the following.

- KdV: $c(u) = \text{const}$, $p(u) = 0$. For the choice $c(u) = \frac{1}{24}$ the D-operator can be represented [13] in terms of the Lax operator and the fractional derivative $\partial_u^{1/2} f(u)$ of the function $f(u)$

$$\begin{aligned} Df &= \frac{1}{\sqrt{2}} \mathrm{res}(\partial^{1/2} f)(L) \\ L &= \frac{\epsilon^2}{2} \partial_x^2 + u(x). \end{aligned} \tag{3.25}$$

- Volterra lattice

$$\epsilon u_t = \mathrm{e}^{u(x+\epsilon)} - \mathrm{e}^{u(x-\epsilon)}$$

  has $c(u) = 2$, $p(u) = -\frac{1}{240}$.
- Camassa–Holm equation

$$u_t - \epsilon^2 u_{txx} = \tfrac{3}{2} u \, u_x - \epsilon^2 \left[u_x u_{xx} + \tfrac{1}{2} u \, u_{xxx} \right] \tag{3.26}$$

  has $c(u) = 8u$, $p(u) = \frac{1}{3} u$.

Summarizing, integrable systems of Hamiltonian PDEs of the form (3.5) are parametrized by the following:

- orthogonal systems of curvilinear coordinates in the pseudo-Euclidean space with metric (2.9). Any such system defines a maximal commutative Lie algebra $\mathcal{K}_0$ of local functionals with the densities given by solutions to to the linear system (3.16);
- a D-operator acting on solutions to (3.16). It defines a deformation of the commutative Lie algebra $\mathcal{K}_0$.

So far we only know very few particular examples of D-operators. For a particular subclass of integrable *biHamiltonian* PDEs it can be proven that a deformation of the commutative Lie algebra $\mathcal{K}_0$ can depend on at most n arbitrary functions of one variable called *central invariants* of a strongly non-degenerate semi-simple biHamiltonian structure [15]. Some partial improvements are available for the particular case of curvilinear orthogonal coordinates given by the so-called canonical coordinates on a semi-simple Frobenius manifold [17]. The general classification of D-operators remains an open problem even in the case $n = 1$. Equation (3.16) is empty in this case, so a D-operator must act on the space of arbitrary smooth functions $f(u)$.

4. Solving integrable PDEs

We begin with considering formal perturbative solutions

$$u^i(x,t;\epsilon) = u_0^i(x,t) + \epsilon\, u_1^i(x,t) + \epsilon^2 u_2^i(x,t) + \cdots, \qquad i = 1,\ldots,n, \tag{4.1}$$

to a system of integrable Hamiltonian PDEs of the form considered in the previous section. The functions

$$v^i(x,t) := u_0^i(x,t), \qquad i = 1,\ldots,n,$$

must solve the leading-order hyperbolic system

$$v_t^i = A_j^i(v) v_x^j. \tag{4.2}$$

(We use different notations $v = v(x,t)$ and $u = u(x,t)$ for the dependent variables of the unperturbed/perturbed systems resp. for a convenience later on.) The generic solution to an integrable hyperbolic system (4.2) can be found by the following procedure, due to Tsarev [29].

Definition 4.1. *A solution $v = v(x,t)$ to (4.2) is called monotone at the point (x_0,t_0) if*

$$\frac{\partial r^i(v(x,t))}{\partial x}|_{x=x_0,t=t_0} \neq 0 \qquad \textit{for all} \quad i = 1,\ldots,n. \tag{4.3}$$

Here $r^1(v),\ldots,r^n(v)$ are Riemann invariants of the integrable hyperbolic system (4.2).

All monotone solutions can be obtained as follows. Let $F_0 = \int f(v)\,\mathrm{d}x$ be any first integral of the hyperbolic system, i.e. the function $f(v)$ satisfies equations (3.3). Denote

$$B_j^i(v) = \eta^{is}\frac{\partial^2 f(v)}{\partial v^s \partial v^j}$$

and consider the following system of n^2 equations:

$$\begin{aligned} &x\cdot \mathrm{id} + tA(v) = B(v)\\ &A(v) = \left(A_j^i(v)\right), \qquad B(v) = \left(B_j^i(v)\right). \end{aligned} \tag{4.4}$$

Since the matrices $A(v)$, $B(v)$ are simultaneously diagonalizable, only n of equations (4.4) are independent. Let $v_0 \in M$ be such a point that equations (4.4) become valid identities after the substitution $(x \to x_0, t \to t_0, v \to v_0)$. Assume that the conditions of the implicit function theorem hold true at the point (x_0, t_0, v_0). Then a smooth vector-function $v = v(x, t)$ such that $v(x_0, t_0) = v_0$ is uniquely defined from the system (4.4) for sufficiently small $|x - x_0|$, $|t - t_0|$.

Theorem 4.2. *Under the above assumptions the vector-function v(x, t) defines a monotone solution to the hyperbolic system (4.2) on some small neighbourhood of the point (x_0, t_0). Moreover, all monotone solutions to (4.2) can be obtained by the above procedure.*

Example 4.3. For $n = 1$ one recognizes the well-known method of characteristics for representing solutions to the Hopf equation

$$v_t + vv_x = 0$$

in the implicit form

$$x = vt + f(v).$$

The function $f(v)$ is determined by the Cauchy data

$$f(v(x, 0)) \equiv x.$$

Example 4.4. The *focusing* nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\,\epsilon\psi_t + \tfrac{1}{2}\epsilon^2\psi_{xx} + |\psi|^2\psi = 0 \tag{4.5}$$

can be reduced to the form (3.5) by the substitution

$$u = |\psi|^2, \qquad v = \frac{\epsilon}{2\mathrm{i}}\left(\frac{\psi_x}{\psi} - \frac{\bar\psi_x}{\bar\psi}\right). \tag{4.6}$$

The resulting system

$$\begin{aligned} &u_t + (uv)_x = 0 \\ &v_t + vv_x - u_x = \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\frac{u_x^2}{u^2}\right)_x \end{aligned} \tag{4.7}$$

is Hamiltonian with the Poisson bracket

$$\{u(x), v(y)\} = \delta'(x - y), \qquad \{u(x), u(y)\} = \{v(x), v(y)\} = 0$$

and the Hamiltonian

$$H = \int \left[\frac{1}{2}(uv^2 - u^2) + \frac{\epsilon^2}{8u}u_x^2\right] \mathrm{d}x. \tag{4.8}$$

The leading-order system

$$\begin{aligned} &u_t + (uv)_x = 0 \\ &v_t + vv_x - u_x = 0 \end{aligned} \tag{4.9}$$

is of *elliptic type* since the eigenvalues $\lambda_\pm = v \pm \mathrm{i}\sqrt{u}$ of the coefficient matrix

$$A = \begin{pmatrix} v & u \\ -1 & v \end{pmatrix}$$

are complex conjugate (observe that $u = |\psi|^2 > 0$). The above procedure is applicable; it locally represents generic solutions to (4.9) in the implicit form

$$\left.\begin{aligned} x &= vt + f_u \\ 0 &= ut + f_v \end{aligned}\right\}, \tag{4.10}$$

where $f = f(u, v)$ is a solution to the linear PDE (3.3):

$$f_{vv} + u f_{uu} = 0. \tag{4.11}$$

Let us now proceed to the study of the subsequent terms of the perturbative expansion (4.1). In principle they can be recursively determined by solving linear inhomogeneous equations of the form

$$\frac{\partial u^i_{m+1}}{\partial t} - \partial_x\left(A^i_j(v)u^j_{m+1}\right) = \mathcal{F}^i_m(v; u_1, \ldots, u_m), \qquad i = 1, \ldots, n, \qquad m \geqslant 0. \tag{4.12}$$

The discrepancies $\mathcal{F}^i_0(v)$, $\mathcal{F}^i_1(v; u_1)$ can be determined by the direct substitution of the formal series (4.1) into the perturbed system

$$\frac{\partial u^i}{\partial t} = \eta^{ij}\partial_x\left[\frac{\delta H_0}{\delta u^j(x)} + \epsilon\frac{\delta H_1}{\delta u^j(x)} + \epsilon^2\frac{\delta H_2}{\delta u^j(x)} + \cdots\right] = A^i_j(u)u^j_x + \mathcal{O}(\epsilon). \tag{4.13}$$

However in many cases there exists a universal perturbative solution to the system (4.13) written in the form (4.1) with

$$u^i_k = F^i_k(v; v_x, \ldots, v^{(m_k)}), \qquad k \geqslant 1, \tag{4.14}$$

where $F^i_k(v; v_x, \ldots, v^{(m_k)})$ is a *rational* function in the jet variables v_x, v_{xx}, etc. The values of these rational functions must be defined on any monotone solution $v(x, t)$ to the unperturbed system. The explicit form of these functions does not depend on the choice of a particular solution. The advantage of such a representation of the perturbative solution is in its *locality*; the values of the perturbed solution and of its derivatives at a given point (x_0, t_0) depend only on the germ of the unperturbed solution at this point.

Example 4.5 ([11]). The canonical transformation

$$\begin{aligned}
v \mapsto u &= v + \epsilon\{v(x), K\} + \frac{\epsilon^2}{2}\{\{v(x), K\}, K\} + \cdots \\
&= v + \frac{\epsilon^2}{24}\partial_x\left(c\frac{v_{xx}}{v_x} + c'v_x\right) + \epsilon^4\partial_x\left[c^2\left(\frac{v_{xx}^3}{360v_x^4} - \frac{7v_{xx}v_{xxx}}{1920v_x^3} + \frac{v_{xxxx}}{1152v_x^2}\right)_x\right. \\
&\quad + cc'\left(\frac{47v_{xx}^3}{5760v_x^3} - \frac{37v_{xx}v_{xxx}}{2880v_x^2} + \frac{5v_{xxxx}}{1152v_x}\right) + c'^2\left(\frac{v_{xxx}}{384} - \frac{v_{xx}^2}{5760v_x}\right) \\
&\quad + cc''\left(\frac{v_{xxx}}{144} - \frac{v_{xx}^2}{360v_x}\right) + \frac{1}{1152}\left(7c'c''v_xv_{xx} + c''^2v_x^3\right. \\
&\quad \left. + 6cc'''v_xv_{xx} + c'c'''v_x^3 + cc^{(4)}v_x^3\right) \\
&\quad \left. + p\left(\frac{v_{xx}^3}{2v_x^3} - \frac{v_{xx}v_{xxx}}{v_x^2} + \frac{v_{xxxx}}{2v_x}\right) + p'v_{xxx} + p''\frac{v_xv_{xx}}{2}\right] + \mathcal{O}(\epsilon^6)
\end{aligned} \tag{4.15}$$

generated by the Hamiltonian

$$K = -\int\left[\frac{1}{24}\epsilon c(v)v_x\log v_x + \epsilon^3\left(\frac{c^2(v)}{5760}\frac{v_{xx}^3}{v_x^3} - \frac{p(v)}{4}\frac{v_{xx}^2}{v_x}\right)\right]\mathrm{d}x + \mathcal{O}(\epsilon^5) \tag{4.16}$$

transforms any monotone solution to the Hopf equation

$$v_t + vv_x = 0$$

to a solution to the perturbed equation

$$\begin{aligned}
&u_t + uu_x + \epsilon^2\partial_x\left\{\tfrac{1}{24}\left(2cu_{xx} + c'u_x^2\right) + \epsilon^2\left[2pu_{xxxx} + 4p'u_xu_{xxx} + 3p'u_{xx}^2\right.\right. \\
&\quad \left.\left. + \left(2p'' - \tfrac{1}{288}cc'''\right)u_x^2u_{xx} - \tfrac{1}{1152}(c'c''' - cc^{IV})u_x^4\right]\right\} + \mathcal{O}(\epsilon^6) = 0,
\end{aligned} \tag{4.17}$$

equivalent to the generic perturbation (2.17). In this formula $c = c(v)$, $p = p(v)$. The same substitution transforms any monotone solution $v(x, s)$ to the equation

$$v_s + f''(v)v_x = 0$$

with an arbitrary $f(v)$ to a solution to the perturbed equation

$$\begin{aligned} &u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0, \qquad H_f = \int h_f \, \mathrm{d}x \\ &h_f = f - \frac{\epsilon^2}{24} c f''' u_x^2 + \epsilon^4 \left[\left(p f''' + \frac{c^2 f^{(4)}}{480} \right) u_{xx}^2 \right. \\ &\qquad \left. - \left(\frac{c c'' f^{(4)}}{1152} + \frac{c c' f^{(5)}}{1152} + \frac{c^2 f^{(6)}}{3456} + \frac{p' f^{(4)}}{6} + \frac{p f^{(5)}}{6} - \frac{c c'''}{3456} f''' \right) u_x^4 \right] \end{aligned} \tag{4.18}$$

five-commuting with (4.17) for an arbitrary function $f(v)$.

Construction of the perturbed solutions in the form (4.1), (4.14) is based on the theory of *quasitriviality transformations* for Hamiltonian PDEs [15, 17].

Definition 4.6. *The substitution*

$$v \mapsto u = v + \sum_{k \geqslant 1} \epsilon^k F_k(v; v_x, \ldots, v^{(m_k)}), \qquad i = 1, \ldots, n, \tag{4.19}$$

is called a quasitriviality transformation for the commutative Lie algebra of perturbed Hamiltonians if for any $H_0^f[v] = \int f(v)\,\mathrm{d}x \in \mathcal{K}_0$, $H^f[u] = \int f(u)\,\mathrm{d}x + \mathcal{O}(\epsilon) \in \mathcal{K}$ *one has*

$$H^f\left[v + \sum_{k \geqslant 1} \epsilon^k F_k(v; v_x, \ldots, v^{(m_k)}) \right] = H_0^f[v]. \tag{4.20}$$

In other words, substitution (4.19) transforms all perturbed Hamiltonian densities to the unperturbed ones, modulo total derivatives. Inverting substitution (4.19) one obtains a practical recipe for computing the perturbed Hamiltonians from the unperturbed one.

If all terms of expansion (4.19) are differential polynomials, then the substitution is a generalized Miura transformation. In this case the perturbed Hamiltonian system together with all its symmetries is equivalent to a hyperbolic system. Such perturbations are called *trivial*. The properties of solutions to trivially perturbed hyperbolic systems essentially do not differ from those for the unperturbed systems.

The existence of a quasitriviality transformation for biHamiltonian integrable PDEs has been established in [15]. In the scalar case $n = 1$ the existence of a quasitriviality transformation was proven in [22] in a very general situation, even without assuming a Hamiltonian structure. For more specific properties of quasitriviality transformations of Hamiltonian scalar PDEs see [23].

Remark 4.7. The perturbative solutions studied in this section are formal power series in ϵ. It is an interesting open problem to prove that for sufficiently small $|t - t_0|$ these series are asymptotic expansions for actual solutions to the perturbed system (4.13) provided analyticity in ϵ on the right-hand side of the system.

5. Critical behaviour of solutions and universality

The solutions to the dispersionless systems (4.2) locally represented in the implicit form (4.4) typically have only finite lifespan. Extending such a solution in x and t from the original point one arrives at points where the implicit function theorem assumptions fail to be true for the system (4.4). At such a point (x_0, t_0) the solution tends to a finite limit v_0 while the derivatives v_x, v_t blow up. These points of weak singularities of solutions to hyperbolic systems are called *critical points* or *points of gradient catastrophe*.

Example 5.1. Let us consider the solution to the Hopf equation

$$v_t + vv_x = 0$$

with a monotone decreasing initial data $\phi(x)(x)$. For small $|t|$ the solution can be determined from the equation

$$x = vt + f(v), \tag{5.1}$$

where $f(v)$ is the function inverse to $\phi(x)$. The point of gradient catastrophe is determined from the system

$$\begin{aligned} x_0 &= v_0 t_0 + f(v_0) \\ 0 &= t_0 + f'(v_0) \\ 0 &= f''(v_0). \end{aligned} \tag{5.2}$$

This is an inflection point on the graph of the solution $v(x, t_0)$. Let us add the *genericity assumption*

$$\kappa := -f'''(v_0) \neq 0 \tag{5.3}$$

(for a monotone decreasing function $f'''(v_0) \leqslant 0$). Near the point of catastrophe for $t < t_0$ the graph of the solution can be approximated by the cubic curve

$$v(x,t) \simeq v_0 + \kappa^{-2/3} V(\kappa[x - x_0 - v_0(t - t_0)], \kappa^{1/3}(t - t_0)), \tag{5.4}$$

where $V(X, T)$ is the real root of the cubic equation

$$X = VT - \tfrac{1}{6}V^3. \tag{5.5}$$

Note that the root is unique for all $X \in \mathbb{R}$ for $T < 0$.

Example 5.2. Solutions to the dispersionless limit

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t + vv_x + u_x &= 0 \end{aligned} \tag{5.6}$$

of the defocusing NLS equation

$$\mathrm{i}\epsilon\, \psi_t + \frac{\epsilon^2}{2}\psi_{xx} - |\psi|^2\psi = 0 \tag{5.7}$$

(cf example 4.4 above) can be found from the system

$$\left.\begin{aligned} x &= vt + f_u \\ 0 &= ut + f_v \end{aligned}\right\}, \tag{5.8}$$

where $f = f(u, v)$ is a solution to the linear PDEs (3.3):

$$f_{vv} - uf_{uu} = 0. \tag{5.9}$$

The Riemann invariants

$$r_\pm = v \pm 2\sqrt{u} \tag{5.10}$$

are real in this case; at a generic critical point only one of them breaks down. Let us assume that the Riemann invariant $r_+(x,t)$ remains smooth at the critical point (x_0, t_0) while the second invariant $r_-(x,t)$ has a generic gradient catastrophe at this point. Then the coordinates of the critical point and the limiting values (u_0, v_0) of the dependent functions can be determined [12] from the system

$$\begin{aligned} x_0 &= v_0 t_0 + f_u^0 \\ 0 &= u_0 t_0 + f_v^0 \\ f_{uv}^0 &- \frac{1}{\sqrt{u_0}} f_{vv}^0 = 0 \\ f_{uvv}^0 &- \frac{1}{\sqrt{u_0}} f_{vvv}^0 - \frac{1}{4u_0} f_{vv}^0 = 0. \end{aligned}$$

Here, as above we denote

$$f_u^0 = f_u(u_0, v_0), \qquad f_{uu}^0 = f_{uu}(u_0, v_0), \quad \text{etc.}$$

For $t < t_0$ the solution can be approximated by the graph of the canonical Whitney singularity

$$\begin{aligned} x_+ &\simeq \alpha \bar{r}_+ \\ x_- &\simeq \beta \bar{r}_+ r_- - \gamma \frac{\bar{r}_-^3}{6}, \end{aligned} \tag{5.11}$$

where

$$x_\pm = (x - x_0) + \left(u_0 + \tfrac{1}{2} v_0^2 \pm v_0 \sqrt{u_0}\right)(t - t_0) \tag{5.12}$$

are the characteristic variables at the critical point

$$\bar{r}_\pm = r_\pm - r_\pm(u_0, v_0). \tag{5.13}$$

The coefficients α, β, γ depend on the choice of the solution (see details in [12]).

Example 5.3. The critical points (x_0, t_0, u_0, v_0) of solution (4.10) to the dispersionless limit of the focusing NLS equation (4.7) are determined from the system

$$\begin{aligned} x_0 &= v_0 t_0 + f_u(u_0, v_0) \\ 0 &= u_0 t_0 + f_v(u_0, v_0) \\ f_{uu}(u_0, v_0) &= f_{vv}(u_0, v_0) = 0, \qquad f_{uv}(u_0, v_0) = -t_0. \end{aligned} \tag{5.14}$$

The critical point is called *generic* if

$$f_{uuv}^0 := f_{uuv}(u_0, v_0) \neq 0. \tag{5.15}$$

Note that the Riemann invariants of the dispersionless system (4.9) are complex conjugate. At the critical point both of them have a 'gradient catastrophe'. Near the generic critical point for $t < t_0$ the solution can be approximated by a complex square root function

$$(u - u_0) + \mathrm{i}\sqrt{u_0}(v - v_0) \simeq -\mathrm{e}^{\mathrm{i}\psi}\left[\sqrt{r^2(t-t_0)^2 + \mathrm{e}^{-\mathrm{i}\psi}(S + \mathrm{i}X)} + r(t - t_0)\right] \tag{5.16}$$

with

$$\begin{aligned} X &= 2r\sqrt{u_0}[(x - x_0) - v_0(t - t_0)], \qquad S = -2ru_0(t - t_0) \\ r\mathrm{e}^{\mathrm{i}\psi} &= \left[f_{uuv}^0 + \mathrm{i}\sqrt{u_0} f_{uuu}^0\right]^{-1}. \end{aligned}$$

The branch of the square root is obtained by the analytic continuation of the positive branch from the positive real half-axis.

We study the behaviour of a generic perturbed solution near a critical point of the unperturbed one. For brevity we will call it *a critical behaviour*.

In what follows it will be assumed that the dispersionless limit (4.2) is integrable, i.e. the eigenvalues of the coefficient matrix $A^i_j(v)$ are pairwise distinct (but not necessarily real) and the system admits Riemann invariants that may occur in complex conjugated pairs. For example, any generic system with $n \leqslant 2$ enter in this class.

In contrast, we do not assume integrability of the perturbed Hamiltonian system. The idea of universality of critical behaviour for solutions to Hamiltonian PDEs first suggested in [11] states that the leading term of the asymptotic expansion of a perturbed solution near the critical point of the unperturbed one is essentially independent from the choice of a generic solution and, moreover, independent from the choice of a generic Hamiltonian perturbation. This leading-order term is described, up to simple affine transformations of independent variables, by certain particular solutions to Painlevé equations and their generalizations.

Example 5.4. For a generic Hamiltonian perturbation (2.17) of the Hopf equation the conjectural universality type is given by the following formula [11]:

$$u(x,t) = v_0 + \gamma\, \epsilon^{2/7} U\left(\frac{x - x_0 - v_0(t-t_0)}{\alpha\, \epsilon^{6/7}}, \frac{t-t_0}{\beta\, \epsilon^{4/7}}\right) + \mathcal{O}(\epsilon^{4/7}), \tag{5.17}$$

where α, β, γ are some nonzero constants and $U = U(X,T)$ is the real smooth solution to the so-called P_I^2 equation (also called the fourth-order analogue of the Painlevé-I)

$$X = TU - \left[\frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2UU'') + \frac{1}{240}U^{IV}\right], \qquad U' = \frac{dU}{dX}, \quad \text{etc.} \tag{5.18}$$

The existence of such a solution has been proven by Claeys and Vanlessen [8]. It satisfies the boundary conditions

$$U \sim -\sqrt[3]{6X}, \qquad |X| \to \infty.$$

Conjecture (5.17) was proven by Claeys and Grava [5] for the particular case of analytic solutions to the KdV equation in the soliton-free sector.

The proof of (5.17) for more general class of perturbations remains open. We also expect the same special function $U(X, T)$ to be involved in the description of critical behaviour of solutions to more general systems of Hamiltonian PDEs near a point of gradient catastrophe of one of the Riemann invariants of the unperturbed system, for example, for defocusing NLS, see [12, 13] for a more extensive discussion of this conjecture.

We do not discuss in the present paper the behaviour of solutions to the KdV equation on the boundary of the oscillatory zone. Important results in this direction bave been obtained in references [6, 7].

Example 5.5. Let us now consider the critical behaviour in the situation of a simultaneous gradient catastrophe of a pair of complex conjugated Riemann invariants $r_+(x,t)$, $r_-(x,t) = r_+^*(x,t)$ of the dispersionless limit (4.2). In this case another special function conjecturally enters into the asymptotic description of the critical behaviour. This function $W = W(Z)$ of a complex variable Z is defined as a particular solution to the Painlevé-I equation

$$W_{ZZ} = 6W^2 - Z. \tag{5.19}$$

This particular solution was discovered by Boutroux in 1913 [3]; it is called the *tritronquée* solution. It is uniquely characterized by the following property: the number of poles of this solution in the sector

$$|\arg Z| < \frac{4\pi}{5} - \delta \tag{5.20}$$

is finite for any positive δ.

A stronger property of the *tritronquée* solution was conjectured in [14]: we expect the *tritronquée* solution to be analytic in any sector of the form (5.20). This conjecture remains open; further arguments supporting this conjecture have been presented in [26–28]. The main motivation of the analyticity conjecture proposed in [14] was in the following asymptotic formula describing the critical behaviour of solutions to the focusing NLS equation. It can be written in a simple form considering the complex Riemann invariant $r_+(x,t)$. Near the critical point the following asymptotic formula was conjectured in [14]

$$\begin{aligned} r_+(x,t) &= r_+(u_0, v_0) + \gamma\epsilon^{2/5} W\left(\frac{x_+}{\alpha\epsilon^{4/5}}\right) + \mathcal{O}(\epsilon^{4/5}) \\ x_+ &= x - x_0 + \left(u_0 + \frac{1}{2}v_0^2 + \mathrm{i}v_0\sqrt{u_0}\right)(t - t_0). \end{aligned} \tag{5.21}$$

The complex constant α depending on the choice of a particular solution is such that the argument of the function W belongs to the analyticity sector (5.20).

The proof of the asymptotic formula (5.21) remains an open problem, although there are further supporting arguments proposed in [2]. The main difficulty in the proof of (5.21) for solutions to the focusing NLS equation is in the asymptotic analysis of the scattering data for the non-self-adjoint Zakharov–Shabat operator playing the crucial role in the inverse spectral transform for the focusing NLS equation.

We expect that the same *tritronquée* solution describes the critical behaviour for more general Hamiltonian perturbed PDEs near a point of a simultaneous gradient catastrophe of a pair of complex conjugated Riemann invariants.

Let us compare the Hamiltonian critical behaviour with the one known [19] from the study of *dissipative* perturbations of hyperbolic PDEs. Let us consider a nonlinear heat equation

$$u_t + a(u)u_x = \epsilon u_{xx}. \tag{5.22}$$

Here as above ϵ is a positive small parameter; $a(u)$ is an arbitrary smooth function. Let $v = v(x,t)$ be a generic solution to the equation

$$v_t + a(v)v_x = 0 \tag{5.23}$$

with a monotone decreasing initial data defined for $x \in \mathbb{R}$ defined for $t < t_0$ (let t_0 be positive). Let us assume that at the critical point $(x = x_0, t = t_0)$ the solution has a generic gradient catastrophe, i.e. the graph of the function $v(x, t_0)$ has a non-degenerate inflection point at $x = x_0$. Denote $v_0 = v(x_0, t_0)$ and put $a_0 = a(v_0)$. Then, near the point of catastrophe the solution to the Cauchy problem

$$u(x, 0; \epsilon) = v(x, 0)$$

to the nonlinear heat equation admits the following asymptotic expansion:

$$u(x,t;\epsilon) = u_0 + \gamma\,\epsilon^{1/4}\Gamma\left(\frac{x - x_0 - a_0(t - t_0)}{\alpha\,\epsilon^{3/4}}, \frac{t - t_0}{\beta\,\epsilon^{1/2}}\right) + \mathcal{O}(\epsilon^{1/2}), \tag{5.24}$$

where $\Gamma(\xi,\tau)$ is the logarithmic derivative of a Pearcey integral

$$\Gamma(\xi,\tau) = -2\frac{\partial}{\partial\xi}\log\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{8}(z^4 - 2z^2\tau + 4z\xi)}\,\mathrm{d}z. \tag{5.25}$$

Here α, β, γ are some nonzero constants depending on the choice of the solution. Note that the function $\Gamma(\xi, \tau)$ is a particular solution to the Burgers equation

$$\Gamma_\tau + \Gamma\Gamma_\xi = \Gamma_{\xi\xi}. \tag{5.26}$$

For $\epsilon \to 0$ solution (5.24) tends to a discontinuous function (shock wave).

We see that the leading term of the asymptotic expansion of the (5.24) is essentially independent on the choice of a generic solution. It will be of interest to derive a similar asymptotics (5.24) for more general dissipative perturbations

$$u_t + a(u)u_x = \epsilon\left[p(u)u_{xx} + q(u)u_x^2\right] + \mathcal{O}(\epsilon^2), \qquad p(u) > 0 \tag{5.27}$$

of the transport equation (5.23).

Remark 5.6. The assumption of integrability of the dispersionless limit is essential for the formulation of the universality conjectures. In order to go beyond the integrable case the nature of weak singularities of a generic hyperbolic system with $n \geqslant 3$ components has to be clarified. To the best of our knowledge, no general picture describing the local structure of singularities is available so far.

Due to volume limitations we do not consider here the behaviour of perturbed solutions of certain Hamiltonian PDEs near the boundary between the regions of regular and oscillatory behaviour. We refer the reader to the papers [1, 6].

In the present paper we did not touch the theory of Hamiltonian PDEs with more than one spatial dimension. This is an immense domain of researches; the observations of the recent papers [4, 21, 24, 25] can perhaps be boiled down to make future foundations of the multi-dimensional theory.

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