SIAM J. APPL. MATH.
Vol. 71, No. 4, pp. 983–1008

NUMERICAL STUDY OF BREAKUP IN GENERALIZED KORTEWEG–DE VRIES AND KAWAHARA EQUATIONS*

B. DUBROVIN†, T. GRAVA‡, AND C. KLEIN§

Abstract. This article is concerned with a conjecture in [B. Dubrovin, *Comm. Math. Phys.*, 267 (2006), pp. 117–139] on the formation of dispersive shocks in a class of Hamiltonian dispersive regularizations of the quasi-linear transport equation. The regularizations are characterized by two arbitrary functions of one variable, where the condition of integrability implies that one of these functions must not vanish. It is shown numerically for a large class of equations that the local behavior of their solution near the point of gradient catastrophe for the transport equation is described by a special solution of a Painlevé-type equation. This local description holds also for solutions to equations where blowup can occur in finite time. Furthermore, it is shown that a solution of the dispersive equations away from the point of gradient catastrophe is approximated by a solution of the transport equation with the same initial data, modulo terms of order ϵ^2, where ϵ^2 is the small dispersion parameter. Corrections up to order ϵ^4 are obtained and tested numerically.

Key words. generalized Korteweg–de Vries equations, Kawahara equations, dispersive shocks, multiscale analysis

AMS subject classifications. Primary, 65M70; Secondary, 65L05, 65M20

DOI. 10.1137/100819783

1. Introduction.

Many wave phenomena in dispersive media with negligible dissipation, in hydrodynamics, nonlinear optics, and plasma physics, are described by nonlinear dispersive partial differential equations (PDEs). These equations are also mathematically challenging since the solutions can have highly oscillatory regions and blowup even for smooth initial data (see, e.g., [33], [19], [21]).

This article is concerned with a conjecture in [10] on the formation of dispersive shocks [19], [27], [31] in a class of Hamiltonian regularizations of the quasi-linear transport equation

$$u_t + a(u)u_x = 0, \quad a'(u) \neq 0, \quad u, x \in \mathbb{R}. \tag{1.1}$$

In the present paper we will consider general Hamiltonian perturbations of (1.1) up to fourth order in a small dispersion parameter $0 < \epsilon \ll 1$. They can be written in the form of a conservation law,

$$\begin{aligned} &u_t + a(u)u_x + \epsilon^2\partial_x\big\{b_1(u)u_{xx} + b_2(u)u_x^2 + \epsilon\left[b_3(u)u_{xxx} + b_4(u)u_{xx}u_x + b_5(u)u_x^3\right] \\ &\quad + \epsilon^2\left[b_6(u)u_{xxxx} + b_7(u)u_{xxx}u_x + b_8(u)u_{xx}^2 + b_9(u)u_{xx}u_x^2 + b_{10}(u)u_x^4\right]\big\} = 0, \end{aligned} \tag{1.2}$$

*Received by the editors January 4, 2011; accepted for publication (in revised form) March 28, 2011; published electronically July 7, 2011. This work was supported by the project FroM-PDE funded by the European Research Council through the Advanced Investigator Grant Scheme.
http://www.siam.org/journals/siap/71-4/81978.html

†Laboratory of Geometric Methods in Mathematical Physics, Moscow State University "M.V. Lomonosov," Russia. Current address: SISSA, Via Bonomea 265, I-34136 Trieste, Italy (dubrovin@sissa.it).

‡SISSA, Via Bonomea 265, I-34136 Trieste, Italy (grava@sissa.it).

§Institut de Mathématiques de Bourgogne, Université de Bourgogne, 9 avenue Alain Savary, 21078 Dijon Cedex, France (christian.klein@u-bourgogne.fr). This author's work was partially supported by the Conseil Régional de Bourgogne via a FABER grant and by the ANR via the program ANR-09-BLAN-0117-01.

where the coefficients $b_1(u), \dots, b_{10}(u)$ are smooth functions satisfying certain constraints following from the existence of a Hamiltonian representation

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0$$

(see Corollary 2.4 below). Here and below we use the notation

$$\partial_x = \frac{\partial}{\partial x}.$$

This class of equations contains important equations such as the Korteweg–de Vries (KdV) equation $u_t+6uu_x+\epsilon^2 u_{xxx} = 0$ and its generalizations, the Kawahara equation, and the Camassa–Holm equation in an asymptotic sense (see [10]).

Up to certain equivalencies the Hamiltonian regularizations of (1.1) are characterized by two free functions $c(u)$ and $p(u)$:

(1.3)
$$u_t + a(u)u_x + \epsilon^2\partial_x\left[c\,a' u_{xx} + \frac{1}{2}(c\,a')' u_x^2\right] + \epsilon^4\partial_x\left[\left(2p\,a' + \frac{3}{5}c^2 a''\right)u_{xxxx} + \cdots\right] = 0.$$

Two equations of the form (1.3) with the same invariants $c(u)$ and $p(u)$ commute, up to order $\mathcal{O}(\epsilon^6)$,

$$(u_t)_{\tilde t} - (u_{\tilde t})_t = \mathcal{O}\left(\epsilon^6\right),$$

where for an arbitrary function $\tilde a = \tilde a(u)$

$$u_{\tilde t} + \tilde a(u)u_x + \epsilon^2\partial_x\left[c\,\tilde a' u_{xx} + \frac{1}{2}(c\,\tilde a')' u_x^2\right] + \epsilon^4\partial_x\left[\left(2p\,\tilde a' + \frac{3}{5}c^2 \tilde a''\right)u_{xxxx} + \cdots\right] = 0.$$

In this paper the analysis of [10] up to order ϵ^4 is extended to higher orders of ϵ. Our analysis suggests that the only obstruction to the functions $c(u)$ and $p(u)$ by the condition of integrability is the condition that $c(u)$ must not vanish.

We then proceed to the study of the *critical behavior* of solutions to (1.2). Namely, let (x_c, t_c, u_c) be a point of *gradient catastrophe* of a solution $u^0(x,t)$ to (1.1) specified by an initial value $u^0(x,0) = \phi(x)$. This means that the solution is a smooth function of (x,t) for sufficiently small $|x - x_c|$ and $t - t_c < 0$. Moreover, there exists the limit

$$\lim_{x\to x_c,\, t\to t_c-0} u^0(x,t) = u_c,$$

but the derivatives $u_x^0(x,t)$, $u_t^0(x,t)$ blow up at the point. The *universality conjecture* of [10] says that, up to shifts, Galilean transformations, and rescalings, the behavior at the point of gradient catastrophe of a solution to (1.2) with the *same* ϵ-independent initial data $\phi(x)$ essentially depends neither on the choice of the generic solution nor on the choice of the generic equation. Moreover, the generic solution near this point (x_c, t_c, u_c) is given by

$$u(x,t,\epsilon) \simeq u_c + \alpha\,\epsilon^{2/7} U\left(\frac{x - x_c - a_0(t-t_c)}{\beta\,\epsilon^{6/7}}; \frac{t-t_c}{\gamma\,\epsilon^{4/7}}\right) + O\left(\epsilon^{4/7}\right), \qquad (1.4)$$

where $a_0 = a(u_c)$ and the constants α, β, γ depend on the choice of the generic equation and the solution

$$\alpha = \left(\frac{12b_1^0}{a_0' k^2}\right)^{1/7},$$

$$\beta = \left(\frac{12^3 k\,(b_1^0)^3}{{a_0'}^3}\right)^{1/7}, \tag{1.5}$$

$$\gamma = \left(\frac{12^2 k^3 (b_1^0)^2}{{a_0'}^9}\right)^{1/7}.$$

Here $a_0' = a'(u_c)$, $b_1^0 = b_1(u_c)$; it is assumed that $b_1^0 \neq 0$. The constant k in these formulae is inversely proportional to the "strength" of the breakup of the dispersionless solution $u^0(x,t)$,

$$k = -6 \lim_{x \to x_c} \frac{x - x_c}{\left(u^0(x,t_c) - u_c\right)^3}, \tag{1.6}$$

where we assume that $k \neq 0$ (another genericity hypothesis) and $a_0' k > 0$.

The function $U = U(X;T)$, $(X,T) \in \mathbb{R}^2$, is defined as the unique real smooth solution to the fourth order ODE [5], [23]

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}U_X^2 + \frac{1}{12}U\,U_{XX} + \frac{1}{240}U_{XXXX}\right], \tag{1.7}$$

which is the second member of the Painlevé I hierarchy. We will call this equation PI2.

The relevant solution is characterized by the asymptotic behavior

$$U(X,T) = -(6X)^{\frac{1}{3}} - \frac{2^{2/3}T}{(3X)^{\frac{1}{3}}} + O(X^{-\frac{5}{3}}), \quad X \to \pm\infty, \tag{1.8}$$

for each fixed $T \in \mathbb{R}$. The existence of a smooth solution of (1.7) for all real X, T satisfying (1.8) has been proved by Claeys and Vanlessen [7].

Observe that the principal term of the asymptotics (1.4) depends only on the order ϵ^2 regularization. In the present paper we will numerically analyze, in particular, the influence of the higher order corrections[1] on the local behavior of solutions to (1.2) near the point of catastrophe.

First numerical tests of the PI2 asymptotic description at the critical point for the class of PDEs in [10] have been presented in [18] for the KdV and the Camassa–Holm equation. In [6] a rigorous proof of the asymptotic behavior (1.4) has been obtained for the KdV equation, namely (1.3) with $a(u) = u$, $c(u) = 1$, and $p(u) = 0$.

In this paper we generalize the numerical investigation of [18] to a larger class of equations which include the generalized KdV equation, the Kawahara equations with a dispersion of fifth order, and the second equation in the KdV hierarchy; see [1], [2] for the importance of these equations in applications.

We comment on the formation of blowup and on the role of integrability in the formation of oscillatory regions. In particular we show the differences in the formation

[1] One should also take into account [10] that the actually small parameter of the expansion (1.4) is $\left(12 b_1^0 \epsilon^2\right)^{1/7}$. In other words, the asymptotic expansion (1.4) makes sense only under the assumption $b_1^0 \epsilon^2 \ll \frac{1}{12}$, in agreement with [3].

of dispersive shock waves between integrable and nonintegrable cases. The KdV equation has been extensively studied numerically in [17].

Then we show numerically that the solution of the dispersive equation (1.2) converges to the solution of the dispersionless equation (1.1) for times much smaller than the time for the point of gradient catastrophe at a rate of order ϵ^2. Finally, we show that the solution of the dispersive equation (1.2) is well approximated as a series in even powers of ϵ in terms of the solution of the dispersionless equation (1.1) up to order ϵ^4 by the so-called quasi triviality transformation [10] away from the point of gradient catastrophe. Such an approximation has already been obtained for conservation laws with positive viscosity [16]. Furthermore the existence of an expansion in even powers of ϵ has already appeared and been proved in the context of large N expansions in Hermitian matrix models [4], [15].

In this paper we consider local Hamiltonian perturbations of the Riemann wave equation. Non-Hamiltonian perturbations, especially of "random" type, introduce in general dissipation. Systems with dissipation and dispersion are known in the literature as the KdV–Burgers equation; see, e.g., [28]. In such systems, there is a competition between dissipative and dispersive effects. There is, in principle, no problem in studying such equations numerically; the problem is mainly which type of effects one is interested in. The focus of the present paper is on local dispersive effects and on their asymptotic description at the point of gradient catastrophe.

The paper is organized as follows. In section 2 we briefly review the results of [10]. In section 3 we discuss higher order in ϵ regularizations of (1.1) and obstructions on the function $c(u)$ by the condition of integrability. In section 4 we study a quasi triviality transformation for the solution before the critical time. A numerical study of the applicability of the conjecture to generalized KdV equations is given in section 5. We also comment on the possibility of blowup. In section 6 the conjecture is tested numerically for equations with high order dispersion such as the Kawahara equation. Differences in the formation of rapid oscillations in the solutions to integrable and nonintegrable equations are studied. In section 7 we add some concluding remarks. Details about the used numerical methods are given in the appendix.

2. Hamiltonian PDEs and their invariants. In this paper we mainly study scalar Hamiltonian PDEs of order at most five. They are written in the form of a conservation law:

$$u_t + \partial_x \varphi(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \epsilon^3 u_{xxx}, \epsilon^4 u_{xxxx}) = 0, \tag{2.1}$$

where

$$\varphi = \frac{\delta H}{\delta u(x)}, \qquad H = \int h(u, \epsilon\, u_x, \epsilon^2 u_{xx})\, dx. \tag{2.2}$$

Recall that the Euler–Lagrange derivative is defined by

$$\frac{\delta H}{\delta u(x)} = \frac{\partial h}{\partial u} - \partial_x \frac{\partial h}{\partial u_x} + \partial_x^2 \frac{\partial h}{\partial u_{xx}} - \cdots . \tag{2.3}$$

Here and in what follows, the integral of a differential polynomial is understood, in the spirit of *formal calculus of variations*, as the equivalence class of the polynomial modulo the image of the operator of the total x-derivative,

$$\partial_x h = u_x \frac{\partial h}{\partial u} + u_{xx} \frac{\partial h}{\partial u_x} + \cdots . \tag{2.4}$$

It worth recalling that a differential polynomial $p(u; u_x, \dots, u^{(m)})$ belongs to $\operatorname{Im} \partial_x$ iff

$$\frac{\delta P}{\delta u(x)} \equiv 0, \qquad P = \int p(u; u_x, \dots, u^{(m)})\, dx. \tag{2.5}$$

The Poisson bracket of two local functionals H, F associated with (2.1), (2.2), is a local functional of the form

$$\{H, F\} = \int \frac{\delta H}{\delta u(x)} \frac{d}{dx} \frac{\delta F}{\delta u(x)}\, dx. \tag{2.6}$$

Applying the criterion (2.5), one obtains the following useful statement.

PROPOSITION 2.1. *Two Hamiltonians*

$$H = \int h(u; u_x, u_{xx}, \dots)\, dx \quad \textit{and} \quad F = \int f(u; u_x, u_{xx}, \dots)\, dx$$

commute,

$$\{H, F\} = 0,$$

iff their densities satisfy the equation

$$\frac{\delta}{\delta u(x)} \int \frac{\delta H}{\delta u(x)} \frac{d}{dx} \frac{\delta F}{\delta u(x)}\, dx \equiv 0. \tag{2.7}$$

Example 2.2. Two Hamiltonians of the form

$$H = \int h(u)\, dx, \qquad F = \int f(u)\, dx$$

always commute. Indeed,

$$\int \frac{\delta H}{\delta u(x)} \frac{d}{dx} \frac{\delta F}{\delta u(x)}\, dx = \int h'(u) f''(u) u_x\, dx.$$

Here $h'(u) = \frac{dh(u)}{du}$, etc. So

$$\frac{\delta}{\delta u(x)} \int \frac{\delta H}{\delta u(x)} \frac{d}{dx} \frac{\delta F}{\delta u(x)}\, dx = \frac{\partial}{\partial u} \left(h'(u) f''(u) u_x\right) - \frac{\partial}{\partial x} \left(h'(u) f''(u)\right) \equiv 0.$$

Let us now derive necessary and sufficient conditions for existence of a Hamiltonian representation (2.1), (2.2) of an equation of the form (1.2).

LEMMA 2.3. *Equation* (2.1) *can be written in the Hamiltonian form* (2.2) *iff the function φ satisfies the following two constraints:*

$$\begin{aligned} \frac{\partial \varphi}{\partial u_x} &= \partial_x \left[\frac{\partial \varphi}{\partial u_{xx}} - \frac{1}{2} \partial_x \frac{\partial \varphi}{\partial u_{xxx}} \right], \\ \frac{\partial \varphi}{\partial u_{xxx}} &= 2 \partial_x \frac{\partial \varphi}{\partial u_{xxxx}}. \end{aligned} \tag{2.8}$$

Proof. According to the classical Helmholtz criterion (see [9]) the function $\varphi(u, u_x, u_{xx}, \dots)$ can be locally represented as the variational derivative of some functional $H = \int h(u, u_x, \dots)\, dx$ iff it satisfies the following system of constraints:

$$\frac{\partial \varphi}{\partial u^{(i)}} = (-1)^i \sum_{m \geq 0} \frac{(m+i)!}{i!\; m!} (-\partial_x)^m \frac{\partial \varphi}{\partial u^{(i+m)}}, \quad i = 0, 1, \dots. \tag{2.9}$$

For the particular case under consideration, the equations (2.9) reduce to (2.8). ∎

Applying the lemma to a PDE (2.1) written in the form of the weak dispersion expansion, one arrives at the following claim.

COROLLARY 2.4. *The equation*

$$\begin{aligned} u_t + a(u)u_x + \partial_x\big\{&\epsilon\, b_0(u)u_x + \epsilon^2 \left[b_1(u)u_{xx} + b_2(u)u_x^2\right] \\ &+ \epsilon^3 \left[b_3(u)u_{xxx} + b_4(u)u_{xx}u_x + b_5(u)u_x^3\right] \\ &+ \epsilon^4 \left[b_6(u)u_{xxxx} + b_7(u)u_{xxx}u_x + b_8(u)u_{xx}^2 + b_9(u)u_{xx}u_x^2 + b_{10}(u)u_x^4\right]\big\} = 0 \end{aligned} \tag{2.10}$$

is Hamiltonian iff the coefficients $b_0, \dots, b_{10}$ *satisfy*

$$\begin{aligned} b_0 &= 0, \\ b_2 &= \frac{1}{2} b_1', \\ b_3 &= 0, \\ b_5 &= \frac{1}{3} b_4', \\ b_7 &= 2 b_6', \\ b_8 &= \frac{3}{2} b_6', \\ b_{10} &= \frac{1}{4} b_9'. \end{aligned}$$

The Hamiltonian equations (2.1) are considered modulo canonical transformations written in the form of a time-ϵ shift,

$$u(x) \mapsto \tilde{u}(x) = u(x) + \epsilon\{u(x), K\} + \frac{\epsilon^2}{2!}\{\{u(x), K\}, K\} + \cdots, \tag{2.11}$$

generated by a Hamiltonian

$$K = \int k(u, \epsilon\, u_x, \dots)\, dx. \tag{2.12}$$

The transformations (2.11) preserve the canonical form of the Poisson bracket (2.6). Two Hamiltonian equations are called *equivalent* if they are related by a canonical transformation of the form (2.11), (2.12). For example, the degree 3 terms in a Hamiltonian PDE of the form (2.10) can be eliminated by a transformation (2.11) if $a'(u) \neq 0$. Indeed, it suffices to choose the generating Hamiltonian in the form

$$K = -\frac{\epsilon^2}{6} \int \frac{b_4(u)}{a'(u)} u_x^2\, dx.$$

The following lemma describes a normal form of Hamiltonians of order 4 (cf. [10]) with respect to transformations (2.11).

LEMMA 2.5. *Any Hamiltonian equation of the form* (2.10) *with* $a'(u) \neq 0$ *is equivalent to*

$$u_t + a(u)u_x + \epsilon^2\partial_x\Big[b_1 u_{xx} + \frac{1}{2}b_1' u_x^2 + \epsilon^2\Big(b_6\, u_{xxxx} + 2b_6' u_{xxx}u_x + \frac{3}{2}b_6' u_{xx}^2 + b_9 u_{xx}u_x^2 + \frac{1}{4}b_9' u_x^4\Big)\Big] = 0. \tag{2.13}$$

The Hamiltonian PDEs (2.13) *and*

$$u_t + \tilde{a}(u)u_x + \epsilon^2\partial_x\Big[\tilde{b}_1 u_{xx} + \frac{1}{2}\tilde{b}_1' u_x^2 + \epsilon^2\Big(\tilde{b}_6\, u_{xxxx} + 2\tilde{b}_6' u_{xxx}u_x + \frac{3}{2}\tilde{b}_6' u_{xx}^2 + \tilde{b}_9 u_{xx}u_x^2 + \frac{1}{4}b_9' u_x^4\Big)\Big] = 0 \tag{2.14}$$

are equivalent iff

$$\tilde{a} = a, \quad \tilde{b}_1 = b_1, \quad \tilde{b}_6 = b_6. \tag{2.15}$$

Proof. We have already proved that the coefficients of degree 3 in ϵ can be eliminated by a canonical transformation of the form (2.11). One can easily see that the coefficients a, b_1, and b_6 are invariant with respect to these transformations. Two Hamiltonians of the form

$$H = \int \left[f - \frac{\epsilon^2}{2} b_1 u_x^2 + \frac{\epsilon^4}{2} b_6 u_{xx}^2 - \frac{\epsilon^4}{12} b_9 u_x^4 \right] dx$$

and

$$\tilde{H} = \int \left[f - \frac{\epsilon^2}{2} b_1 u_x^2 + \frac{\epsilon^4}{2} b_6 u_{xx}^2 - \frac{\epsilon^4}{12} \tilde{b}_9 u_x^4 \right] dx,$$

generating the flows (2.13) and (2.14) with the same coefficients $\tilde{a} = a$, $\tilde{b}_1 = b_1$, $\tilde{b}_6 = b_6$ but with different $\tilde{b}_9 \neq b_9$, are related by a canonical transformation (2.11) with

$$K = \frac{\epsilon^3}{24} \int \frac{\tilde{b}_9 - b_9}{a'} u_x^3 dx. \qquad \square$$

Thus the coefficients a, b_1, b_6 are *invariants* of the Hamiltonian PDE (2.13).

As it was discovered in [10], any Hamiltonian PDE of the form (2.13) is integrable at the order ϵ^4 approximation. More precisely, assuming $a' \neq 0$, let us replace the invariants $b_1 = b_1(u)$ and $b_6 = b_6(u)$ with

$$c = \frac{b_1}{a'}, \quad p = \frac{b_6}{2a'} - \frac{3}{10} b_1^2 \frac{a''}{a'^3}. \tag{2.16}$$

Then (2.13) is equivalent to the PDE

$$u_t + a(u)u_x + \epsilon^2\partial_x\left[c\, a' u_{xx} + \frac{1}{2}(c\, a')' u_x^2\right] + \epsilon^4\partial_x\left[\left(2p\, a' + \frac{3}{5}c^2 a''\right) u_{xxxx} + \cdots\right] = 0 \tag{2.17}$$

with the Hamiltonian

$$H_f = \int \Big\{ f - \frac{\epsilon^2}{2} c\, f''' u_x^2 + \epsilon^4 \Big[\Big(p\, f''' + \frac{3}{10} c^2 f^{(4)} \Big) u_{xx}^2 \tag{2.18}$$
$$- \frac{1}{6} \Big(\frac{3c\, c'' f^{(4)} + 3c\, c' f^{(5)} + c^2 f^{(6)}}{4} + p' f^{(4)} + p\, f^{(5)} \Big) u_x^4 \Big] \Big\} \, dx,$$

where, as above,

$$f''(u) = a(u).$$

The approximate integrability means that, fixing the functional parameters $c = c(u)$, $p = p(u)$, one obtains a family of Hamiltonians satisfying

$$\{H_f, H_g\} = \mathcal{O}\left(\epsilon^6\right) \tag{2.19}$$

for an arbitrary pair of smooth functions $f = f(u)$, $g = g(u)$. In particular, choosing $f(u) = \frac{1}{6}u^3$, one obtains the Hamiltonian

$$H = \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{2} u_x^2 + \epsilon^4 p(u) u_{xx}^2 \right] dx \tag{2.20}$$

of a general order 4 dispersive regularization of the Hopf equation

$$u_t + u\, u_x + \epsilon^2 \partial_x \left[c\, u_{xx} + \frac{1}{2} c' u_x^2 \right] \tag{2.21}$$
$$+ \epsilon^4 \partial_x \left[2p\, u_{xxxx} + 4p' u_{xxx} u_x + 3p' u_{xx}^2 + 2p'' u_{xx} u_x^2 \right] = 0$$

introduced in [10].[2]

More generally [11], [12], we call a perturbation

$$H = H_0 + \epsilon\, H_1 + \epsilon^2 H_2 + \cdots$$

of the Hopf Hamiltonian

$$H_0 = \int \frac{u^3}{6} \, dx$$

N-integrable if, for any smooth function $f = f(u)$, there exists a perturbed Hamiltonian

$$H_f = H_f^0 + \sum_{k \geq 1} \epsilon^k H_f^k$$

such that for $f = \frac{u^3}{6}$ the Hamiltonian H_f coincides with H and, moreover, for any pair of functions f, g the Hamiltonians H_f, H_g satisfy

$$\{H_f, H_g\} = \mathcal{O}\left(\epsilon^{N+1}\right).$$

For example, the perturbed Hamiltonian (2.20) is 5-integrable. The commuting Hamiltonians have the form (2.18). In the next section we will discuss the problem of constructing higher integrable perturbations of (2.20).

[2]In the present paper we use a different normalization, $c(u) \mapsto 12c(u)$.

3. On obstacles to integrability. We will now study the possibility of extending the commuting Hamiltonians (2.18) to the next order of the perturbative expansion. In particular, we will check whether obstructions appear on the functions $c(u)$ and $p(u)$ by the condition of integrability in higher orders.[3]

THEOREM 3.1. (1) *Any order 6 perturbation of the cubic Hamiltonian* $H_0 = \int \frac{u^3}{6}\, dx$ *can be represented in the form*

$$H = \int \left[\frac{u^3}{6} - \frac{\epsilon^2}{2} c(u) u_x^2 + \epsilon^4 p(u) u_{xx}^2 - \epsilon^6 \left(\alpha(u) u_{xxx}^2 + \beta(u) u_{xx}^3 \right) \right] dx. \tag{3.1}$$

Such a perturbation is 7-integrable for arbitrary functional parameters $c = c(u)$, $p = p(u)$, $\alpha = \alpha(u)$, $\beta = \beta(u)$.

(2) *The perturbation* (3.1) *can be extended to a 9-integrable one iff* $c(u) \neq 0$ *and*

$$\alpha = \frac{1}{28} \left[80 \frac{p^2}{c} - 67\, p\, c' + 33\, c\, p' + 12\, c\, {c'}^2 - 9\, c^2\, c'' \right]. \tag{3.2}$$

Proof. A general order 6 perturbation of the cubic Hamiltonian H_0 must have the form

$$H = \int \left\{ \frac{u^3}{6} - \frac{\epsilon^2}{2} c(u) u_x^2 + \epsilon^4 p(u) u_{xx}^2 \right.$$
$$\left. - \epsilon^6 \left[\alpha(u) u_{xxx}^2 + \beta(u) u_{xx}^3 + \gamma(u) u_{xx}^2 u_x^2 + \delta(u) u_x^6 \right] \right\} dx.$$

The last two terms can be eliminated by a canonical transformation,

$$H \mapsto H - \epsilon \{H, F\} + \cdots,$$

with

$$F = \int \epsilon^5 \left(\frac{1}{6} \gamma(u) u_{xx}^2 u_x + \frac{1}{4} \delta(u) u_x^5 \right) dx.$$

For an arbitrary function $f = f(u)$ the density of a Hamiltonian

$$H_f = \int h_f\, dx$$

commuting with (3.1) modulo $\mathcal{O}\left(\epsilon^6\right)$ must have the form

$$h_f = f - \frac{\epsilon^2}{2} c\, f''' u_x^2 + \epsilon^4 \left[\left(p\, f''' + \frac{3}{10} c^2 f^{(4)} \right) u_{xx}^2 - \frac{1}{6} \left(p' f^{(4)} + \frac{3}{4} c\, c'' f^{(4)} + p\, f^{(5)} \right.\right.$$
$$\left.\left. + \frac{3}{4} c\, c' f^{(5)} + \frac{1}{4} c^2 f^{(6)} \right) u_x^4 \right] - \epsilon^6 \left[\alpha_f(u) u_{xxx}^2 + \beta_f(u) u_{xx}^3 + \gamma_f(u) u_{xx}^2 u_x^2 + \delta_f(u) u_x^6 \right] \tag{3.3}$$

with some smooth functions $\alpha_f = \alpha_f(u)$, $\beta_f = \beta_f(u)$, $\gamma_f = \gamma_f(u)$, $\delta_f = \delta_f(u)$ depending on f. From the commutativity

$$\{H, H_f\} = \mathcal{O}\left(\epsilon^7\right)$$

[3]Cf. [26] where an alternative approach to asymptotic integrability has been developed.

one uniquely determines these coefficients:

$$\alpha_f = \alpha\, f''' + \left(\frac{8}{7}c\,p + \frac{3}{70}c^2c'\right) f^{(4)} + \frac{9}{70}c^3 f^{(5)},$$

$$\beta_f = \beta\, f''' - \left(\frac{3}{2}\alpha + \frac{253p\,c' + 169c\,p'}{168} + \frac{c\,c'^2}{35} + \frac{5}{56}c^2c''\right) f^{(4)}$$
$$- \left(\frac{29}{21}c\,p + \frac{31}{70}c^2c'\right) f^{(5)} - \frac{c^3 f^{(6)}}{7},$$

$$\gamma_f = \left(\frac{3}{7}\beta - \frac{6}{7}\alpha' + \frac{3}{35}(c'^3 - c^2c''' - 3c\,c'c'') + c'p' - \frac{47}{14}p\,c'' - c\,p''\right) f^{(4)}$$
$$- \left(2\alpha + \frac{37}{14}p\,c' + \frac{3}{35}(c\,c'^2 + 11c^2c'') + \frac{8}{7}c\,p'\right) f^{(5)}$$
$$- \frac{1}{14}\left(23c\,p + 9c^2c'\right) f^{(6)} - \frac{3}{20}c^3 f^{(7)},$$

$$\delta_f = \left(\frac{1}{10}p'c''' + \frac{10c\,c''c''' + 7c\,c'c^{(4)} + c^2c^{(5)}}{40} + \frac{2}{15}p\,c^{(4)} + \frac{1}{60}c\,p^{(4)}\right) f^{(4)}$$
$$+ \left(\frac{1}{15}\alpha'' + \frac{1}{5}p'c'' + \frac{3}{40}c\,c''^2 + \frac{3}{10}p\,c''' + \frac{c\,c'c'''}{10} + \frac{1}{15}c\,p''' + \frac{c^2c^{(4)}}{15}\right) f^{(5)}$$
$$+ \left(\frac{2}{15}\alpha' + \frac{2}{15}c'p' + \frac{1}{3}p\,c'' + \frac{7c\,c'c'' + 3c^2c'''}{40} + \frac{1}{10}c\,p''\right) f^{(6)}$$
$$+ \left(\frac{1}{15}\alpha + \frac{1}{6}p\,c' + \frac{c\,c'^2}{16} + \frac{1}{10}c\,p' + \frac{3}{40}c^2c''\right) f^{(7)} + \left(\frac{1}{20}c\,p + \frac{3}{80}c^2c'\right) f^{(8)} + \frac{c^3}{240}f^{(9)}.$$

Thus the resulting Hamiltonian H_f satisfies

$$\{H, H_f\} = \mathcal{O}\left(\epsilon^8\right).$$

It is not difficult to also verify the commutativity

$$\{H_f, H_g\} = \mathcal{O}\left(\epsilon^8\right)$$

for an arbitrary pair of functions $f = f(u)$, $g = g(u)$.

Let us now analyze the possibility of extension to a commutative family of order 8. We add to (3.1) terms of the form

$$H \mapsto \tilde{H} = H + \int \epsilon^8 \big[A_1 u_x^8 + A_2 u_x^4 u_{xx}^2 + A_3 u_x^2 u_{xx}^3 + A_4 u_{xx}^4$$
$$+ A_5 u_x^2 u_{xxx}^2 + A_6 u_{xx} u_{xxx}^2 + A_7 u_{xxxx}^2\big]\, dx,$$

and to (3.3) a similar expression,

$$H_f \mapsto \tilde{H}_f = H_f + \int \epsilon^8 \big[B_1 u_x^8 + B_2 u_x^4 u_{xx}^2 + B_3 u_x^2 u_{xx}^3 + B_4 u_{xx}^4$$
$$+ B_5 u_x^2 u_{xxx}^2 + B_6 u_{xx} u_{xxx}^2 + B_7 u_{xxxx}^2\big]\, dx.$$

Here $A_1, \ldots, A_7$, $B_1, \ldots, B_7$ are some functions of u. The goal is to meet the condition

$$\{\tilde{H}, \tilde{H}_f\} = \mathcal{O}\left(\epsilon^9\right). \tag{3.4}$$

The order 8 terms in the brackets in (3.4) are represented by a differential polynomial of degree 9. From the vanishing of the coefficient of $u^{(8)}u_x$ it follows that

$$B_7 = A_7\, f''' + \left(\frac{10}{9}\,\alpha\, c + \frac{10}{9}p^2 + \frac{10}{63}\,c\, c'p - \frac{1}{210}c^2\, c'^2 + \frac{1}{21}c^2\, p' + \frac{1}{70}c^3\, c''\right) f^{(4)} + \left(\frac{5}{7}\, c^2\, p + \frac{3}{70}c^3\, c'\right) f^{(5)} + \frac{3}{70}c^4\, f^{(6)}.$$

Next, from the vanishing of the coefficient of $u^{(6)}u_{xxx}$ we get (3.2).

Further calculations allow one to determine B_6 from the coefficient of $u^{(6)}u_{xx}u_x$, B_5 from the coefficient of $u^{(6)}u_x^3$, B_4 from the coefficient of $u^{(4)}u_{xx}^2u_x$, B_3 from the coefficient of $u^{(4)}u_{xx}u_x^3$, B_2 from the coefficient of $u^{(4)}u_x^5$, and, finally, B_1 from the coefficient of $u_{xx}u_x^7$. All these coefficients are represented by linear differential operators of order at most 12 acting on the arbitrary function $f = f(u)$. The coefficients of these operators depend linearly on $A_1, \dots, A_7$ and their u-derivatives and also on $c(u)$ and $p(u)$ and their derivatives. The explicit formulae are rather long; they will not be given here. As above, one can verify validity of the identity

$$\left\{\tilde H_f, \tilde H_g\right\} = \mathcal{O}\left(\epsilon^{10}\right)$$

for any pair of functions $f(u)$, $g(u)$. ☐

Corollary 3.2. *Let $p(u)$ be an arbitrary nonvanishing function. Then the Hamiltonian*

$$H = \int\left[\frac{u^3}{6} + \epsilon^4 p(u) u_{xx}^2\right] dx \tag{3.5}$$

cannot be included in a 9-integrable family.

4. Quasi triviality transformations and perturbative solutions. In this section we will develop a perturbative technique for constructing monotone solutions to equations of the form (2.21) for sufficiently small time t. This technique is based on the so-called *quasi triviality transformation* [10] expressing solutions to the perturbed equation (2.21) in terms of solutions to the unperturbed equation.

To explain the basic idea let us consider

$$u_t + u\, u_x + \epsilon^2 \partial_x\left(c\, u_{xx} + \frac{1}{2}c' u_x^2\right) + \cdots = u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0, \tag{4.1}$$
$$H = \int\left[\frac{1}{2}u^3 - \frac{\epsilon^2}{2}c\, u_x^2 + \cdots\right] dx.$$

The quasi triviality transformation for this equation,

$$v \to u = v + \epsilon^2\left[\frac{c}{2}\left(\frac{v_{xxx}}{v_x} - \frac{v_{xx}^2}{v_x^2}\right) + c' v_{xx} + \frac{1}{2}c'' v_x^2\right] + \mathcal{O}(\epsilon^4), \tag{4.2}$$

is generated by the Hamiltonian

$$K = -\frac{\epsilon}{2}\int c\, v_x \log v_x\, dx + \mathcal{O}(\epsilon^3), \tag{4.3}$$

$$u = v + \epsilon\{v(x), K\} + \frac{\epsilon^2}{2!}\{\{v(x), K\}, K\} + \cdots.$$

Substituting into (4.1), one obtains a function $u(x,t;\epsilon)$ satisfying (4.1) up to terms of order ϵ^4. Indeed, one can easily derive the following expression for the discrepancy:

$$
\begin{aligned}
&\epsilon^{-4}\left[u\, u_x + \epsilon^2 \partial_x\left(c\, u_{xx} + \frac{1}{2}c' u_x^2\right) - u_t\right]\\
&\quad = c^2\Bigg(\frac{23 {v_{xx}}^5}{2{v_x}^5} - \frac{115 {v_{xx}}^3 v_{xxx}}{4\, {v_x}^4} + \frac{39 {v_{xx}}^2 v_{xxxx}}{4{v_x}^3} + \frac{57\ v_{xx} {v_{xxx}}^2}{4{v_x}^3}\\
&\qquad\qquad - \frac{5 v_{xx} v_{xxxxx}}{2\ {v_x}^2} - \frac{19 v_{xxx} v_{xxxx}}{4{v_x}^2} + \frac{v_{xxxxxx}}{2\ v_x}\Bigg)\\
&\quad + c\,c'\left(-\frac{35 {v_{xx}}^4}{4{v_x}^3} + \frac{19 {v_{xx}}^2\ v_{xxx}}{{v_x}^2} - \frac{7 v_{xx} v_{xxxx}}{v_x} - \frac{23\ {v_{xxx}}^2}{4 v_x} + \frac{7 v_{xxxxx}}{2}\right)\\
&\quad + c\,c''\left(\frac{3 {v_{xx}}^3}{2 v_x} + \frac{13 v_x v_{xxxx}}{2} + 3\ v_{xx} v_{xxx}\right) + c\,c'''\left(\frac{15 {v_x}^2 v_{xxx}}{2} + 8 v_x {v_{xx}}^2\right)\\
&\quad + \frac{11}{2} c\, c^{(4)} v_x^3 v_{xx} + \frac{1}{2} c\, c^{(5)} v_x^5 + c'^2\left(\frac{3 {v_{xx}}^3}{2 v_x} + 4 v_x v_{xxxx} + \frac{v_{xx}\ v_{xxx}}{2}\right)\\
&\quad + c' c''\left(\frac{21 {v_x}^2 v_{xxx}}{2} + 10 v_x {v_{xx}}^2\right) + 9 c' c''' v_x^3 v_{xx} + c' c^{(4)} v_x^5\\
&\quad + 5 c''^2 v_x^3 v_{xx} + \frac{5}{4} c'' c''' v_x^5 + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

Note that the *same* quasi triviality transformation works for solutions $v = v(x,t)$ to the nonlinear transport equation

$$v_t + a(v) v_x = 0,$$

transforming it to solutions, modulo $\mathcal{O}(\epsilon^4)$, to the perturbed equation (1.1).

Denote by $\phi(x) = v(x,0)$ the initial data for the Hopf equation. The initial value of solution $u(x,t;\epsilon)$ given by formula (4.2) differs from $\phi(x)$:

$$u(x,0;\epsilon) = \phi + \epsilon^2\left[\frac{c}{2}\left(\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_{xx}^2}{\phi_x^2}\right) + c'\phi_{xx} + \frac{1}{2}c''\phi_x^2\right] + \mathcal{O}(\epsilon^4). \tag{4.4}$$

In order to solve the Cauchy problem for (4.1) with the *same* initial data $u(x,0;\epsilon) = \phi(x)$ one can use the following trick. Let us consider the solution $\tilde v = \tilde v(x,t;\epsilon)$ to the Hopf equation with the ϵ-dependent initial data

$$\tilde v(x,0;\epsilon) = \phi - \epsilon^2\left[\frac{c}{2}\left(\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_{xx}^2}{\phi_x^2}\right) + c'\phi_{xx} + \frac{1}{2}c''\phi_x^2\right]. \tag{4.5}$$

Such a solution can be represented in the form

$$\tilde v(x,t;\epsilon) = v(x,t) + \epsilon^2 w(x,t) + \mathcal{O}(\epsilon^4), \tag{4.6}$$

where the function $w(x,t)$ has to be determined from the equation

$$\Phi'(w) - w\,t = \left[\frac{c(v)}{2}\frac{2\Phi''^2(v) - \Phi'(v)\Phi'''(v)}{\Phi'^3(v)} - c'(v)\frac{\Phi''(v)}{\Phi'^2(v)} + \frac{c''(v)}{2\Phi'(v)}\right]_{v=v(x,t)}. \tag{4.7}$$

Here $\Phi(v)$ is the function inverse to $\phi(x)$. Applying the quasi triviality transformation to the solution $\tilde v(x,t;\epsilon)$, one obtains a function

$$u(x,t;\epsilon) = v + \epsilon^2 w + \epsilon^2\left[\frac{c}{2}\left(\frac{v_{xxx}}{v_x} - \frac{v_{xx}^2}{v_x^2}\right) + c' v_{xx} + \frac{1}{2}c'' v_x^2\right] \tag{4.8}$$

satisfying (4.1) modulo terms of order ϵ^4 with the initial data

$$u(x,0;\epsilon) = \phi(x) + \mathcal{O}(\epsilon^4). \tag{4.9}$$

5. Generalized KdV equations. In this section we will first study the role of the function $a(u)$ in (1.1) on the validity of the conjecture. This is done for the generalized KdV equations having the form

$$u_t + a(u)u_x + \epsilon^2 u_{xxx} = 0. \tag{5.1}$$

We will assume that $a(u)$ is monotonic in an open neighborhood of each critical point. The functional parameters $c(u)$ and $p(u)$ in (2.18) are given by

$$c(u) = \frac{1}{a'(u)}, \qquad p(u) = -\frac{3}{10}\frac{a''(u)}{a'(u)^3}. \tag{5.2}$$

The basic idea of the PI2 approach to the breakup behavior is that the equation behaves in this case approximately like the KdV equation. We will test this assumption first for $a(u)$ of the form $a(u) = 6u^n$, $n \in \mathbb{N}$.

5.1. Breakup. To begin we will study the solutions to generalized KdV equations close to the breakup of the corresponding dispersionless equation. A generic critical point (x_c, t_c, u_c) is given by

$$\begin{aligned} a(u_c)t_c + \Phi(u_c) &= x_c, \\ a'(u_c)t_c + \Phi'(u_c) &= 0, \\ a''(u_c)t_c + \Phi''(u_c) &= 0, \end{aligned} \tag{5.3}$$

where $\Phi(u)$ is the inverse of the initial data $\phi(x)$ (which might consist of several branches). We will always study the initial data $\phi(x) = \mathrm{sech}^2 x$, which imply $\Phi(u) = \ln((1 \pm \sqrt{1-u})/\sqrt{u})$. For the critical values we obtain

$$\begin{aligned} u_c &= \frac{2n}{2n+1}, \\ t_c &= \frac{(1+2n)^{n+1/2}}{6(2n)^{n+1}}, \\ x_c &= \frac{\sqrt{2n+1}}{2n} + \ln\left(\frac{\sqrt{2n+1}+1}{\sqrt{2n}}\right), \\ k &= -\frac{1}{6}(a'''(u_c)t_c + \Phi'''(u_c)) = \frac{(2n+1)^{9/2}}{96n^2}. \end{aligned} \tag{5.4}$$

We first study the difference between the numerical solution to the generalized KdV equation and the solution to the dispersionless equation, a generalized Hopf equation, on the whole computational domain. For values of $\epsilon = 10^{-1}, 10^{-1.25}, \ldots, 10^{-3}$ we find that this difference scales for $n = 1$ (KdV) roughly as ϵ^α with $\alpha = 0.299$ (correlation coefficient $r = 0.99997$ in linear regression, standard deviation $\sigma_\alpha = 0.0018$), for $n = 3$ we have $\alpha = 0.317$ ($r = 0.9998$, $\sigma_\alpha = 0.0046$), for $n = 4$ we have $\alpha = 0.324$ ($r = 0.9998$, $\sigma_\alpha = 0.005$), and for $n = 5$ we have $\alpha = 0.325$ ($r = 0.9998$, $\sigma_\alpha = 0.0053$). The predicted value is $2/7 = 0.2857$. It can be seen that the above values are all higher, and that the scaling for the generalized KdV equations is close to $\epsilon^{1/3}$. Thus the decrease is at least of the predicted order. It is not surprising that higher values

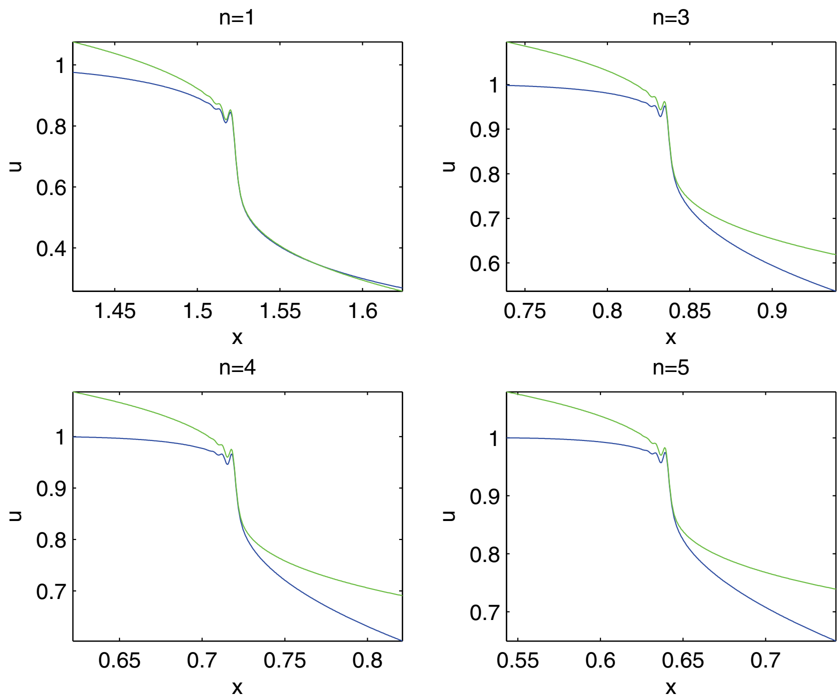

FIG. 5.1. *The lighter line is the solution of the generalized KdV equation* $u_t + 6u^n u_x + \epsilon^2 u_{xxx} = 0$ *for different values of* n *for the initial data* $\phi(x) = 1/\cosh^2 x$ *and* $\epsilon = 10^{-3}$ *at the time* t_c *and near the point of gradient catastrophe* x_c *of the Hopf solution (center of the figure). The darker line is the multiscale approximation in terms of the PI2 solution.*

for the exponent are found since we consider comparatively large values of ϵ for which the contributions of higher order in the difference still play a considerable role.

As discussed in the previous section, it is conjectured that the behavior of the solutions to the generalized KdV equation in the vicinity of the critical point is given in terms of the special solution to the PI2 equation. Expanding $a(u)$ for $u \sim u_c$ as in [10], one finds the behavior shown in Figure 5.1 for different values of n. It can be seen that the asymptotic description is much better for the KdV equation due the fact that the PI2 transcendent gives an exact solution. For other values of n this transcendent gives the conjectured description in the vicinity of the critical point. The quality of the asymptotic description shows the expected scaling for smaller values of ϵ, as can be seen on the left in Figure 5.2.

The quality of this PI2 approximation is not limited to functions $a(u)$ in (5.1) which are polynomial in u. If we consider the case $a(u) = 6 \sinh u$, we obtain the right-hand panel of Figure 5.2. It can be seen that the PI2 asymptotics gives the same excellent description as for KdV.

5.2. Oscillatory regimes and blowup. It is known that solutions to initial value problems with sufficiently smooth initial data for the generalized KdV equation with $n < 4$ are globally regular in time. This is not the case for for $n \geq 4$, where blowup can occur at finite time with $n = 4$ being the critical case. For this case a theorem by Martel and Merle [29] states that solutions on the real line, with negative

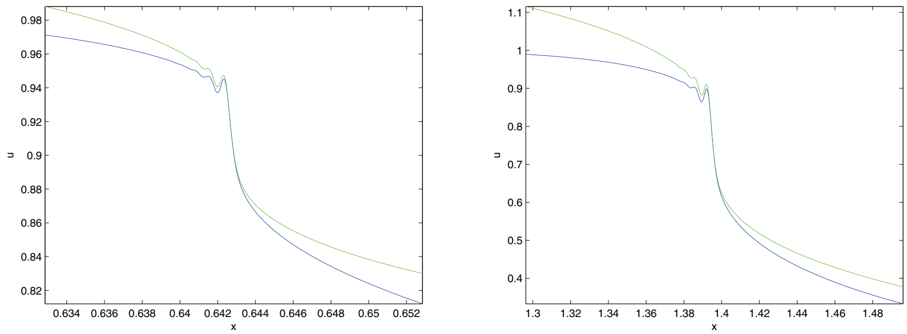

FIG. 5.2. *The lighter line is the solution of* $u_t + a(u)u_x + \epsilon^2 u_{xxx} = 0$ *with* $a(u) = 6u^5$ *(left) or* $a(u) = 6\sinh u$ *(right) for the initial data* $\phi(x) = 1/\cosh^2 x$ *and* $\epsilon = 10^{-4}$ *at the time* t_c *near the point of gradient catastrophe* x_c *of the Hopf solution (center of each figure). The darker line is the multiscale approximation in terms of the PI2 solution.*

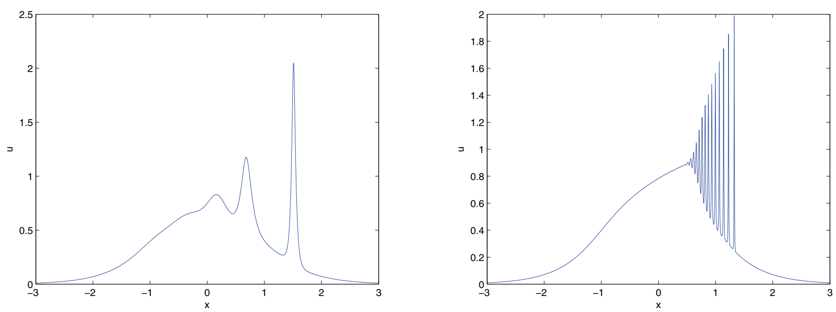

FIG. 5.3. *Solution of the generalized KdV equation with* $n = 4$ *for the initial data* $\phi(x) = 1/\cosh^2 x$, $\epsilon = 10^{-1}$, *and time* $t = 0.3180 \gg t_c$ *(left);* $\epsilon = 10^{-2}$ *and time* $t = 0.2235 \gg t_c$ *(right).*

energy, blow up in finite or infinite time. For the general case $n > 4$ and periodic settings considered here, the question is still open. Since the energy has the form

$$E = \int_{\mathbb{R}} \left[\frac{\epsilon^2}{2} u_x^2 - \frac{6u^{n+2}}{(n+1)(n+2)} \right] dx,$$

it will be always negative for sufficiently small ϵ and positive u.

Here we address numerically the question of whether the formation of dispersive shocks, i.e., of a region of rapid modulated oscillations, precedes a potential blowup. We expect that the breakup of the solution to the dispersionless equation is regularized by the dispersion in the form of oscillations, which then develop into blowup if the latter exists. This is exactly what we see in the following. Notice that the breakup time is given by the dispersionless equation and is thus independent of ϵ. We first study the case $n = 4$. For $\epsilon = 1$, the energy is positive and no indication of blowup is observed. For $\epsilon = 0.1$ we obtain for the initial data $\phi(x) = \text{sech}^2 x$; see Figure 5.3 (left). For smaller ϵ ($\epsilon = 0.01$) the behavior is similar, but there are as expected more oscillations, and the size of the oscillations reaches higher values earlier, as can be seen in Figure 5.3. For obvious reasons it is numerically difficult to decide whether the dispersive shock will lead to a blowup. In practice we run out of resolution before the

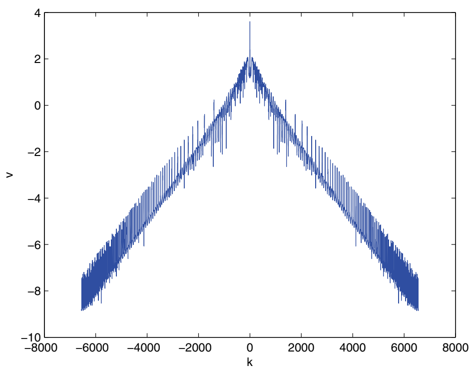

FIG. 5.4. *Logarithm with base* 10 *of the modulus of the Fourier coefficients of the function shown in Figure* 5.3(right).

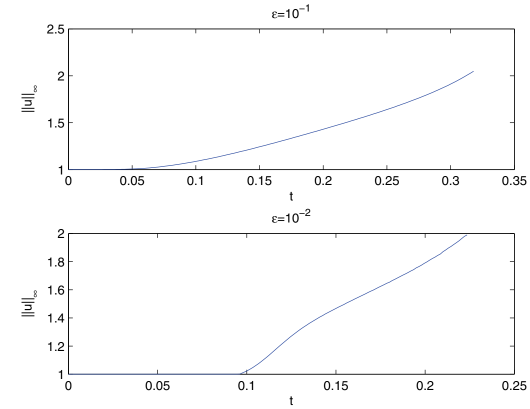

FIG. 5.5. *L^∞-norm of the solutions in Figure* 5.3.

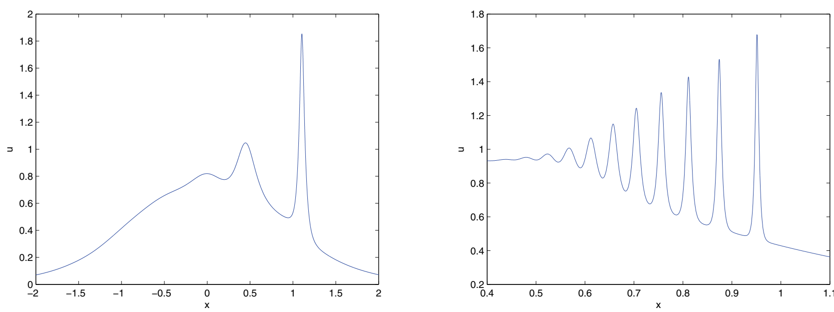

FIG. 5.6. *Solution of the generalized KdV equation with $n = 5$ for the initial data $\phi(x) = 1/\cosh^2 x$, $\epsilon = 10^{-1}$, and $t = 0.2362 \gg t_c$ (left); $\epsilon = 10^{-2}$ and $t = 0.2235 \gg t_c$ (right).*

code breaks down because of a blowup. This is due to oscillations in Fourier space, as can be seen in Figure 5.4. Though there is in principle enough resolution to approach $u(x,t)$, the oscillations of the Fourier coefficients make an accurate approximation via a Fourier transform impossible.

The reason for this behavior is as discussed in [32]: singularities of the form $(z-z_j)^{\mu_j}$ in the complex plane lead asymptotically to Fourier coefficients with modulus of the form $Ck^{-(\mu_j+1)}\exp(-\delta k)$, $\delta > 0$. If there are several such singularities, there will be oscillations in the Fourier coefficients. In the present case there are at least two such singularities, the breakup—which is, strictly speaking, singular only for $\epsilon = 0$, but has an effect already for finite ϵ—and the blowup—which leads to the behavior seen in (5.4). In Figure 5.5 we give the L^∞-norm of the solutions in Figure 5.3. It cannot be decided on the basis of these numerical data whether there is finite time blowup in this case; if it exists, it is clearly preceded by a dispersive shock.

In the supercritical case $n = 5$ we obtain a similar picture. In Figure 5.6 we see the solution in the case $\epsilon = 0.1$. Again it appears as if the rightmost peak evolves into a singularity. For smaller ϵ ($\epsilon = 0.01$) there are again more oscillations, which stresses the importance of dispersive regularization before a potential blowup. Studying the L^∞-norm of the solutions in Figure 5.7, we can see that the case $\epsilon = 0.1$ indeed seems

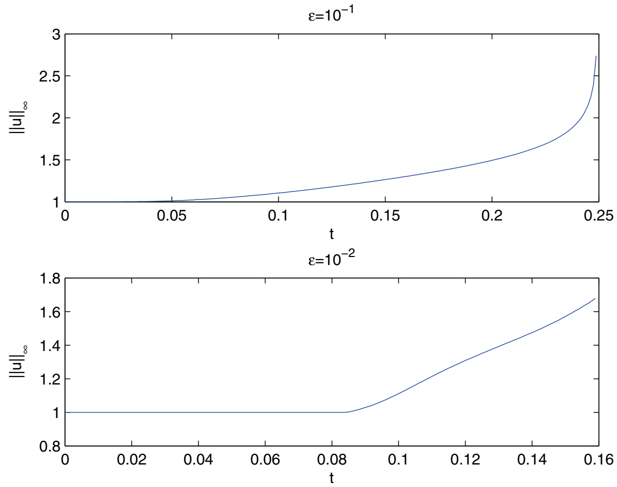

FIG. 5.7. L^∞-*norm of the solutions in Figure* 5.6.

to approach an L^∞ blowup in finite time. Because of the above-mentioned resolution problems, we could not reach a similar point for $\epsilon = 0.01$.

6. Kawahara equations. The Kawahara equations [24], which appear in general dispersive media where the effects of the third order derivative is weak as in certain hydrodynamic or magneto-hydrodynamic settings, can be written in the form

$$u_t + \frac{1}{2}\partial_x f(u, \epsilon\, u_x, \epsilon^2 u_{xx}) + \beta\, \epsilon^4 u_{xxxxx} = 0. \tag{6.1}$$

Here we will mainly study the case

$$f(u, \epsilon\, u_x, \epsilon^2 u_{xx}) = 6u^2 + 2\alpha\, \epsilon^2 u_{xx} \tag{6.2}$$

with $\alpha = 1$ and $\beta = \pm 1$. The global well-posedness of solutions of (6.1) in a suitable Sobolev space has been proved in [30].

The functional parameters $c(u)$, $p(u)$ in (2.18) are constants

$$c(u) = \frac{1}{6}\alpha, \qquad p(u) = \frac{1}{12}\beta. \tag{6.3}$$

At the critical point we obtain for $\beta = -1$ that the breakup behavior is well described by PI2 in lowest order, as can be seen in Figure 6.1.

It can be seen that the PI2 solution gives close to the breakup point a much better description of the Kawahara solution than the corresponding Hopf solution. The oscillation closest to the breakup point is too far away from the latter to be correctly reproduced, but the PI2 solution qualitatively catches the oscillatory behavior of the Kawahara solution near the critical point. With smaller ϵ, the agreement gets better, as expected; see Figure 6.1(right).

For $\beta = 1$, the breakup behavior of solutions to the Kawahara changes, as can be seen from Figure 6.2. In this case the oscillations in the Kawahara solution appear on the other side of the critical point and around it with small amplitude. This behavior cannot be captured by the PI2 solution but is a higher order effect. Close to the

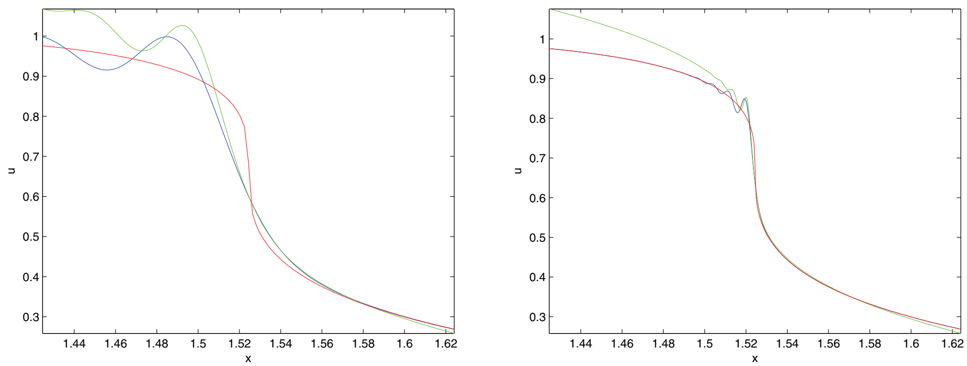

FIG. 6.1. *The blue line is the solution of the Kawahara equation ($\beta = -1$) for the initial data $\phi(x) = 1/\cosh^2 x$ and $\epsilon = 10^{-2}$ (left) or $\epsilon = 10^{-3}$ (right). The plot is taken at the time t_c near the point of gradient catastrophe x_c of the Hopf solution (center of the figure). Here $x_c \simeq 1.524$, $t_c \simeq 0.216$. The red line is the corresponding Hopf solution, and the green line is the multiscale approximation in terms of the PI2 solution.*

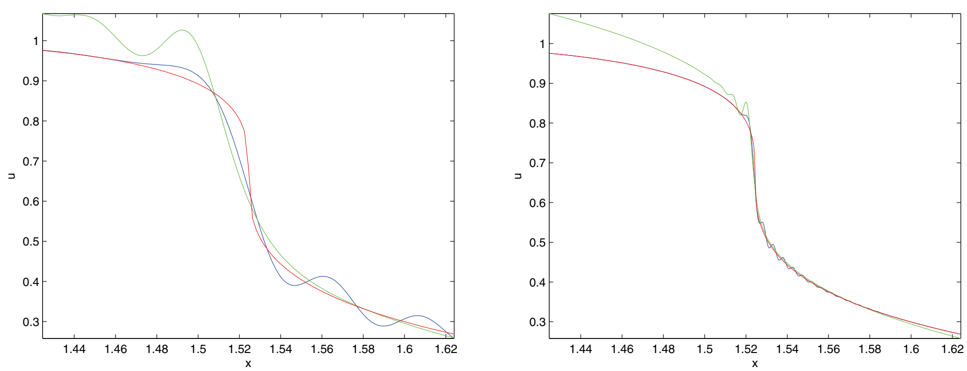

FIG. 6.2. *The blue line is the solution of the Kawahara equation ($\beta = 1$) for the initial data $\phi(x) = 1/\cosh^2 x$ and $\epsilon = 10^{-2}$ (left) or $\epsilon = 10^{-3}$ (right). The plot is taken at the time t_c near the point of gradient catastrophe x_c of the Hopf solution (center of the figure). Here $x_c \simeq 1.524$, $t_c \simeq 0.216$. The red line is the corresponding Hopf solution, and the green line is the multiscale approximation in terms of the PI2 solution.*

critical point, the multiscale solution gives, as before, a much better description of the Kawahara solution than the Hopf solution.

For smaller values of ϵ, both asymptotic solutions become more satisfactory, as can be seen from Figure 6.2(right). It is interesting to note that with decreasing ϵ, the oscillations become smaller in amplitude in this case but appear closer to the critical point. It can also be seen that the solution has the tendency to form one oscillation on the other side of the critical point close to the corresponding PI2 oscillation. The fact that the oscillations to the right of the critical point disappear more rapidly with ϵ as $\epsilon \to 0$ than the oscillations captured by the PI2 asymptotics is obvious and can also be clearly seen in Figure 6.3 for an even smaller value of ϵ.

Tracing the solution for larger times, it can be recognized that this will be the only oscillation to the left of the critical point, whereas a zone of high-frequency

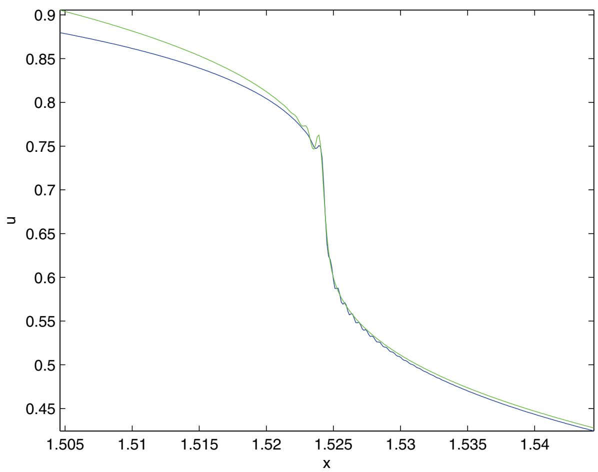

FIG. 6.3. *Solution to the Kawahara equation ($\beta = 1$) with $\epsilon = 10^{-4}$ (lighter line) and its approximation by the PI2 solution (darker line).*

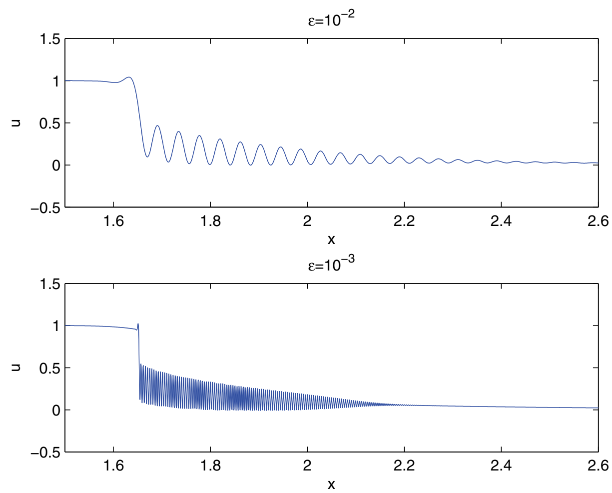

FIG. 6.4. *Oscillatory zone of the solutions to the Kawahara equation ($\beta = 1$) for the initial data $\phi(x) = 1/\cosh^2 x$ and two values of ϵ at time $t = 0.25 > t_c$.*

oscillations which appears to be essentially unbounded (see [14], [22]) develops to the right; see Figure 6.4. The oscillations appear to be, as in the KdV case, more and more confined to a zone similar to the Whitham zone, though no asymptotic description of the oscillations (see, e.g., [13]) exists since the equation is not integrable. The different behavior of the dispersive effects in the cases $\beta = \pm 1$ can be qualitatively understood as follows: the dispersive effects are due to the term $u_{xxx} + \beta u_{xxxxx}$, or in Fourier space by the multiplier $-ik^3(1 - \beta k^2)$. Thus the case $\beta = -1$ is qualitatively as for KdV since the dispersion has the same sign, whereas it will change for $\beta = 1$; see also [14].

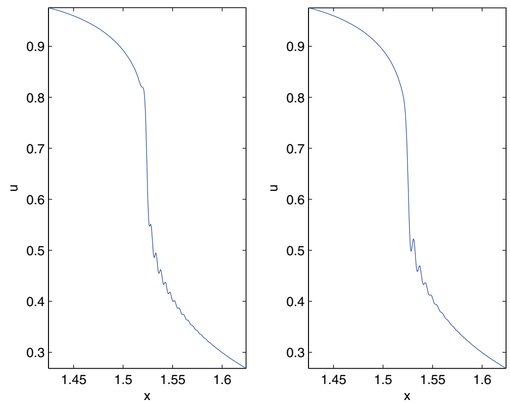

FIG. 6.5. *Solutions of the Kawahara equation* (6.1) *with* $\beta = 1$ *and* $\alpha = 1$ *(left) or* $\alpha = 0$ *(right). The solution is given for the initial data* $\phi(x) = 1/\cosh^2 x$ *and* $\epsilon = 10^{-3}$ *at the time* t_c *near the point of gradient catastrophe* x_c *of the Hopf solution.*

It seems also that this one oscillation to the left is really due to the third order derivative in the Kawahara equation, as can be seen in Figure 6.5, where one has to the left the Kawahara solution from Figure 6.2 and to the right the analogous solution for $\alpha = 0$, i.e., Kawahara without a third order derivative, a nonintegrable case even in the considered order of ϵ. The oscillations to the right of the critical point, being due to the fifth order derivative, are present in both cases and have only slightly different forms.

6.1. PDE with nonlinear dispersion. To show that the breakup behavior discussed in the previous sections is typical, we will now consider equations of the form (2.17) with nonlinear dispersion, i.e., with functions $c(u)$ and $p(u)$ not constant. The Camassa–Holm equation (CH) falls into this class if the nonlocal term is expanded in a von Neumann series (see [10]) for functions $c(u) \sim u$ and $p(u) \sim u$. The applicability of the PI2 asymptotics to CH was studied numerically in [18]. For simplicity we restrict our analysis to the case $c(u)$ and $p(u)$ both proportional to u^2 with the Hopf equation as the dispersionless equation and initial data of the form $\phi(x) = \text{sech}^2 x$. More complicated functions c and p can be considered, but the results are qualitatively the same.

In Figure 6.6 the behavior at the critical time can be seen for $c(u) = u^2$ and $p(u) = 0$. The situation is obviously like that in the KdV case.

As for the Kawahara equation, the relative sign between the third and the fifth derivatives is important for the form of the oscillations. The situation with the opposite signs of c and p can be seen in Figure 6.7. It is qualitatively the same as in the KdV case. New features appear as in the case of the Kawahara equation for the same sign in front of the third and fifth derivatives. As can be seen in Figure 6.8, oscillations of small amplitude appear as in the Kawahara equation on the other side of the critical point. Thus nonconstant functions $c(u)$ and $p(u)$ as expected do not change the picture from the case of constant functions as long as they do not vanish at the critical point.

6.2. Quasi triviality transformation. In this subsection we study numerically the validity of the expansion given in section 4. For times $t \ll t_c$ the behavior

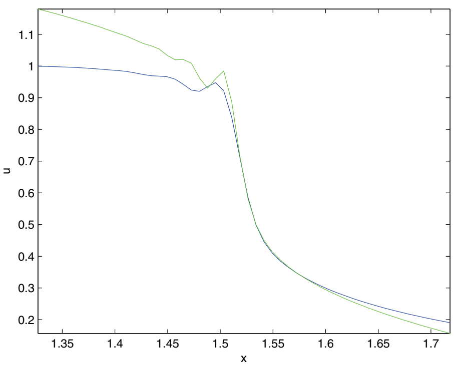

FIG. 6.6. *Solution to* (2.17) *for* $a(u) = u$, $c(u) = u^2$, $p(u) = 0$, *and initial data* $\phi(x) = sech^2 x$ *at the critical time, and the corresponding multiscale solution in terms of the PI2 transcendent.*

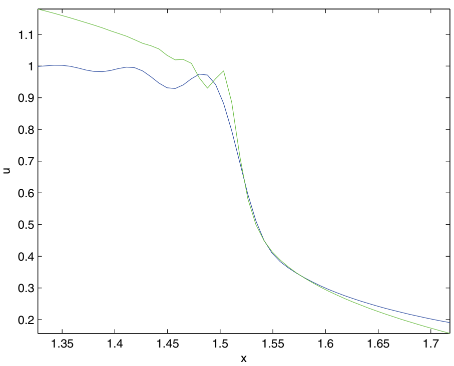

FIG. 6.7. *Solution to* (2.17) *for* $a(u) = u$, $c(u) = -p(u) = u^2$, *and initial data* $\phi(x) = sech^2 x$ *at the critical time, and the corresponding multiscale solution in terms of the PI2 transcendent.*

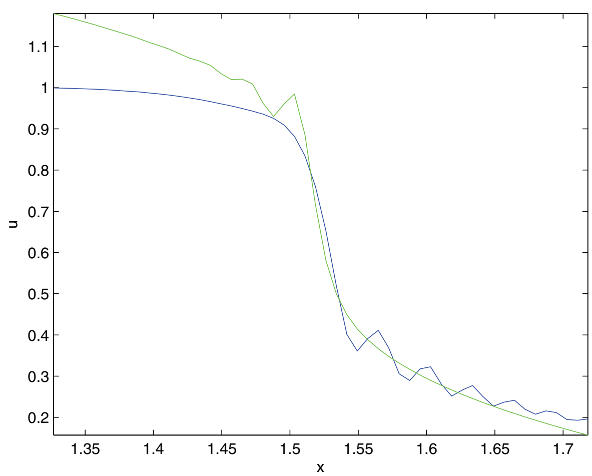

FIG. 6.8. *Solution to* (2.17) *for* $a(u) = u$, $c(u) = p(u) = u^2$, *and initial data* $\phi(x) = sech^2 x$ *at the critical point, and the corresponding multiscale solution in terms of the PI2 transcendent.*

of the solution of (4.1) should be described to order ϵ^2 by the solution of the Hopf equation $u_t + uu_x = 0$ with the same initial data. In fact we find that the difference between the Hopf solution and the solution to Kawahara equation (6.1) with $\beta = 1$, $\alpha = 1$ for the initial data $\phi(x) = \mathrm{sech}^2 x$ at $t = t_c/2$ scales as ϵ^γ with $\gamma = 1.94$ (values of $\epsilon = 10^{-1}, 10^{-1.125}, \ldots, 10^{-3}$, correlation coefficient $r = 0.9997$ in linear regression, standard deviation $\sigma_\alpha = 0.027$). For the same setting in the interval $x \in [0.8, 2]$, the difference between the quasi triviality solution as described in section 4 and the Kawahara solution scales as ϵ^γ with $\gamma = 3.77$ (correlation coefficient $r = 0.999$ in linear regression, standard deviation $\sigma_\alpha = 0.088$). This confirms the theoretical expectations. In Figure 6.9 the difference between the Kawahara and the Hopf solutions and the quasi triviality transform in order ϵ^2 can be seen for this case. A similar scaling

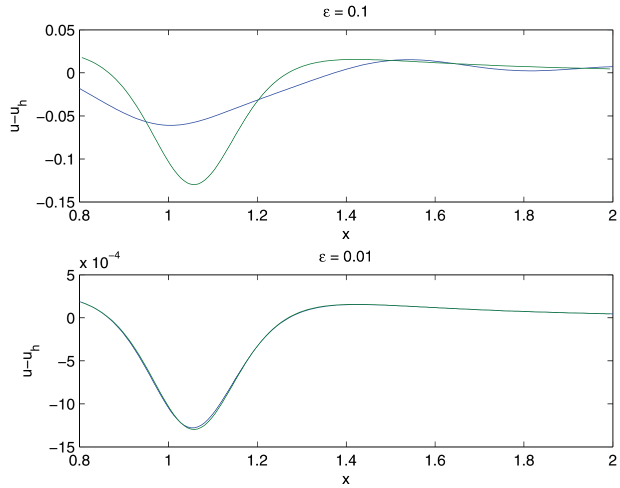

FIG. 6.9. *Solution to the Kawahara equation with $\alpha = 1$ for the initial data $\phi(x) = 1/\cosh^2 x$ at the time $t = t_c/2$ for two values of ϵ; in blue the difference between the Kawahara and the corresponding Hopf solution, in green the order ϵ^2 term of the quasi triviality transformation.*

is observed for $\alpha = -1$ and $\alpha = 0$ (KdV).

The difference between the solution of generalized KdV equation $u_t + u^5 u_x + \epsilon^2 u_{xxx} = 0$ and the solution of the corresponding conservation law scales, at $t = t_c/2$ as ϵ^γ with $\gamma = 1.9890$ (values of $\epsilon = 10^{-1}, 10^{-1.125}, \ldots, 10^{-3}$, correlation coefficient $r = 0.99998$ in linear regression, standard deviation $\sigma_\alpha = 0.0068$).

6.3. Second equation in the KdV hierarchy. In this subsection we study the formation of dispersive shock waves for a family of equations that includes integrable and nonintegrable PDEs. Interestingly the family of equations

$$u_t + 30u^2 u_x + 10\alpha\epsilon^2(u u_{xxx} + 2u_x u_{xx}) + \epsilon^4 u_{xxxxx} = 0, \tag{6.4}$$

having the invariants

$$c(u) = \frac{1}{6}\alpha, \qquad p(u) = \frac{1-\alpha^2}{120u},$$

is completely integrable for $\alpha = \pm 1$ and coincides with the second equation in the KdV hierarchy (KdVII). Varying this factor, one can study the transition to the Kawahara equation. As expected, KdVII shows oscillations similar to those of KdV [17], as can be seen in Figure 6.10.

For larger values of α one can recognize in Figure 6.10 also a formation of oscillations on the other side of the inflection point, as in the Kawahara equation. These effects become smaller for larger values of α ($\alpha > 1.2$), but it shows that the phenomenon of integrability with the appearance of KdV-type oscillations is rather subtle. Thus it seems that the decisive factor for the appearance and the size of these oscillations is the relative sign and size of the factors in front of the third and the fifth derivatives in the equation. Notice that for smaller α (6.4) is closer to the nonintegrable (in higher orders in ϵ) equation

$$u_t + 30u^2 u_x + \epsilon^4 u_{xxxxx} = 0$$

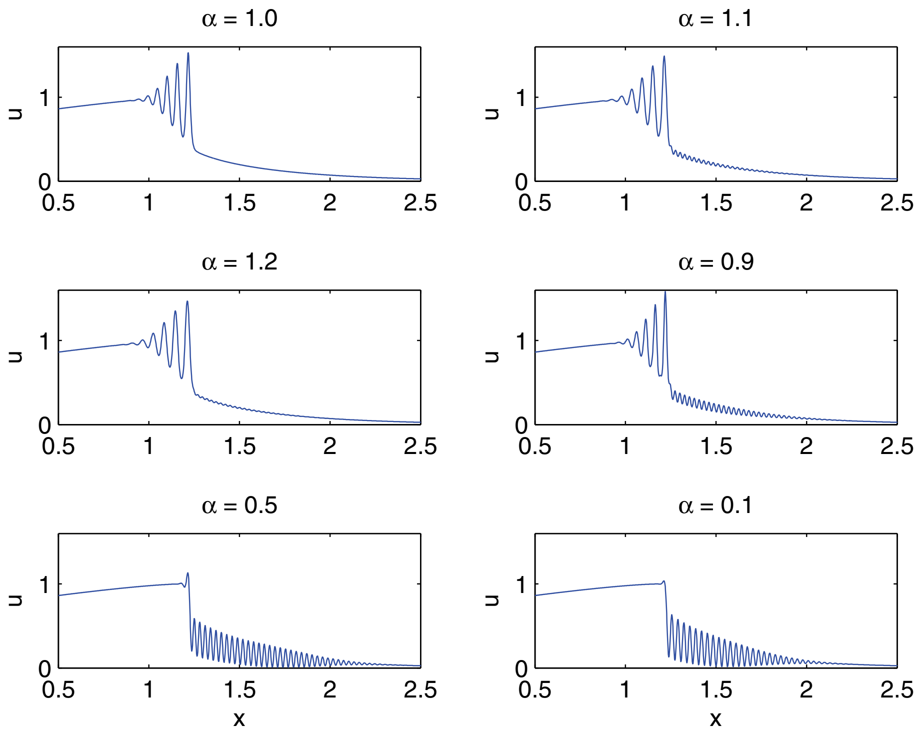

FIG. 6.10. *Oscillatory part of the solution to the KdVII equation for the initial data* $\phi(x) = 1/\cosh^2 x$ *and* $\epsilon = 10^{-2}$ *at a time* $t = 0.04 > t_c = 0.029$ *for several values of* α.

(see section 3 above). It would be interesting to elaborate upon this observation in order to develop numerical tests of (approximate) integrability based on the study of the phase transition from regular to oscillatory behavior.

7. Outlook. In the present paper we have presented local Hamiltonian perturbations of the Riemann wave equation that are integrable up to order ϵ^{10}. This allowed us to identify possible obstructions on the free functions appearing in order ϵ^2 by the condition of integrability in higher order. We were able to test the conjectured asymptotic description near the point of gradient catastrophe and before. Since we were working with double precision numerics, we could access only values of $\epsilon \geq 10^{-4}$ and the first order correction in the quasi-triviality approximation for small times. To test higher order corrections, we will use multiprecision computations in the future.

A similar analysis will be presented for Hamiltonian perturbations for systems of equations. The case of nonlocal Hamiltonian perturbations seems to belong to a qualitatively different class of phenomena and will be subject of a subsequent investigation.

Appendix A. Numerical methods. In this appendix we will briefly review the methods used in the numerical study of the PDE in the small dispersion limit and of the PI2 solution, and we will give references in which details can be found.

Since critical phenomena are generally believed to be independent of specific boundary conditions, we restrict our analysis to essentially periodic functions. Typically we consider Schwarzian functions on a domain at the boundaries of which the functions are smaller than machine precision (10^{-16} in double precision, which is used throughout this paper). Such functions can be periodically continued and are smooth with numerical precision. This allows a Fourier discretization of the spatial variables and an approximation of the solutions via truncated Fourier series. The use of Fourier spectral methods is especially efficient for the studied dispersive PDE because of the excellent approximation of smooth functions and the only minimal introduction of numerical dissipation. The latter is especially important if one is interested in the study of dispersive effects.

After discretization of the spatial coordinates, the PDE is equivalent to a typically large system of ODEs in the time variable. Because of the high order of the spatial derivatives and because of the strong gradients we want to study, these systems will be typically stiff. If the stiff part is linear, as is the case for the generalized KdV equations and for the Kawahara equations, the system of ODEs has the form

$$Lv + N[v] = 0,$$

where v is the discrete Fourier transform of the solution, L is the stiff linear operator, and the nonlinear term $N[v]$ contains only derivatives of lower order. For such systems, efficient integration schemes exist. We use a fourth order *exponential time differencing scheme* [8]; see [25] for a comparison of fourth order schemes for KdV. The numerical accuracy is controlled by sufficient spatial resolution, i.e., Fourier coefficients decreasing to at least 10^{-8}, and by numerically checking energy conservation. Since all equations studied here are Hamiltonian, energy is a conserved quantity. Due to unavoidable numerical errors, it will be weakly time-dependent in numerical time integrations. As discussed in [25], conservation of the numerically computed energy typically overestimates the accuracy of a solution by two orders of magnitude. We always compute with an error in energy conservation smaller than 10^{-6}, which implies that the error is well below plotting accuracy.

The situation is different for the equations with nonlinear dispersion in section 6.2. For these PDEs we use an implicit fourth order Runge–Kutta method (the Hammer and Hollingsworth method [20]). These equations are numerically much more demanding. Therefore we compute with lower spatial resolution and an energy conservation of the order of 10^{-4}.

The special solution of the PI2 equation is generated with the code *bvp*4 distributed with MATLAB; for details see [18]. The Hopf solution is obtained from the implicit form $u(x,t) = \phi(\xi)$, $x = t\phi(\xi) + \xi$ with a fixed point iteration to machine precision. The derivatives of the Hopf solution are obtained by evaluating the analytic expressions following from the characteristic method.

REFERENCES

[1] I. B. Bakholdin, *Non-dissipative and low-dissipative shocks with regular and stochastic structures in non-linear media with dispersion*, in Proceedings of the IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures, Turbulence, A. V. Borisov, V. V. Kozlov, I. S. Mamaev, and M. A. Sokolovskiy, eds., IUTAM Bookseries 6, Springer, New York, 2008.

[2] I. Bakholdin and A. Il'ichev, *Radiation and modulational instability described by the fifth-order Korteweg–DeVries equation*, in Mathematical Problems in the Theory of Water Waves, F. Dias, J.-M. Ghidaglia, and J.-C. Saut, eds., Contemp. Math. 200, AMS, Providence, RI, 1996.

[3] Yu. A. Berezin and V. I. Karpman, *On the nonlinear evolution of perturbation in plasma and other dispersive media*, Zh. Elek. Tekh. Fiz., 51 (1966), 1557–1568 (translated as Soviet Phys. JETP, 24 (1967)).

[4] P. Bleher and A. Its, *Asymptotics of the partition function of a random matrix model*, Ann. Inst. Fourier (Grenoble), 55 (2005), pp. 1943–2000.

[5] É. Brézin, E. Marinari, and G. Parisi, *A nonperturbative ambiguity free solution of a string model*, Phys. Lett. B, 242 (1990), pp. 35–38.

[6] T. Claeys and T. Grava, *Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann–Hilbert approach*, Comm. Math. Phys., 286 (2009), pp. 979–1009.

[7] T. Claeys and M. Vanlessen, *The existence of a real pole-free solution of the fourth order analogue of the Painlevé* I *equation*, Nonlinearity, 20 (2007), pp. 1163–1184.

[8] S. Cox and P. Matthews, *Exponential time differencing for stiff systems*, J. Comput. Phys., 176 (2002), pp. 430–455.

[9] P. Dedecker and W. M. Tulczyjev, *Spectral sequences and the inverse problem of the calculus of variations*, in Differential Geometrical Methods in Mathematical Physics, Geometry, Topology, and Mathematical Physics, Lecture Notes in Math. 836, Springer, New York, 1980, pp. 498–503.

[10] B. Dubrovin, *On Hamiltonian perturbations of hyperbolic systems of conservation laws,* II: *Universality of critical behaviour*, Comm. Math. Phys., 267 (2006), pp. 117–139.

[11] B. Dubrovin, *On universality of critical behaviour in Hamiltonian PDEs*, in Amer. Math. Soc. Transl. 224, AMS, Providence, RI, 2008, pp. 59–109.

[12] B. Dubrovin, *Hamiltonian perturbations of hyperbolic PDEs: From classification results to the properties of solutions*, in New Trends in Mathematical Physics. Selected Contributions of the XVth International Congress on Mathematical Physics, V. Sidoravicius, ed., Springer, Dordrecht, The Netherlands, 2009, pp. 231–276.

[13] B. Dubrovin and S. P. Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory*, Russian Math. Surv., 44 (1989), pp. 29–98.

[14] G. El, *Resolution of a shock in hyperbolic systems modified by weak dispersion*, Chaos, 15 (2005), paper 037103.

[15] N. Ercolani and K. McLaughlin, *Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration*, Internat. Math. Res. Notices, 14 (2003), pp. 755–820.

[16] J. Goodman and Z. P. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Ration. Mech. Anal., 121 (1992), pp. 235–265.

[17] T. Grava and C. Klein, *Numerical solution of the small dispersion limit of the KdV equation and Whitham equations*, Comm. Pure Appl. Math., 60 (2007), pp. 1623–1664.

[18] T. Grava and C. Klein, *Numerical study of a multiscale expansion of KdV and Camassa-Holm equation*, in Integrable Systems and Random Matrices, J. Baik, T. Kriecherbauer, L.-C. Li, K. D. T.-R. McLaughlin, and C. Tomei, eds., Contemp. Math. 458, AMS, Providence, RI, 2008, pp. 81–99.

[19] A. Gurevich and L. Pitaevski, *Nonstationary structure of a collisionless shock wave*, Sov. Phys. JETP Lett., 38 (1974), pp. 291–297.

[20] P. C. Hammer and J. W. Hollingsworth, *Trapezoidal methods of approximating solutions of differential equations*, Math. Tables Aids Comput., 9 (1955), pp. 92–96.

[21] T. Y. Hou and P. D. Lax, *Dispersive approximations in fluid dynamics*, Comm. Pure Appl. Math., 44 (1991), pp. 1–40.

[22] J. Hunter and J. Scheurle, *Existence of perturbed solitary wave solutions to a model equation for water waves*, Phys. D, 32 (1988), pp. 253–268.

[23] A. A. Kapaev, *Weakly nonlinear solutions of the equation* P_1^2, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 187 (1991); English translation in J. Math. Sci., 73 (1995), pp. 468–481.

[24] T. Kawahara, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Japan, 33 (1972), pp. 260–264.

[25] C. Klein, *Fourth order time-stepping for low dispersion Korteweg-de Vries and nonlinear Schrödinger equation*, Electron. Trans. Numer. Anal., 29 (2008), pp. 116–135.

[26] Y. Kodama and A. Mikhailov, *Obstacles to asymptotic integrability*, in Algebraic Aspects of Integrable Systems, Progr. Nonlinear Differential Equations Appl. 26, Birkhäuser, Boston, MA, 1997, pp. 173–204.

[27] V. Kudashev and B. Suleimanov, *A soft mechanism for the generation of dissipationless shock waves*, Phys. Lett. A, 221 (1996), pp. 204–208.

[28] P. G. LeFloch and R. Natalini, *Conservation laws with vanishing nonlinear diffusion and dispersion*, Nonlinear Anal., 36 (1999), pp. 213–230.

[29] Y. Martel and F. Merle, *Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation*, Ann. of Math. (2), 155 (2002), pp. 235–280.

[30] G. Ponce, *Lax pairs and higher order models for water waves*, J. Differential Equations, 102 (1993), pp. 360–381.

[31] B. Suleĭmanov, *Onset of nondissipative shock waves and the "nonperturbative" quantum theory of gravitation*, J. Exp. Theor. Phys., 78 (1994), pp. 583–587.

[32] C. Sulem, P.-L. Sulem, and H. Frisch, *Tracing complex singularities with spectral methods*, J. Comput. Phys., 50 (1983), pp. 138–161.

[33] N. Zabusky and M. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., 15 (1965), pp. 240–243.