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On the Critical Behavior in Nonlinear Evolutionary PDEs with Small Viscosity

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Abstract. The problem of general dissipative regularization of the quasilinear transport equation is studied. We argue that the local behavior of solutions to the regularized equation near the point of gradient catastrophe for the transport equation is described by the logarithmic derivative of the Pearcey function; this statement generalizes a result of A. M. Il'in [12]. We provide some analytic arguments supporting the conjecture and test it numerically.

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1. INTRODUCTION

In this article we address the problem of shock formation in a general dissipative regularization

$$u_t + a(u)u_x = \varepsilon\left[b(u)u_{xx} + c(u)u_x^2\right] + \varepsilon^2\left[b_1(u)u_{xxx} + c_1(u)u_{xx}u_x + d_1(u)u_x^3\right] + \dots \tag{1}$$

of the quasilinear transport equation

$$u_t + a(u)u_x = 0, \quad a'(u) \neq 0, \quad u, x \in \mathbb{R}. \tag{2}$$

Here ε is a small positive parameter, and the coefficient $b(u)$ does not vanish. In this study, we were inspired by the Universality Conjecture of [3] concerning the universal shape of *dispersive* shock waves at the point of phase transition from the regular behavior to the oscillatory one. This universal dispersive shock profile is described in terms of a particular solution of a generalization of the Painlevé-I equation (the importance of this particular solution in 2D quantum gravity and in the theory of Korteweg–de Vries equation was also observed in [9, 1, 14]). The universality conjecture for solutions to the Korteweg–de Vries equation with analytic initial data was proved in [2]. Further numerical evidences supporting the universality conjecture of [3] can be found in [4]. Another starting point for the present research was a remarkable result of A. M. Il'in (see [12] and the references therein) describing the asymptotics of the generic solution to the equation

$$u_t + a(u)u_x = \varepsilon\, u_{xx} \tag{3}$$

at the point of shock formation in terms of the logarithmic derivative of the so-called Pearcey integral (see below the precise formulation of Il'in's asymptotic formula). Both in dispersive and dissipative cases, the leading term of the asymptotic formula is essentially independent, up to few constants, on the choice of both a particular generic solution and a particular generic perturbation.

Our main goal is to generalize Il'in's universality result from the equations (3) to the more general case of equations of the form (1). In the present paper, we present a conjectural form of such a generalization and describe results of numerical experiments supporting its validity.

The paper is organized as follows. In Sec. 1, we explain simple arguments suggesting that, for sufficiently small ε, the solutions to the perturbed equation (1) can be approximated by solutions to the nonlinear transport equation (2) up to the time of gradient catastrophe of (2). To save the space, we omit the terms of order ϵ^2 and higher in our formulas; the contribution of these terms to the asymptotic expansions is of higher order anyway. We then proceed to a precise formulation of the dissipative universality conjecture (see Conjecture 3 below) which describes the leading term of the asymptotic expansion at the point of shock formation. We also give heuristic motivations for this main conjecture. In the last section, we present results of numerical experiments supporting the main conjecture. To this end, we begin with the standard Burgers equation, in order to test the numerical codes based on the finite-element analysis. After this, we proceed to a particular case of generalized Burgers equation, comparing the numerical solution with the asymptotic formula.

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1. CRITICAL BEHAVIOR IN THE GENERALIZED BURGERS EQUATION

Consider the following class of nonlinear PDEs depending on a small parameter $\varepsilon > 0$:

$$u_t + a(u)u_x = \varepsilon \left[b(u)u_{xx} + c(u)u_x^2\right]. \tag{4}$$

The coefficients $a(u)$, $b(u)$, and $c(u)$ are smooth functions, and $a'(u) \neq 0$. This class of equations is invariant with respect to arbitrary changes of the dependent variable of the form $u \mapsto \tilde{u} = f(u)$, $f'(u) \neq 0$. Using these transformations, one can reduce (4) to one of the following two normal forms:

$$u_t + u\, u_x = \varepsilon \left[b(u)u_{xx} + c(u)u_x^2\right] \tag{5}$$

or

$$u_t + a(u)u_x = \varepsilon\, b(u)u_{xx}. \tag{6}$$

We study solutions $u = u(x, t; \varepsilon)$ to the Cauchy problem

$$u(x, 0; \varepsilon) = F(x) \tag{7}$$

with ε-independent smooth initial data. In the particular case of $b(u) = 1$ and $c(u) = 0$, one arrives at the generalized Burgers equation

$$u_t + a(u)u_x = \varepsilon\, u_{xx}, \tag{8}$$

which was thoroughly studied by A. M. Il'in (see the book [12] and the references therein). Let us briefly summarize the main results of [12].

For simplicity, the initial data function $F(x)$ is assumed to be monotone on the entire real line $x \in \mathbb{R}$. The first issue is the comparison of the solution $u(x, t; \varepsilon)$ to the Cauchy problem (4), (7) with the solution $v = v(x, t)$ to the inviscid equation obtained by equating ε to 0 equipped with the *same* initial data,

$$v_t + a(v)v_x = 0, \qquad v(x, 0) = F(x). \tag{9}$$

The solutions asymptotically coincide on finite intervals of the x axis for sufficiently small time, $|u(x, t; \varepsilon) - v(x, t)| \to 0$ as $\varepsilon \to 0+$, $x \in [x_1, x_2]$, for $0 \leqslant t \leqslant t_1$. However, the lifespan of the solution $v(x, t)$ is finite, due to nonlinear steepening, if the function $a\,(F(x))$ is monotone decreasing on some interval of the real axis. In this case, the solution to the inviscid equation is defined only on the interval $[0, t_0]$ where

$$t_0 = \min_{x \in \mathbb{R}} \left(-1/([a\,(F(x))]_x)\right). \tag{10}$$

Assuming the minimum in (10) attained at an isolated point $x = x_0$ to be non-degenerate, one arrives at a point of *gradient catastrophe* of the solution $v(x, t)$, i.e., the limit

$$\lim_{t \to t_0,\ t < t_0} v(x_0, t) =: v_0 \tag{11}$$

exists, but the derivatives $v_x(x, t)$ $v_t(x, t)$ blow up at the point (x_0, t_0). Thus, the solution $u(x, t; \varepsilon)$ to the Cauchy problem (4)–(7), if exists, cannot be approximated by the inviscid solution. For the equation (9), a correct asymptotic formula can be found in [12]. In the present section, we derive a suitable modification of this asymptotic formula and present some heuristic arguments justifying its validity. In the next section, we also give numerical evidences supporting our conjectures.

Let us begin with recollecting some basics from the method of characteristics to solve the inviscid equation (9). For $t < t_0$, the solution to the inviscid equation can be represented in the following implicit form:

$$x = a(u)t + f(u) \tag{12}$$

where the function $f(u)$ is inverse[1] to the initial data $v(x,0)$,

$$f\left(v(x,0)\right) \equiv x. \tag{13}$$

Let (x_0, t_0) be the point of gradient catastrophe of the solution. As above, denote by $v_0 = v(x_0, t_0)$ the value of the solution at the point of catastrophe. The triple (x_0, t_0, v_0) satisfies the following system of equations:

$$x_0 = a_0 t_0 + f_0, \qquad 0 = a_0' t_0 + f_0', \qquad 0 = a_0'' t_0 + f_0''. \tag{14}$$

Here and below, we use the notation $a_0 = a(v_0)$, $f_0 = f(v_0)$, $a_0' = \left(da(v)/dv\right)_{v=v_0}$, $a_0'' = \left(d^2a(v)/dv^2\right)_{v=v_0}$, etc. In the subsequent considerations, we always assume that

$$a_0' \neq 0. \tag{15}$$

The genericity assumption

$$\kappa := -(1/6)(a_0''' t_0 + f_0''') \neq 0 \tag{16}$$

ensures that the graph of the solution $v(x, t_0)$ has a nondegenerate inflection point at $x = x_0$. Such a solution is said to be *generic*. The generic solution can locally be approximated by a cubic curve. For our subsequent considerations, this well-known statement can be presented in the following form (cf. [3]).

Lemma 1. *Near the point of gradient catastrophe, a generic solution* (12) *to the inviscid equation* (9) *admits the representation*

$$v(x,t) = v_0 + k^{1/3}\bar{v}(\bar{x},\bar{t}) + \mathcal{O}\left(k^{2/3}\right), \qquad k \to 0, \quad t < t_0, \tag{17}$$

$$\bar{x} = \frac{x - x_0 - a_0(t - t_0)}{k}, \qquad \bar{t} = \frac{t - t_0}{k^{2/3}}. \tag{18}$$

where the function $\bar{v}(\bar{x},\bar{t})$ for $\bar{t} < 0$ is given by the (unique) root of the cubic equation

$$\bar{x} = a_0' \bar{v}\,\bar{t} - \kappa\,\bar{v}^3. \tag{19}$$

The proof can readily be obtained by substituting (17), (18) into the implicit equation (12) of the method of characteristics and then by expanding with respect to the small parameter $k^{1/3}$. Observe that the uniqueness of the root of the cubic equation (19) for $\bar{t} < 0$ is ensured by the condition

$$a_0' \kappa > 0, \tag{20}$$

which holds due to monotone decreasing of the superposition $a\left(v(x, t_0)\right)$.

Remark 2. Note that the cubic equation (19) has a unique root for $t > t_0$ as well provided that

$$\frac{|\bar{x}|}{\bar{t}^{3/2}} > \frac{2}{3\sqrt{3}} \left(\frac{{a_0'}^3}{\kappa}\right)^{1/2}. \tag{21}$$

[1] If the initial data function is not globally monotone, then the representation (12) works on every interval of monotonicity.

This observation also readily implies the existence and uniqueness of a solution $v(x,t)$ to (9) for sufficiently small $t-t_0>0$ outside a cuspidal neighborhood

$$\frac{|x-x_0-a_0(t-t_0)|}{(t-t_0)^{3/2}}<C \quad \text{for some positive constant} \quad C \tag{22}$$

of the point of catastrophe.

We are now in a position to formulate the main statement of the present paper.

Conjecture 3. *Let $v(x,t)$ be the solution to the inviscid equation* (9) *with a smooth monotone initial data $v(x,0)$ defined on $\mathbb{R}\times[0,t_0)$ and let v have a gradient catastrophe at the point (x_0,t_0) satisfying* (15) *and* (20). *Assume that the smooth function $b(u)$ satisfies the condition*

$$b_0:=b(v_0)>0. \tag{23}$$

Then
1) *for sufficiently small $\varepsilon>0$, there exists a unique solution $u(x,t;\varepsilon)$ to the generalized Burgers equation* (4) *with the same ε-independent initial condition $u(x,0;\varepsilon)=v(x,0)$, $x\in\mathbb{R}$ defined on $\mathbb{R}\times[0,t_0+\delta(\varepsilon))$ for some sufficiently small $\delta(\varepsilon)>0$;*
2) *outside a cuspidal neighborhood of the point of catastrophe, the solution $u(x,t;\varepsilon)$ can be approximated by the inviscid solution $v(x,t)$ $|u(x,t;\varepsilon)-v(x,t)|=\mathcal{O}(\varepsilon)$.*
For arbitrary X and T, the limit

$$\lim_{\varepsilon\to 0+}\frac{u\left(x_0+a_0\beta\,\varepsilon^{1/2}T+\alpha\,\varepsilon^{3/4},t_0+\beta\,\varepsilon^{1/2}T\right)-v_0}{\gamma\,\varepsilon^{1/4}}=:U(X,T) \tag{24}$$

exists, where

$$\alpha=\left(\kappa\, b_0^3 {a_0'}^{-3}\right)^{1/4},\quad \beta=\left(\kappa\, b_0 {a_0'}^{-3}\right)^{1/2},\quad \gamma=\left(b_0\kappa^{-1}{a_0'}^{-1}\right)^{1/4}. \tag{25}$$

Moreover, the limit depends neither on the choice of solution nor on the choice of the ε-terms in the generalized Burgers equation (4) *and is given by the logarithmic derivative of the Pearcey function*

$$U(X,T)=-2\frac{\partial}{\partial X}\log\int_{-\infty}^{\infty}e^{-\frac{1}{8}\left(z^4-2z^2T+4z\,X\right)}dz. \tag{26}$$

A somewhat stronger version of the last statement of the Main Conjecture can be given in the form of the following asymptotic formula:

$$u(x,t;\varepsilon)=v_0+\gamma\,\varepsilon^{1/4}U\Big(\frac{x-x_0-a_0(t-t_0)}{\alpha\,\varepsilon^{3/4}},\frac{t-t_0}{\beta\,\varepsilon^{1/2}}\Big)+\mathcal{O}(\varepsilon^{1/2}), \tag{27}$$

which is expected to be valid on some neighborhood of the catastrophe point. For the particular case $b(u)\equiv 1$ and $c(u)\equiv 0$, the asymptotic formula (main3) coincides with that obtained by A. M. Il'in (see [12]).

Let us add few heuristic motivations for the Main Conjecture. First, consider a small-time behavior of the solution $u(x,t;\epsilon)$. Since the function $v(x,t)$ satisfies (4) modulo terms of order ε, one can seek a solution to the generalized Burgers equation in the form of a perturbative expansion $u(x,t;\epsilon)=v(x,t)+\varepsilon\, v^{(1)}(x,t)+\varepsilon^2 v^{(2)}(x,t)+\dots$ The terms of the expansion are to be determined from linear inhomogeneous equations (for details, see [12]). For example, the first correction can be found from the PDE $v_t^{(1)}+\left(a(v)v^{(1)}\right)_x=b(v)v_{xx}+c(v)v_x^2$. Instead, one can apply the method of the so-called *quasitriviality transformations* [5, 13] finding a universal substitution

$$v\mapsto u=v+\sum_{k\geqslant 1}\varepsilon^k\frac{f_k(v;v_x,v_{xx},\dots,v^{(4k-2)},\log|v_x|)}{v_x^{3k-2}} \tag{28}$$

transforming any monotone solution of the inviscid equation (9) to a formal asymptotic solution to the perturbed equation (4). Here $f_k(v; v_x, v_{xx}, \dots, v^{(4k-2)}, \log|v_x|)$ are some polynomials in the variables v_x, v_{xx}, $\dots$, $v^{(4k-2)}$, $\log|v_x|$ whose coefficients are smooth functions of v. They satisfy the following homogeneity condition:

$$f_k\big(v; \lambda v_x, \lambda^2 v_{xx}, \dots, \lambda^{4k-2} v^{(4k-2)}, \log|v_x|\big) = \lambda^{4k-2} f_k(v; v_x, v_{xx}, \dots, v^{(4k-2)}, \log|v_x|), \quad k \geqslant 1, \tag{29}$$

for any $\lambda \neq 0$. An advantage of the perturbative expansion written in the form (28) is the *locality* principle, namely, changing the unperturbed solution within a small neighborhood of a point (x^*, t^*) does not change the value of the perturbed solution outside the neighborhood.

For convenience of the reader, let us explain the computational algorithm for the derivation of the perturbative expansion (28). For simplicity, consider a perturbed equation of the form

$$u_t + u\, u_x = \varepsilon\, \Phi(u; u_x, u_{xx}, \dots), \tag{30}$$

where $\Phi(u; u_x, u_{xx}, \dots)$ is a smooth function of its variable polynomial in the jets u_x, u_{xx}, etc. Represent (30) as an equation for the function $x(u,t)$ inverse to $u(x,t)$,

$$x_t = u - \varepsilon\, x_u \Phi\big(u; 1/x_u, -x_{uu} x_u^{-3}, \dots\big). \tag{31}$$

The clue is in the following statement (see [13]) describing the perturbative solution to (31).

Lemma 4. *Define the function* $\Psi(u; x_u, x_{uu}, \dots)$ *by the rule*

$$\Psi(u; x_u, x_{uu}, \dots) = \int x_u \Phi\big(u; 1/x_u, -x_{uu} x_u^{-3}, \dots\big)\, dx_u. \tag{32}$$

Then the function

$$x(u,t) = x^{(0)}(u,t) - \varepsilon\, x^{(1)}(u,t) \tag{33}$$

such that

$$x_t^{(0)} = u, \quad x^{(1)} = \Psi\big(u; x_u^{(0)}, x_{uu}^{(0)}, \dots\big), \tag{34}$$

satisfies the perturbed equation (31) *modulo terms of order* ε^2.

The proof is immediate because the higher u-derivatives of $x^{(0)}$ do not depend on t, namely, $\partial/\partial t\, \partial^m x^{(0)}/\partial u^m = \partial^m/\partial u^m\, \partial x^{(0)}/\partial t = \delta_{m,1}$ for $m \geqslant 1$.

Inverting the series (33) gives the desired algorithm.

Corollary 5. *Let* $v = v(x,t)$ *be a solution to the PDE* $v_t + v\, v_x = 0$ *satisfying* $v_x \neq 0$. *Then the function*

$$u = v + \varepsilon\, v_x \Psi\left(v; \frac{1}{v_x}, -\frac{v_{xx}}{v_x^3}, \dots\right) \tag{35}$$

satisfies the perturbed equation (30) *modulo terms of order* ε^2.

For the particular case of the generalized Burgers equation (4), the first terms of the quasitriviality expansion read

$$u = v - \varepsilon\left[\frac{b}{a'}\frac{v_{xx}}{v_x} + \frac{c\, a' - b\, a''}{a'^2} v_x \log|v_x|\right] + \mathcal{O}\left(\varepsilon^2\right). \tag{36}$$

It would be of interest to rigorously justify that, for sufficiently small ε, the above algorithm produces an asymptotic expansion of an actual solution to the generalized Burgers equation.

Consider now the solution to (4) in a neighborhood of the point of catastrophe. After a change of variables in (4),

$$x - x_0 - a_0(t - t_0) = \varepsilon^{3/4}\bar{x}, \qquad t - t_0 = \varepsilon^{1/2}\bar{t}, \qquad u - v_0 = \varepsilon^{1/4}\bar{u}, \tag{37}$$

one arrives at the equation

$$\bar{u}_{\bar{t}} + a_0'\bar{u}\,\bar{u}_{\bar{x}} = b_0\bar{u}_{\bar{x}\bar{x}} + \mathcal{O}\left(\varepsilon^{1/4}\right). \tag{38}$$

Another substitution.

$$\bar{x} = \alpha X, \quad \bar{t} = \beta T, \quad \bar{u} = \gamma U, \tag{39}$$

reduces the leading term of (38) to the standard form of the Burgers equation $U_T + U\,U_X = U_{XX}$ provided that the constants α, β, and γ satisfy the constraints

$$a_0'\frac{\beta\gamma}{\alpha} = 1, \quad b_0\frac{\beta}{\alpha^2} = 1. \tag{40}$$

The Burgers equation is solved by the Cole–Hopf substitution $U(X,T) = -2\partial/\partial X \log W(X,T)$, where $W = W(X,T)$ solves the heat equation $W_T = W_{XX}$. The Pearcey function

$$W(X,T) = \int_{-\infty}^{\infty} e^{-\frac{1}{8}\left(z^4 - 2z^2T + 4z\,X\right)}dz$$

clearly satisfies the heat equation. We claim that, using this function in the substitution $\bar{u} = -2\gamma\frac{\partial}{\partial X}W(X,T)$, one arrives at a correct asymptotic expression for the function $\bar{u}$ near the point of catastrophe,

$$\bar{x} = a_0'\bar{u}\,\bar{t} - \kappa\bar{u}^3 + \mathcal{O}\left(\varepsilon^{1/4}\right) \tag{41}$$

(cf. (19) above). Indeed, rescaling the integration variable $\zeta = \varepsilon^{1/4}z$, we represent the expression for $\bar{u}$ in the form

$$\bar{u} = -2\alpha\,\gamma\,\varepsilon^{3/4}\frac{\partial}{\partial x}\log\int_{-\infty}^{\infty} e^{-\frac{S(\zeta;x,t)}{\varepsilon}}d\zeta,$$

where

$$S(\zeta;x,t) = \frac{1}{8}\left(\zeta^4 - 2\zeta^2\frac{t-t_0}{\beta} + 4\zeta\,\frac{x - x_0 - a_0(t-t_0)}{\alpha}\right). \tag{42}$$

For $t < t_0$, the phase function has a unique minimum at the point $\zeta_0 = \zeta_0(x,t)$; this minimum is determined by the cubic equation

$$x - x_0 - a_0(t - t_0) = \frac{\alpha}{\beta}(t - t_0)\,\zeta_0 - \alpha\zeta_0^3. \tag{43}$$

Applying the Laplace formula to the Pearcey integral

$$\int_{-\infty}^{\infty} e^{-\frac{S(\zeta;x,t)}{\varepsilon}}d\zeta = \frac{2\sqrt{\pi\,\varepsilon}}{\sqrt{3\zeta_0^2 - \frac{t-t_0}{\beta}}}e^{-\frac{S(\zeta_0;x,t)}{\varepsilon}}\left(1 + \mathcal{O}\left(\varepsilon\right)\right)$$

and using the obvious formula $\partial S\left(\zeta_0(x,t);x,t\right)/\partial x = \zeta_0(x,t)/(2\alpha)$, one arrives at the expansion $\bar{u} = \gamma\,\varepsilon^{-1/4}\zeta_0\left(1 + \mathcal{O}\left(\varepsilon\right)\right)$. The substitution into the cubic equation (43) yields (41) provided that the constants α, β, and γ satisfy another constraint,

$$\alpha\gamma^{-3} = \kappa. \tag{44}$$

This, together with the constraints (40), gives (25).

2. SOLVING NUMERICALLY THE GENERALIZED BURGERS EQUATION. COMPARISON WITH THE ASYMPTOTIC FORMULA

In order to test the numerical algorithms, we begin with the standard Burgers equation. First, consider the Cauchy problem for the inviscid equation

$$u_t + uu_x = 0, \qquad u(x,0) = F(x). \tag{45}$$

At the point of catastrophe, one has

$$x_0 = a_0 + F(a_0)t_0, \quad t_0 = -\frac{1}{F'(a_0)}, \quad u_0 = F(a_0), \quad F''(a_0) = 0 \tag{46}$$

(cf. (14) above). For the particular choice of the initial data $F(x) = 1/(1+x^2)$, the point of the catastrophe can be located as follows:

$$x_0 = \sqrt{3}, \qquad t_0 = 8\sqrt{3}/9, \qquad u_0 = 3/4. \tag{47}$$

For $t > t_0$, the solution to the Cauchy problem is close to a discontinuous one. Indeed, as is well known (see, e.g., [15]), the limit as $\epsilon \to 0$ of a smooth solution to the Burgers equation

$$u_t + uu_x = \varepsilon u_{xx} \tag{48}$$

is described by a discontinuous function on the (x,t)-plane. The curve of discontinuity $x = s(t)$ of the limiting function is referred to as the *shock front* (the solid line on Fig. 1). We shall try to find a numerical solution to the Cauchy problem in a neighborhood of the shock front and compare it with Il'in's asymptotic formula. Let us explain the algorithm used to find the shock front.

Fixing a point $t = t^*$, we select an array of values $\{x_i^*\}$ in some neighborhood of the curve $x = s(t)$. We evaluate the function $u = u_i^*$ at the points (t^*, x_i^*), applying Il'in's asymptotic formula and using Maple to compute the Pearcey function.

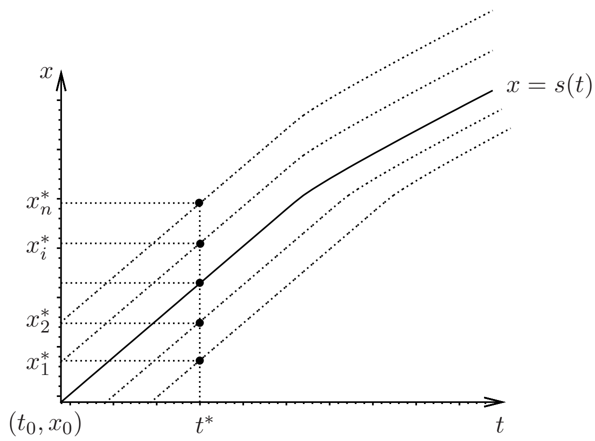

Fig. 1. Shock front.

To find the shock front (see [15]), we use the Rankin–Hugoniot conditions

$$\frac{dx}{dt} = \frac{1}{2}(F(a_1) + F(a_2)), \tag{49}$$

where $a_1(t)$ and $a_2(t)$ are determined by the equations of characteristics

$$x(t) = a_1 + F(a_1)t, \quad x(t) = a_2 + F(a_2)t. \tag{50}$$

Differentiating (50) with respect to t and taking into account (49), one arrives at a system of differential equations for the functions $x(t)$, $a_1(t)$, and $a_2(t)$,

$$dx/dt = (1/2)(F(a_1) + F(a_2)), \quad da_1/dt = \frac{1}{2}\frac{F(a_2) + F(a_1)}{1 + F'(a_1)t}, \quad da_2/dt = \frac{1}{2}\frac{F(a_1) - F(a_2)}{1 + F'(a_2)t}. \tag{51}$$

The initial data for these equations have the form

$$a_1(t_0) = a_0, \qquad a_2(t_0) = a_0, \qquad x(t_0) = x_0, \tag{52}$$

where x_0, t_0, and u_0 are given by (46).

If the solution to the Cauchy problem (51)–(52) can be represented in an explicit analytic form, then the shock front can also be computed explicitly. Otherwise, system (51)–(52) can be solved numerically. Note that, at $t = t_0$, one faces an "indeterminate form" $\frac{0}{0}$. It can be resolved with the help of asymptotic expansions of the functions $x(t)$, $a_1(t)$, and $a_2(t)$ near the point $t = t_0$. If $a_1(t) < a_0 < a_2(t)$ for $t > t_0$, then, for the characteristics $a_1(t)$ and $a_2(t)$, we have

$$a_1(t) = a_0 - \left(\frac{2F'(a_0)^2}{F'''(a_0)}(t - t_0)\right)^{1/2}, \qquad a_2(t) = a_0 + \left(\frac{2F'(a_0)^2}{F'''(a_0)}(t - t_0)\right)^{1/2}. \tag{53}$$

The expansion of $x(t)$ near $t = t_0$ is of the form

$$x = x_0 + F(a_0)(t - t_0). \tag{54}$$

Thus, to solve the Cauchy problem (51)–(52), we solve the system of differential equations (51), where we set $t = t_0 + \Delta t$. Here Δt is the time step. We use the asymptotic values (53) and (54) as the initial data, i.e.,

$$a_1(t_0+\Delta t) = a_0 - \left(\frac{2F'(a_0)^2}{F'''(a_0)}\Delta t\right)^{1/2}, \; a_2(t+\Delta t) = a_0 + \left(\frac{2F'(a_0)^2}{F'''(a_0)}\Delta t\right)^{1/2}, \; x(t_0+\Delta t) = x_0 + F(a_0)\Delta t.$$

In order to control the computation, the following identity is used (see, e.g., [15]):

$$\frac{1}{2}(F_1(a_1) + F_2(a_2))(a_1 - a_2) = \int_{a_2}^{a_1} F(a)da.$$

Finite element analysis

To solve the standard Burgers equation (48), we use the finite element method (see [6–8, 10, 11]) realized in the package FreeFem++ [11]. Since this package, strictly speaking, is not designed for solving spatially one-dimensional problems, one can reformulate the original problem in the 2D form by considering solutions depending on one spatial variable only. Assume that the 2D domain has the rectangular form, $\Omega = \{(x, y) : 0 \leqslant x \leqslant L_x, 0 \leqslant y \leqslant L_y\}$, of the size $L_x \times L_y$ and $L_x \gg L_y$.

We impose the no-flux boundary conditions at $y = 0$ and $y = L_y$ but do not specify the values of u at $x = 0$ and $x = L_x$, assuming that the boundary values of u are fixed at some fictitious boundary of a wider region,

$$\left.\frac{\partial u}{\partial n}\right|_{y=0,L_y} = 0 \qquad \left.\frac{du}{dt}\right|_{x=0,L_x} = 0. \tag{55}$$

Here n is the exterior normal to the boundary $\partial\Omega$ and $d/dt = \partial/\partial t + u\partial/\partial x$.

In the numerical experiments, we use the following initial data:

$$u|_{t=0} = \frac{1}{1 + x^2} \tag{56}$$

For the time approximation, the semi-explicit Euler scheme is be used. To this end, we multiply the equation by a test function θ, integrating the resulting expression over the domain Ω,

$$\iint_\Omega \left(\frac{u^{m+1} - u^m}{\tau}\theta + u^m u_x^{m+1}\theta\right) dxdy = \iint_\Omega \varepsilon u_{xx}^{m+1}\theta \, dxdy,$$

or, taking into account the boundary conditions, write

$$\iint_\Omega \left(\frac{u^{m+1} - u^m}{\tau}\theta + u^m u_x^{m+1}\theta + \varepsilon u_x^{m+1}\theta_x\right) dxdy = \int_{\partial\Omega} \theta\frac{\partial u}{\partial n}ds. \tag{57}$$

Problem (57) in the weak formulation, along with the initial conditions (56), is solved by means of the FreeFem++ package.

Comparison of the numerical solution with Il'in's asymptotic formula near the shock front $x = s(t)$ for $t = 1.54$ and $\varepsilon = 0.01$ is shown on Fig. 2. The solid line shows the numerical solution obtained by the finite-element method, while the dashed line corresponds to the asymptotic solution (27).

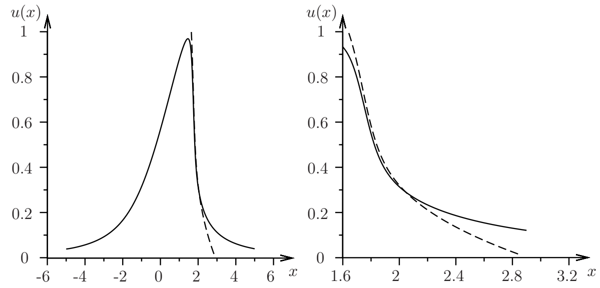

Fig. 2. Comparison of a numerical solution to Burgers equation with Il'in's asymptotic formula for $\varepsilon = 0.01$.

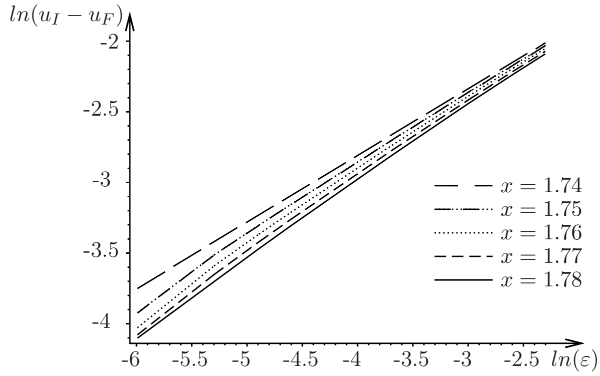

Fig. 3. Numerical estimate of the truncation error in the asymptotic formula.

On the right-hand part of the figure, the region near the catastrophe point x_0, t_0, u_0 (see (47)) is zoomed in.

In Fig. 3, the difference between the asymptotic solution u_I given by (27) and the numerical solution u_F is shown in the logarithmic scale. The evaluation of u_I and u_F is carried out for $t^* = 1.54$, $x^* = \{1.74, 1.75, 1.76, 1.77, 1.78\}$, and $\varepsilon = \{0.0025, 0.005, 0.0075, 0.01, 0.025, 0.05, 0.075, 0.1\}$. The average slope is 0.5175, with the expected value 0.5.

During the computation, we control the total mass as a function of time. With the boundary conditions under consideration, the total mass is a conserved quantity. Thus, the conservation of the total mass is a good test for the quality of numerical simulations. The results for $\varepsilon = \{0.1, 0.01, 0.0025\}$ are shown in Fig. 4. On the interval $[0, 1.8]$ with $\varepsilon = 0.1$, the relative error is 0.0024, for $\varepsilon = 0.01$, the relative error is 0.0006, while, for $\varepsilon = 0.0025$, it drops to 0.0006.

Generalized Burgers equation

Let us now proceed to a particular example of the generalized Burgers equation (4),

$$u_t + uu_x = \varepsilon(uu_x)_x. \tag{58}$$

completed by the boundary conditions (55) and the initial data (56). As in the case of the standard Burgers equation (48), the semi-explicit Euler scheme is used for the time approximation. The variational reformulation of the problem along with the boundary conditions reads

$$\iint_\Omega \left(\frac{u^{m+1} - u^m}{\tau}\theta + u^m u_x^{m+1}\theta + \varepsilon u^m u_x^{m+1}\theta_x \right) dxdy = \int_{\partial\Omega} \theta \frac{\partial u}{\partial n} ds \tag{59}$$

The numerical solution of the problem (58) (in the weak formulation) with the initial data (56) is computed with the help of the FreeFem++ package.

As above, let us compare the results of the numerical simulations with the predictions given by the asymptotic formula (27). In Fig. 5, the solid curve shows the numerical solution, while the

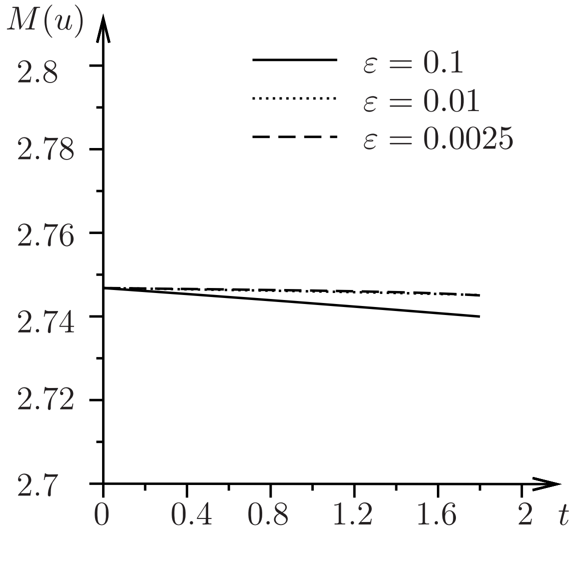

Fig. 4. Testing the conservation of the total mass.

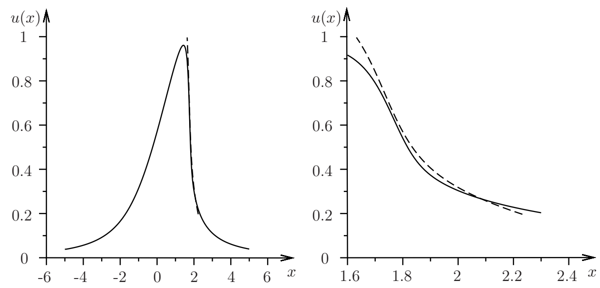

Fig. 5. Comparison of a numerical solution to the generalized Burgers equation with the asymptotic formula (27) for $\varepsilon = 0.01$.

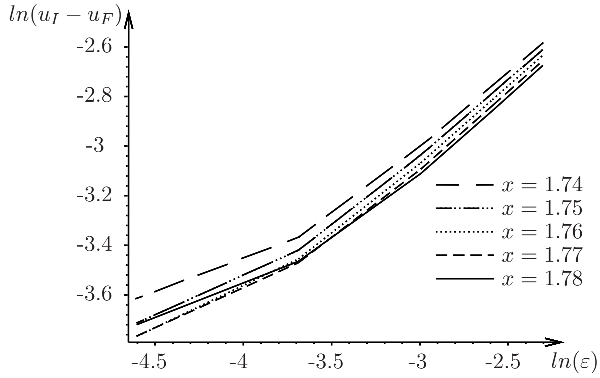

Fig. 6. Numerical estimate of the truncation error in the asymptotic formula (27) for solutions to the generalized Burgers equation.

dashed curve is the graph of the asymptotic solution (27). On the right-hand part of the figure, a neighborhood of the point of catastrophe x_0, t_0, u_0 is zoomed in.

In Fig. 6, the difference between the asymptotic formula u_I and the numerical solution u_F is shown in the logarithmic scale, for the values of (x, t, ε) given by $t^* = 1.54$, $x^* = \{1.74, 1.75, 1.76, 1.77, 1.78\}$, and $\varepsilon = \{0.01, 0.025, 0.05, 0.075, 0.1\}$. One can observe the average slope of 0.5221, against the expected value 0.5.

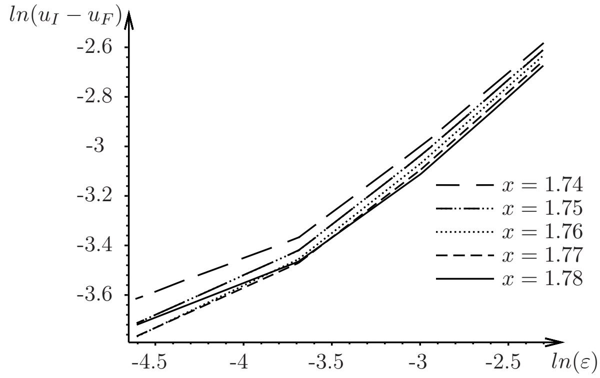

Fig. 7. Testing conservation of the total mass for the numerical solution to the generalized Burgers equation (58)**.**

As above, we have used the conservation of the total mass (which holds for our particular case (58) of the generalized Burgers equation) as a test for the validity of the numerical scheme. The results are shown on Fig. 7 for the values $\varepsilon = \{0.1, 0.01\}$. On the interval $[0, 1.8]$, for $\varepsilon = 0.1$, the relative decay is 0.0025; for $\varepsilon = 0.01$, it drops to 0.0044.

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