Generating series for GUE correlators

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Abstract

We extend to the Toda lattice hierarchy the approach of [3, 4] to computation of logarithmic derivatives of tau-functions in terms of the so-called *matrix resolvents* of the corresponding difference Lax operator. As a particular application we obtain explicit generating series for connected GUE correlators. On this basis an efficient recursive procedure for computing the correlators in full genera is developed.

1 Introduction

1.1 Formulation of the main result

Denote $\mathcal{H}(N)$ the space of $N \times N$ Hermitean matrices. The Gaussian Unitary Ensemble (GUE) correlators of observables $\operatorname{tr} M^i$, $i = 1, 2, \dots$ with respect to the Gaussian probability measure on $\mathcal{H}(N)$ are defined by

$$\left\langle \operatorname{tr} M^{i_1} \cdots \operatorname{tr} M^{i_k} \right\rangle := \frac{\int\limits_{\mathcal{H}(N)} \operatorname{tr} M^{i_1} \dots \operatorname{tr} M^{i_k} \, e^{-\frac{1}{2}\operatorname{tr} M^2} \, dM}{\int\limits_{\mathcal{H}(N)} e^{-\frac{1}{2}\operatorname{tr} M^2} dM}. \tag{1.1.1}$$

They are certain polynomials in N that can be computed by the Wick rule. By $\left\langle \operatorname{tr} M^{i_1} \cdots \operatorname{tr} M^{i_k} \right\rangle_c$ we denote the *connected* correlators. That means that, applying the Wick rule to the computation of Gaussian integrals (1.1.1) we keep summation over connected Feynman diagrams only. For example, $\langle \left(\operatorname{tr} M^2\right)^2 \rangle = 2N^2 + N^4$ but $\langle \left(\operatorname{tr} M^2\right)^2 \rangle_c = 2N^2$; see more details in Appendix A. According to [17, 18, 5] the connected GUE correlators have an important application to the problem of enumeration of ribbon graphs on two-dimensional oriented surfaces, see details in Appendix A below. This was one of the motivations for a significant interest to the problem of computation of the GUE correlators, see e.g. [2, 14, 19, 23].

For every $k \geq 1$ we will consider generating series of the k-point correlators of the form

$$C_k(N; \lambda_1, \dots, \lambda_k) := \sum_{i_1, \dots, i_k = 1}^{\infty} \frac{\left\langle \operatorname{tr} M^{i_1} \cdots \operatorname{tr} M^{i_k} \right\rangle_c}{\lambda_1^{i_1+1} \dots \lambda_k^{i_k+1}} \tag{1.1.2}$$

where $\lambda_1, \dots, \lambda_k$ are independent variables, N refers to the size of the Hermitean matrices. Our main result is the following explicit expressions for the generating series (1.1.2).

Theorem 1.1.1 *1) The generating series for* 1*-point correlators has the form*

$$C_1(N;\lambda) = N\sum_{j\geq 0}\frac{(2j-1)!!}{\lambda^{2j+1}}\left[{}_2F_1(-j,-N;2;2) - j\cdot {}_2F_1(1-j,1-N;3;2)\right]. \tag{1.1.3}$$

2) Introduce a 2×2 *matrix-valued series*

$$\mathcal{R}_n(\lambda) := \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + n\sum_{j=0}^{\infty}\frac{(2j-1)!!}{\lambda^{2j+2}}\begin{pmatrix}(2j+1)A_{n,j} & -\lambda\, B_{n+1,j}\\ \\ \frac{\lambda}{n}\, B_{n,j} & -(2j+1)A_{n,j}\end{pmatrix} \in \mathrm{Mat}\left(2,\mathbb{Z}[n]\left[[\lambda^{-1}]\right]\right) \tag{1.1.4}$$

where

$$\begin{aligned} A_{n,j} &= \frac{1}{n}\sum_{i=0}^{j} 2^i\begin{pmatrix} j\\ i\end{pmatrix}\begin{pmatrix} n\\ i+1\end{pmatrix} = {}_2F_1(-j,1-n;2;2)\\ \\ B_{n,j} &= \sum_{i=0}^{j} 2^i\begin{pmatrix} j\\ i\end{pmatrix}\begin{pmatrix} n-1\\ i\end{pmatrix} = {}_2F_1(-j,1-n;1;2).\end{aligned} \tag{1.1.5}$$

Then

$$C_2(N;\lambda_1,\lambda_2) = \frac{\mathrm{tr}\,\mathcal{R}_N(\lambda_1)\mathcal{R}_N(\lambda_2) - 1}{(\lambda_1-\lambda_2)^2} \tag{1.1.6}$$

$$C_k(N;\lambda_1,\dots,\lambda_k) = -\frac{1}{k}\sum_{\sigma\in S_k}\frac{\mathrm{tr}\,\left[\mathcal{R}_N(\lambda_{\sigma_1})\dots\mathcal{R}_N(\lambda_{\sigma_k})\right]}{(\lambda_{\sigma_1}-\lambda_{\sigma_2})\dots(\lambda_{\sigma_{k-1}}-\lambda_{\sigma_k})(\lambda_{\sigma_k}-\lambda_{\sigma_1})},\quad k\geq 3. \tag{1.1.7}$$

In the above formulae

$${}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!} = 1 + \frac{a\,b}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots$$

is the Gauss hypergeometric function. Recall that it truncates to a polynomial if a or b are non-positive integers.

Remark 1.1.2 *To the best of our knowledge the first generating series for the one-point connected correlators was obtained by J. Harer and D. Zagier in [16]. In [22] A. Morozov and Sh. Shakirov constructed a generating function of the Harer–Zagier type for the two-point correlators. These generating functions are different from ours but produce identical results for the correlators.*

1.2 Matrix resolvent of a second order difference operator and tau-function of the Toda lattice

Consider a second order difference operator L acting on functions ψ_n, $n\in\mathbb{Z}$ by

$$\left(L\,\psi\right)_n = \psi_{n+1} + v_n\,\psi_n + w_n\,\psi_{n-1}. \tag{1.2.1}$$

The standard realization of the Toda lattice hierarchy is given by the Lax representation

$$\frac{\partial L}{\partial t_j} = [A_j, L], \quad j \geq 0 \tag{1.2.2}$$

$$A_j = \left(L^{j+1}\right)_+. \tag{1.2.3}$$

Introduce the matrix

$$U_n(\lambda) = \begin{pmatrix} v_n - \lambda & w_n \\ -1 & 0 \end{pmatrix}. \tag{1.2.4}$$

Observe that the second order difference equation for the eigenfunctions of the Lax operator

$$L\,\psi = \lambda\,\psi$$

can be written in the matrix form

$$\Delta\,\Psi_n + U_n(\lambda)\,\Psi_n = 0, \quad \Psi_n = \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \tag{1.2.5}$$

where Δ is the shift operator

$$\Delta\,\Psi_n = \Psi_{n+1}.$$

Introduce the ring $\mathbb{Z}[\mathbf{v}, \mathbf{w}]$ of polynomials with integer coefficients in the infinite set of variables $\mathbf{v} = (v_n)$, $\mathbf{w} = (w_n)$, $n \in \mathbb{Z}$.

Lemma 1.2.1 *There exists a unique* 2×2 *matrix series*

$$R_n(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}\left(\lambda^{-1}\right) \in \mathrm{Mat}\left(2, \mathbb{Z}[\mathbf{v}, \mathbf{w}]\left[[\lambda^{-1}]\right]\right)$$

satisfying equation

$$R_{n+1}(\lambda)U_n(\lambda) - U_n(\lambda)R_n(\lambda) = 0 \tag{1.2.6}$$

along with the normalization conditions

$$\mathrm{tr}\,R_n(\lambda) = 1, \quad \det R_n(\lambda) = 0. \tag{1.2.7}$$

Definition 1.2.2 *The series* $R_n(\lambda)$ *is called the* matrix resolvent *of the difference operator* L.

Let the difference operator L depend on the times of the Toda lattice hierarchy (1.2.2), (1.2.3). Then so does the matrix resolvent, $R_n = R_n(\mathbf{t}, \lambda)$. Here and below $\mathbf{t} := (t_0, t_1, \dots)$. We will now present an algorithm for computing the so-called tau-function of an arbitrary solution $v_n(\mathbf{t})$, $w_n(\mathbf{t})$. The tau-function of a solution will be determined uniquely, up to a simple factor, by an explicitly written collection of its second order logarithmic derivatives in the continuous variables t_k and the discrete variable n.

Lemma 1.2.3 *For any solution $v_n = v_n(\mathbf{t})$, $w_n = w_n(\mathbf{t})$ to the Toda lattice hierarchy there exists a function $\tau_n(\mathbf{t})$ such that*

$$\sum_{i,\,j\geq 0} \frac{1}{\lambda^{i+2}\mu^{j+2}} \frac{\partial^2 \log \tau_n(\mathbf{t})}{\partial t_i\, \partial t_j} = \frac{\operatorname{tr} R_n(\mathbf{t},\lambda) R_n(\mathbf{t},\mu) - 1}{(\lambda-\mu)^2} \tag{1.2.8}$$

$$\sum_{i\geq 0} \frac{1}{\lambda^{i+2}} \frac{\partial}{\partial t_i} \log \frac{\tau_{n+1}(\mathbf{t})}{\tau_n(\mathbf{t})} = [R_{n+1}(\mathbf{t},\lambda)]_{21} \tag{1.2.9}$$

$$\frac{\tau_{n+1}(\mathbf{t})\tau_{n-1}(\mathbf{t})}{\tau_n^2(\mathbf{t})} = w_n. \tag{1.2.10}$$

The function $\tau_n(\mathbf{t})$ is determined uniquely by the solution $v_n(\mathbf{t})$, $w_n(\mathbf{t})$ up to

$$\tau_n(\mathbf{t}) \mapsto e^{a_0 + a_1 n + \sum_{j\geq 0} b_j t_j} \tau_n(\mathbf{t})$$

for some constants a_0, a_1, b_0, b_1, $\dots$ independent of n.

Definition 1.2.4 *The function $\tau_n(\mathbf{t})$ defined by eqs.* (1.2.8)–(1.2.10) *is called the* tau-function *of the solution $v_n(\mathbf{t})$, $w_n(\mathbf{t})$.*

Remarkably, the higher logarithmic derivatives of the tau-function can also be expressed in terms of the matrix resolvent.

Theorem 1.2.5 *The order $k \geq 3$ logarithmic derivatives of the tau-function of a solution to the Toda lattice hierarchy can be computed from the following generating series*

$$\sum_{i_1,\dots,i_k=0}^{\infty} \frac{1}{\lambda_1^{i_1+2}\dots\lambda_k^{i_k+2}} \frac{\partial^k \log \tau_n(\mathbf{t})}{\partial t_{i_1}\dots\partial t_{i_k}} = -\frac{1}{k}\sum_{\sigma\in S_k} \frac{\operatorname{tr}\,[R_n(\mathbf{t},\lambda_{\sigma_1})\dots R_n(\mathbf{t},\lambda_{\sigma_k})]}{(\lambda_{\sigma_1}-\lambda_{\sigma_2})\dots(\lambda_{\sigma_{k-1}}-\lambda_{\sigma_k})(\lambda_{\sigma_k}-\lambda_{\sigma_1})}. \tag{1.2.11}$$

It is well-known [24], [15] that the GUE partition function (see eq. (A.1.1) below) is the tau-function of a particular solution to the Toda lattice, identifying $s_k = t_{k-1}$. Logarithmic derivatives of the tau-function evaluated at $\mathbf{t} = 0$ coincide with the connected correlators (1.1.1). This solution is specified by the initial data

$$v_n(\mathbf{t}=0) = 0, \quad w_n(\mathbf{t}=0) = n. \tag{1.2.12}$$

Hence it remains to compute the matrix resolvent of the operator

$$\Delta - \begin{pmatrix} \lambda & -n \\ 1 & 0 \end{pmatrix}. \tag{1.2.13}$$

Theorem 1.2.6 *The matrix resolvent of the operator* (1.2.13) *coincides with series $\mathcal{R}_n(\lambda)$ given by eq.* (1.1.4).

Theorem 1.1.1 readily follows from Theorems 1.1.1 and 1.2.5.

The approach of the present paper can be generalized to the integrable systems associated with higher order difference Lax operators. Such a generalization will be developed in a subsequent publication.

Organization of the paper. In Sect. 2 we prove Lem. 1.2.1, Lem. 1.2.3 and Thm. 1.2.5. In Sect. 3 we prove Thm. 1.2.6 and give an algorithm of computing connected GUE correlators in a recursive way. In Sect. 4 we outline an algorithm for computing the genus expansion of the GUE free energy based on [9, 7, 8]. A short review on the Hermitean matrix model, mainly following [5, 21, 15], is given in Appendix A.

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2 Computing tau-functions of Toda lattice hierarchy

2.1 Matrix resolvents. Proof of Lemma 1.2.1

In this section we will remind basic constructions of the Toda lattice hierarchy. We will also prove the Lemma 1.2.1.

The Toda lattice is a system of particles on the line with exponential interaction of neighbors. The Hamiltonian is written as a formal infinite sum

$$H(q,p) = \sum_{n\in\mathbb{Z}} \frac{p_n^2}{2} + e^{q_n - q_{n+1}}.$$

After the substitution

$$v_n = -\dot{q}_n, \quad w_n = e^{q_{n-1}-q_n} \tag{2.1.1}$$

the equations of motion

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}, \quad n \in \mathbb{Z}$$

take the form

$$\begin{aligned} \dot{v}_n &= w_{n+1} - w_n \\ \dot{w}_n &= w_n(v_n - v_{n-1}). \end{aligned} \tag{2.1.2}$$

Eqs. (2.1.2) are considered as a differential-difference evolution system with the time variable t and discrete spatial variable $n \in \mathbb{Z}$. Integrability of the Toda equations was discovered by H. Flaschka [13] and S. Manakov [20]. The corresponding Lax operator is a second order difference operator (1.2.1). The standard realization of the commuting flows of the Toda lattice hierarchy is given by the Lax representation (1.2.2). The first flow of the hierarchy coincides with (2.1.2), $t = t_0$, then

$$\frac{\partial v_n}{\partial t_1} = w_{n+1}(v_{n+1}+v_n) - w_n(v_n + v_{n-1}), \quad \frac{\partial w_n}{\partial t_1} = w_n(w_{n+1} - w_{n-1} + v_n^2 - v_{n-1}^2)$$

etc. They are Hamiltonian equations of the form

$$\frac{\partial v_n}{\partial t_j} = \{v_n, H_j\}, \qquad \frac{\partial w_n}{\partial t_j} = \{w_n, H_j\}, \qquad j \geq 0$$

with respect to the Poisson brackets

$$\{q_n, p_m\} = \delta_{mn} \quad \Rightarrow \quad \{v_n, w_n\} = -w_n, \quad \{v_n, w_{n+1}\} = w_{n+1} \tag{2.1.3}$$

(use the substitution (2.1.1)) with the Hamiltonians

$$H_j = \sum_n h_j(n), \quad h_j(n) = \frac{1}{j+2}\left(L^{j+2}\right)_{nn}, \qquad j \geq -1. \tag{2.1.4}$$

Here $\left(L^{j+2}\right)_{nn}$ means taking the n-th diagonal entry of the infinite matrix L^{j+2}. The first several Hamiltonian "densities" read

$$h_{-1}(n) = v_n, \qquad h_0(n) = \frac{1}{2}(v_n^2 + w_n + w_{n+1})$$

$$h_1(n) = \frac{1}{3}[v_n^3 + 2v_n(w_n + w_{n+1}) + w_n v_{n-1} + w_{n+1}v_{n+1}].$$

Note that $h_{-1}(n)$ is the density of one of the Casimirs of the Poisson bracket. The Hamiltonian densities $h_j(n)$, $j \geq -1$ satisfy

$$(i+1)\frac{\partial h_{j-1}}{\partial t_i} = (j+1)\frac{\partial h_{i-1}}{\partial t_j}, \qquad \forall\, i, j \geq 0.$$

Observe that, changing the normalization

$$\tilde{h}_j = \frac{1}{(j+1)!}h_j \quad \Rightarrow \quad \tilde{t}_j = (j+1)!\, t_j$$

one arrives at the *tau-symmetry* property

$$\frac{\partial \tilde{h}_{j-1}}{\partial \tilde{t}_i} = \frac{\partial \tilde{h}_{i-1}}{\partial \tilde{t}_j}. \tag{2.1.5}$$

Such a normalization of the Hamiltonians/time variables of the hierarchy is used when working with the Gromov–Witten invariants of $\mathbf{P}^1$ [10, 6].

It will be more convenient to work with the λ-dependent first order matrix version (1.2.5), (1.2.4) of the Lax operator acting on two-component vector-valued functions on the lattice

$$\Psi_n = \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}.$$

Notice that the equation (1.2.6) for what we call matrix resolvent $R_n(\lambda)$ can be written in the form

$$[\Delta + U_n(\lambda), R_n(\lambda)] = 0. \tag{2.1.6}$$

Clearly the normalization conditions (1.2.7) are compatible with eq. (1.2.6).

Let us proceed with the proof of Lemma 1.2.1. Write

$$R_n(\lambda) = \begin{pmatrix} 1 + \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}.$$

Substituting this expression in (1.2.6) we obtain

$$\beta_n = -w_n\, \gamma_{n+1} \tag{2.1.7}$$

$$\alpha_{n+1} + \alpha_n + 1 = \gamma_{n+1}\,(\lambda - v_n) \tag{2.1.8}$$

$$(\lambda - v_n)(\alpha_n - \alpha_{n+1}) = w_n\, \gamma_n - w_{n+1}\, \gamma_{n+2}. \tag{2.1.9}$$

Expand

$$\gamma_n = \sum_{j\ge 0} \frac{c_{n,j}}{\lambda^{j+1}}, \qquad \alpha_n = \sum_{j\ge 0} \frac{a_{n,j}}{\lambda^{j+1}}.$$

It follows immediately from (2.1.8)–(2.1.9) the recursion relations:

$$c_{n,j+1} = v_{n-1}\, c_{n,j} + a_{n,j} + a_{n-1,j}. \tag{2.1.10}$$

$$a_{n,j+1} - a_{n+1,j+1} + v_n\,[a_{n+1,j} - a_{n,j}] + w_{n+1}\, c_{n+2,j} - w_n\, c_{n,j} = 0. \tag{2.1.11}$$

The normalization conditions (1.2.7) imply

$$a_{n,0} = 0, \qquad c_{n,0} = 1$$

along with another recursion relation

$$a_{n,\ell} = w_n\,(c_{n+1,\ell-1} + c_{n,\ell-1}) + \sum_{i+j=\ell-1} [a_{n,i}a_{n,j} + c_{n,i}c_{n+1,j}] \tag{2.1.12}$$

The Lemma 1.2.1 readily follows from the recursion relations (2.1.10) and (2.1.12).

2.2 Matrix resolvents and Toda flows. Proof of Lemma 1.2.3 and Thm. 1.2.6

We will now represent equations of the Toda flows in terms of the matrix resolvent R_n.

Lemma 2.2.1 *The Toda flows* (1.2.2) *can be written in terms of* $c_{n,j}$, $a_{n,j}$ *as follows:*

$$\frac{\partial v_n}{\partial t_j} = a_{n+1,j+1} - a_{n,j+1}, \qquad j \ge 0 \tag{2.2.1}$$

$$\frac{\partial w_n}{\partial t_j} = w_n\,\left(c_{n+1,j+1} - c_{n,j+1}\right), \qquad j \ge 0. \tag{2.2.2}$$

Proof. By using the recursion relations (2.1.10), (2.1.11) and by comparing them with (1.2.2). □

Lemma 2.2.1 implies the following

Corollary 2.2.2 *For any* $j \ge -1$*, the following formula holds true*

$$h_j(n) = \frac{1}{j+2} c_{n+1,j+2}.$$

Lemma 2.2.3 *The functions* $\alpha_n = \alpha_n(\lambda), \gamma_n = \gamma_n(\lambda)$ *satisfy*

$$\gamma_{n+3} = \frac{\gamma_{n+2}}{w_{n+2}}(\lambda - v_{n+1})^2 + \frac{(\lambda - v_{n+1})(\gamma_n w_n - \gamma_{n+2} w_{n+1})}{w_{n+2}\,(\lambda - v_n)} + \frac{\gamma_{n+1}\,[w_{n+1} - (\lambda - v_n)(\lambda - v_{n+1})]}{w_{n+2}} \tag{2.2.3}$$

$$\begin{aligned}(\lambda - v_{n-1})\,&[w_{n+1}(\alpha_{n+2} + \alpha_{n+1} + 1) - (\lambda - v_n)(\lambda - v_{n+1})\alpha_{n+1}] \\ &= (\lambda - v_{n+1})\,[w_n(\alpha_n + \alpha_{n-1} + 1) - (\lambda - v_n)(\lambda - v_{n-1})\alpha_n]\,.\end{aligned} \tag{2.2.4}$$

Proof. By using (2.1.8)–(2.1.9) and by eliminating one of the series $\alpha_n(\lambda)$, $\gamma_n(\lambda)$. □

For any $j \geq 0$, define the matrix-valued function

$$V_{n,j}(\lambda) = \left[\lambda^{j+1} R_n(\lambda)\right]_+ + \begin{pmatrix} 0 & 0 \\ 0 & c_{n,j+1} \end{pmatrix}$$

where $[\ \ ]_+$ means taking the polynomial part in λ. The flows of the Toda lattice hierarchy can be represented as the following Lax equation

$$\frac{\partial U_n(\lambda)}{\partial t_j} = V_{n+1,j}(\lambda)\, U_n(\lambda) - U_n(\lambda)\, V_{n,j}(\lambda), \qquad j \geq 0$$

which are the compatibility conditions between eq. (1.2.5) and

$$\frac{\partial \Psi_n}{\partial t_j} = V_{n,j}(\lambda)\, \Psi_n, \qquad j = 0, 1, 2, \dots. \tag{2.2.5}$$

Introduce an operator $\nabla(\lambda)$ depending on a parameter λ by

$$\nabla(\lambda) := \sum_{j \geq 0} \frac{1}{\lambda^{j+2}} \frac{\partial}{\partial t_j}. \tag{2.2.6}$$

From eq. (2.2.5) it readily follows that

$$\nabla(\mu)\, \Psi_n(\lambda) = \left[\frac{R_n(\mu)}{\mu - \lambda} + Q_n(\mu)\right] \Psi_n(\lambda)$$

where

$$Q_n(\mu) := -\frac{\mathrm{id}}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma_n(\mu) \end{pmatrix}.$$

We arrive at

Lemma 2.2.4 *The following equation holds true*

$$\nabla(\mu)\, R_n(\lambda) = \frac{1}{\mu - \lambda} [R_n(\mu), R_n(\lambda)] + [Q_n(\mu), R_n(\lambda)]. \tag{2.2.7}$$

We are now in a position to prove Lemma 1.2.3.

Proof of Lemma 1.2.3. First, let us check that

$$\mathrm{tr}\, R_n(\lambda) R_n(\mu) - 1 \tag{2.2.8}$$

is divisible by $(\lambda - \mu)^2$. Indeed, from the normalization conditions (1.2.7) it readily follows that $\mathrm{tr}\, R_n^2(\lambda) = 1$. So (2.2.8) is divisible by $(\lambda - \mu)$. Due to symmetry in λ, μ this implies divisibility by $(\lambda - \mu)^2$. Thus the r.h.s. of the first eq. in (1.2.8) is a formal series in negative powers of λ, μ,

$$\frac{\mathrm{tr}\, R_n(\lambda) R_n(\mu) - 1}{(\lambda - \mu)^2} = \sum_{i,j \geq 0} \frac{\Omega_{i;j}(n)}{\lambda^{i+2}\, \mu^{j+2}} \quad \text{for some coefficients} \quad \Omega_{i;j}(n) \in \mathbb{Z}[\mathbf{v}, \mathbf{w}].$$

The first few of them are

$$\Omega_{0;0}(n) = w_n, \quad \Omega_{0;1}(n) = w_n(v_n + v_{n-1}), \quad \Omega_{1;1}(n) = w_n\left[w_{n+1} + w_{n-1} + (v_n + v_{n-1})^2\right].$$

Clearly

$$\Omega_{i;j} = \Omega_{j;i}, \qquad \forall\, i, j \geq 0.$$

Let us compute the time-derivatives of these coefficients. We have

$$\begin{aligned}
\sum_{k,\ell,m\geq 0} \frac{\partial_{t_m}\Omega_{k;\ell}}{\nu^{m+2}\lambda^{k+2}\mu^{\ell+2}} &= \nabla(\nu)\sum_{k,\ell\geq 0}\frac{\Omega_{k;\ell}}{\lambda^{k+2}\mu^{\ell+2}} \\
&= \frac{\operatorname{tr}\left(R(\mu)\,\nabla(\nu)R(\lambda)\right)}{(\lambda-\mu)^2} + \frac{\operatorname{tr}\left(R(\lambda)\,\nabla(\nu)R(\mu)\right)}{(\lambda-\mu)^2} \\
&= \frac{\operatorname{tr}\left(\left[R(\nu), R(\lambda)\right]R(\mu)\right)}{(\lambda-\mu)^2(\nu-\lambda)} - \frac{\operatorname{tr}\left(\left[Q(\nu), R(\lambda)\right]R(\mu)\right)}{(\lambda-\mu)^2} \\
&\quad + \frac{\operatorname{tr}\left(R(\lambda)\left[R(\nu), R(\mu)\right]\right)}{(\lambda-\mu)^2(\nu-\mu)} - \frac{\operatorname{tr}\left(R(\lambda)\left[Q(\nu), P(\mu)\right]\right)}{(\lambda-\mu)^2} \\
&= -\frac{\operatorname{tr}\left(\left[R(\nu), R(\lambda)\right]R(\mu)\right)}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}.
\end{aligned}$$

It is easy to see that the last expression is symmetric in λ, μ, ν. Hence

$$\frac{\partial\Omega_{k;\ell}}{\partial t_m} = \frac{\partial\Omega_{m;\ell}}{\partial t_k} \quad \forall\, k, l, m \geq 0.$$

We are now to prove compatibility between eq. (1.2.9) and eq. (1.2.8). On one hand, we have

$$\begin{aligned}
&\sum_{i,\,j\geq 0}\frac{1}{\lambda^{i+2}\mu^{j+2}}\left[\Omega_{i;j}(n+1) - \Omega_{i;j}(n)\right] \\
=\ & \frac{\operatorname{tr} R_{n+1}(\lambda)R_{n+1}(\mu) - \operatorname{tr} R_n(\lambda)R_n(\mu)}{(\lambda-\mu)^2} \\
=\ & \frac{(1+2\alpha_n(\lambda))\,\gamma_n(\lambda) - (1+2\alpha_n(\mu))\,\gamma_{n+1}(\lambda)}{\lambda-\mu} - \gamma_{n+1}(\lambda)\,\gamma_{n+1}(\mu) \\
=\ & \frac{(1+2\alpha_{n+1}(\mu))\,\gamma_{n+1}(\lambda) - (1+2\alpha_{n+1}(\lambda))\,\gamma_{n+1}(\mu)}{\lambda-\mu} + \gamma_{n+1}(\lambda)\,\gamma_{n+1}(\mu)
\end{aligned}$$

where the last equality uses the relation (2.1.8). On another hand, it follows from eq. (2.2.7) that

$$\nabla(\mu)\gamma_{n+1}(\lambda) = \frac{\gamma_{n+1}(\lambda)(1+2\alpha_{n+1}(\mu)) - \gamma_{n+1}(\mu)(1+2\alpha_{n+1}(\lambda))}{\lambda-\mu} + \gamma_{n+1}(\lambda)\gamma_{n+1}(\mu). \qquad (2.2.9)$$

Hence

$$\sum_{i,\,j\geq 0}\frac{1}{\lambda^{i+2}\mu^{j+2}}\left[\Omega_{i;j}(n+1) - \Omega_{i;j}(n)\right] = \nabla(\mu)\gamma_{n+1}(\lambda).$$

Finally we show the compatibility between eq. (1.2.10) and eqs. (1.2.8), (1.2.9). Indeed,

$$
\begin{aligned}
&\sum_{i,\,j\geq 0} \frac{1}{\lambda^{i+2}\mu^{j+2}} \left[\Omega_{i;j}(n+1) + \Omega_{i;j}(n-1) - 2\,\Omega_{i;j}(n)\right] \\
= \; & \frac{\operatorname{tr} R_{n+1}(\lambda)R_{n+1}(\mu) + \operatorname{tr} R_{n-1}(\lambda)R_{n-1}(\mu) - 2 \operatorname{tr} R_n(\lambda)R_n(\mu)}{(\lambda-\mu)^2} \\
= \; & \frac{(1+2\alpha_{n+1}(\mu))\,\gamma_{n+1}(\lambda) - (1+2\alpha_{n+1}(\lambda))\,\gamma_{n+1}(\mu)}{\lambda-\mu} + \gamma_{n+1}(\lambda)\,\gamma_{n+1}(\mu) \\
& - \frac{(1+2\alpha_n(\mu))\,\gamma_n(\lambda) - (1+2\alpha_n(\lambda))\,\gamma_n(\mu)}{\lambda-\mu} - \gamma_n(\lambda)\,\gamma_n(\mu).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla(\mu)\nabla(\lambda)\,\log w_n &= \nabla(\mu)\left[\gamma_{n+1}(\lambda) - \gamma_n(\lambda) - \lambda^{-1}\right] \\
&= \left[\frac{\gamma_{n+1}(\lambda)(1+2\alpha_{n+1}(\mu)) - \gamma_{n+1}(\mu)(1+2\alpha_{n+1}(\lambda))}{\lambda-\mu} + \gamma_{n+1}(\lambda)\gamma_{n+1}(\mu)\right] \\
&\quad - \left[\frac{\gamma_n(\lambda)(1+2\alpha_n(\mu)) - \gamma_n(\mu)(1+2\alpha_n(\lambda))}{\lambda-\mu} + \gamma_n(\lambda)\gamma_n(\mu)\right].
\end{aligned}
$$

Hence

$$
\sum_{i,\,j\geq 0} \frac{1}{\lambda^{i+2}\mu^{j+2}} \left[\Omega_{i;j}(n+1) + \Omega_{i;j}(n-1) - 2\Omega_{i;j}(n)\right] = \nabla(\mu)\nabla(\lambda)\,\log w_n.
$$

This proves compatibility between (1.2.10) and (1.2.8). The compatibility between (1.2.10) and (1.2.9) is equivalent to eq. (2.2.2).

As a result, for an arbitrary solution $v_n(\mathbf{t}), w_n(\mathbf{t})$ to the Toda lattice hierarchy, there exists a function $\tau_n(\mathbf{t})$ of this solution satisfying (1.2.8)–(1.2.10). It is easy to see that the freedom of τ_n satisfying (1.2.8)–(1.2.10) is only an arbitrary factor of the form

$$
e^{a_0 + a_1 n + \sum_{j\geq 0} b_j t_j}
$$

where $a_0, a_1, b_0, b_1, b_2, \ldots$ are constants independent of n. The lemma is proved. □

Definition 2.2.5 *For a given $k \geq 2$ and a given set of integers $i_1, \ldots, i_k \geq 0$, we call*

$$
\frac{\partial^k \log \tau_n(\mathbf{t})}{\partial t_{i_1} \ldots \partial t_{i_k}}
$$

the k-point correlation functions of the given solution $v_n(\mathbf{t})$, $w_n(\mathbf{t})$ of the Toda lattice hierarchy.

Also the first logarithmic derivatives ("one-point correlation functions") of the tau-function will be under consideration. They are determined by a solution $v_n(\mathbf{t})$, $w_n(\mathbf{t})$ up to additive constants.

Clearly the above proof of Lemma 1.2.3 already shows that

Corollary 2.2.6 *The generating series of three-point correlation functions of an arbitrary solution to the Toda lattice hierarchy has the following expression*

$$
\sum_{i,j,l\geq 0} \frac{\partial_{t_i}\partial_{t_j}\partial_{t_l} \log \tau_n}{\lambda^{i+2}\mu^{j+2}\nu^{l+2}} = -\frac{\operatorname{tr} R_n(\lambda)R_n(\mu)R_n(\nu) - \operatorname{tr} R_n(\lambda)R_n(\nu)R_n(\mu)}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}.
$$

Proof of Thm. 1.2.5   For any permutation $\sigma = [\sigma_1, \dots, \sigma_q] \in S_q$, $q \geq 2$, define

$$P(\sigma) := -\prod_{j=1}^{q} \frac{1}{\lambda_{\sigma_j} - \lambda_{\sigma_{j+1}}}, \qquad \sigma_{q+1} := \sigma_1.$$

We use mathematical induction for the proof. For $k = 3$, the formula (1.1.7) is already obtained in Corollary 2.2.6. Suppose (1.1.7) is true for $k = p$, $p \geq 3$. Then for $k = p+1$, we have[1]

$$\begin{aligned}
&\sum_{i_1,\dots,i_{p+1}=0}^{\infty} \frac{1}{\lambda_1^{i_1+2} \dots \lambda_{p+1}^{i_{p+1}+2}} \frac{\partial^k \log \tau(\mathbf{t})}{\partial t_{i_1} \dots \partial t_{i_{p+1}}} = -\frac{1}{p} \nabla(\lambda_{p+1}) \sum_{\sigma \in S_p} \frac{\operatorname{tr}\left(R(\lambda_{\sigma_1}) \dots R(\lambda_{\sigma_p})\right)}{\prod_{j=1}^{p}(\lambda_{\sigma_j} - \lambda_{\sigma_{j+1}})} \\
= &-\frac{1}{p} \sum_{\sigma \in S_p} \sum_{q=1}^{p} \frac{\operatorname{tr}\left(R(\lambda_{\sigma_1}) \dots \left[\frac{R(\lambda_{p+1})}{\lambda_{p+1}-\lambda_{\sigma_q}} + Q(\lambda_{p+1}), R(\lambda_{\sigma_q})\right] \dots R(\lambda_{\sigma_p})\right)}{\prod_{j=1}^{p}(\lambda_{\sigma_j} - \lambda_{\sigma_{j+1}})} \\
= &\frac{1}{p} \sum_{\sigma \in S_p} P(\sigma) \sum_{q=1}^{p} (\lambda_{\sigma_q} - \lambda_{\sigma_{q-1}}) \frac{\operatorname{tr} R(\lambda_{p+1}) R(\lambda_{\sigma_q}) \dots R(\lambda_{\sigma_p}) R(\lambda_{\sigma_1}) \dots R(\lambda_{\sigma_{q-1}})}{(\lambda_{p+1} - \lambda_{\sigma_q})(\lambda_{p+1} - \lambda_{\sigma_{q-1}})} \\
= &\frac{1}{p} \sum_{q=1}^{p} \sum_{\sigma \in S_p} P([p+1, s_q, \dots, s_p, s_1, \dots, s_{q-1}]) \operatorname{tr} R(\lambda_{p+1}) R(\lambda_{\sigma_q}) \dots R(\lambda_{\sigma_p}) R(\lambda_{\sigma_1}) \dots R(\lambda_{\sigma_{q-1}}) \\
= &\sum_{\sigma \in S_p} P([p+1, \sigma]) \operatorname{tr} R(\lambda_{p+1}) R(\lambda_{\sigma_1}) \dots R(\lambda_{\sigma_p}).
\end{aligned}$$

The theorem is proved. □

The resulting expressions for the generating series can be used for developing efficient algorithms for computing the GUE correlators. To this end it is convenient to represent the multipoint formula (1.1.7) in a slightly modified way.

Corollary 2.2.7 *The generating series of order $k \geq 3$ logarithmic derivatives of tau-function of a solution to the Toda lattice hierarchy has the expression*

$$\sum_{i_1,\dots,i_k=0}^{\infty} \frac{1}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} \frac{\partial^k \log \tau_n(\mathbf{t})}{\partial t_{i_1} \dots \partial t_{i_k}} = -\sum_{\sigma \in S_{k-2}} \frac{\left\langle R_n(\mathbf{t}, \lambda_k)\,,\, \mathrm{ad}_{R_n(\mathbf{t}, \lambda_{\sigma_1})} \cdots \mathrm{ad}_{R_n(\mathbf{t}, \lambda_{\sigma_{k-2}})} R_n(\mathbf{t}, \lambda_{k-1}) \right\rangle}{(\lambda_{\sigma_{k-2}} - \lambda_{k-1})(\lambda_{k-1} - \lambda_k)(\lambda_k - \lambda_{\sigma_1}) \prod_{j=1}^{k-3}(\lambda_{\sigma_j} - \lambda_{\sigma_{j+1}})} \tag{2.2.10}$$

where $\mathrm{ad}_a\, b := [a, b]$, and $\langle a, b \rangle := \operatorname{tr} a\, b$.

Remark 2.2.8 *The same type formula as* (2.2.10) *holds true also for the generating series of [3, 25] for the Witten–Kontsevich correlators and for the correlators of Drinfeld–Sokolov hierarchies [4].*

3 Computing GUE correlators

In this section we prove Thm. 1.2.6, Thm. 1.1.1 and present some examples.

[1] In this calculation we omit the index n of τ_n and R_n.

3.1 Proof of Thms. 1.2.6 and 1.1.1

We are to compute the matrix resolvent of the operator

$$\Delta - \begin{pmatrix} \lambda & -n \\ 1 & 0 \end{pmatrix}.$$

It follows from eq. (2.2.3) that the formal series

$$\gamma_n = \gamma(n, \lambda; \mathbf{t} = 0) = \sum_{j \geq 0} \frac{(2j-1)!!}{\lambda^{2j+1}} \gamma_{n,j}$$

satisfies

$$\gamma_{n+3} = \frac{\lambda^2 - (n+1)}{n+2} (\gamma_{n+2} - \gamma_{n+1}) + \frac{n}{n+2} \gamma_n \tag{3.1.1}$$

along with the boundary conditions

$$\gamma_{n,0} = 1, \quad \forall\, n \geq 0 \tag{3.1.2}$$

$$\gamma_{0,j} = (-1)^j (2j-1)!!, \quad \forall\, j \geq 0. \tag{3.1.3}$$

Clearly, solution to (3.1.1)–(3.1.3) if exists must be unique. We are to show that

$$\gamma^*_n := \sum_{j \geq 0} \frac{(2j-1)!!}{\lambda^{2j+1}} \, {}_2F_1(-j, 1-n, 1; 2), \qquad n \geq 0$$

satisfies (3.1.1)–(3.1.3). Indeed, from

$$\gamma^*_{n,j} = {}_2F_1(-j, 1-n; 1; 2) = \sum_{i=0}^{j} 2^i \begin{pmatrix} j \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ i \end{pmatrix}, \qquad \forall\, n, j \geq 0 \tag{3.1.4}$$

it is easy to see that

$$\gamma^*_{n,0} = 1, \qquad \gamma^*_{0,j} = (-1)^j (2j-1)!!.$$

So eqs. (3.1.2)–(3.1.3) are verified. Eq. (3.1.1) is equivalent the following recursion on $\gamma_{n,j}$

$$(n+2)\gamma_{n+3,j} = (2j+1)(\gamma_{n+2,j+1} - \gamma_{n+1,j+1}) - (n+1)(\gamma_{n+2,j} - \gamma_{n+1,j}) + n\, \gamma_{n,j}.$$

To show γ^* (see (3.1.4)) is a solution to the above equation, it suffices to show

$$\begin{aligned}(n+2)\, {}_2F_1(-j, -n-2; 1; 2) &= (2j+1) \left[{}_2F_1(-j-1, -n-1; 1; 2) - {}_2F_1(-j-1, -n; 1; 2) \right] \\ &\quad -(n+1) \left[{}_2F_1(-j, -n-1; 1; 2) - {}_2F_1(-j, -n; 1; 2) \right] + n\, {}_2F_1(-j, 1-n; 1; 2).\end{aligned} \tag{3.1.5}$$

We now use the following contiguous relations of Gauss: $\forall\, a, b, c, z \in \mathbb{C}$

$$\begin{aligned}&(c-a)\, {}_2F_1(a-1, b; c; z) + (a-c+bz) {}_2F_1(a, b; c; z) \\ &\qquad = (c-b)\, {}_2F_1(a, b-1; c; z) + (b-c+az) {}_2F_1(a, b; c; z)\end{aligned} \tag{3.1.6}$$

$$a \left[{}_2F_1(a+1, b; c; z) - {}_2F_1(a, b; c; z) \right] = b \left[{}_2F_1(a, b+1; c; z) - {}_2F_1(a, b; c; z) \right]. \tag{3.1.7}$$

Taking in (3.1.6) $c = 1$, $a = -j$, $b = -n-1$ we obtain

$$(n+2)\,{}_2F_1(-j,-n-2;1;2) = (1+j)\,{}_2F_1(-j-1,-n-1;1;2) - (n+1-j)\,{}_2F_1(-j,-n-1;1;2).$$

Taking in (3.1.6) $c = 1$, $a = -j$, $b = -n$ we obtain

$$(n+1)\,{}_2F_1(-j,-n-1;1;2) = (1+j)\,{}_2F_1(-j-1,-n;1;2) - (n-j)\,{}_2F_1(-j,-n;1;2).$$

Taking in (3.1.6) $c = 1$, $a = -j$, $b = -n+1$ we obtain

$$n\,{}_2F_1(-j,-n;1;2) = (1+j)\,{}_2F_1(-j-1,1-n;1;2) - (n-1-j)\,{}_2F_1(-j,1-n;1;2). \tag{3.1.8}$$

Taking in (3.1.7) $c = 1$, $a = -j-1$, $b = -n-1$ we obtain

$$(n-j)\,{}_2F_1(-j-1,-n-1;1;2) = (n+1)\,{}_2F_1(-j-1,-n;1;2) - (j+1)\,{}_2F_1(-j,-n-1;1;2).$$

So we have

$$\begin{aligned}
& \text{l.h.s. of (3.1.5)} \\
= \; & (1+j)\,{}_2F_1(-j-1,-n-1;1;2) - (n+1-j)\,{}_2F_1(-j,-n-1;1;2) \\
= \; & \frac{(1+j)(n+1)}{n-j}\,{}_2F_1(-j-1,-n;1;2) - \frac{j^2+n+j+1+(n-j)^2}{n-j}\,{}_2F_1(-j,-n-1;1;2) \\
= \; & \frac{(1+j)(1+2j)}{n+1}\,{}_2F_1(-j-1,-n;1;2) + \frac{j^2+n+j+1+(n-j)^2}{n+1}\,{}_2F_1(-j,-n;1;2),
\end{aligned}$$

and we have

$$\begin{aligned}
& \text{r.h.s. of (3.1.5)} \\
= \; & \frac{(2j+1)(j+1)}{n-j}\,{}_2F_1(-j-1,-n;1;2) - \left(\frac{(2j+1)(j+1)}{n-j} + n + 1\right)\,{}_2F_1(-j,-n-1;1;2) \\
& \quad + (n+1)\,{}_2F_1(-j,-n,1;2) + n\,{}_2F_1(-j,1-n,1;2) \\
= \; & \frac{(2j-n)(j+1)}{n+1}\,{}_2F_1(-j-1,-n;1;2) + \left(\frac{(2j+1)(j+1)}{n+1} + 2n - j + 1\right)\,{}_2F_1(-j,-n;1;2) \\
& \quad + n\,{}_2F_1(-j,1-n;1;2).
\end{aligned}$$

Comparing the above equations and using (3.1.8) we find it suffices to show

$${}_2F_1(-j-1,-n;1,2) = {}_2F_1(-j,-n;1,2) + {}_2F_1(-j-1,1-n;1;2) + {}_2F_1(-j,1-n;1;2)$$

which can be verified easily.

Finally, using (2.1.9) and considering $v_n = 0, w_n = n$ we find

$$\lambda\,(\alpha_n - \alpha_{n+1}) = n\,\gamma_n^* - (n+1)\,\gamma_{n+2}^*. \tag{3.1.9}$$

Similarly as above it can be verified that α^* defined by

$$\alpha_n^* = n \sum_{j\geq 0} \frac{(2j+1)!!}{\lambda^{2j+2}}\,{}_2F_1(-j,1-n;2;2)$$

is a solution to (3.1.9). Moreover, α^* obviously satisfies the boundary condition

$$\alpha_0^*(\lambda) = 0.$$

The theorem is proved. □

Proof. of Thm. 1.1.1. The GUE partition function is a particular tau-function of the Toda lattice hierarchy; see Prop. A.2.3 in Appendix A for a detailed proof. The initial data of the corresponding solution satisfy $v_n = 0$, $w_n = n$. As a result, Part 2) of the theorem readily follows from Theorems 1.2.6 and 1.2.5. It remains to prove Part 1). By definition we have

$$c_{n+1,j+1} = \frac{\partial}{\partial t_j} \log \frac{\tau_{n+1}}{\tau_n} = \frac{\partial}{\partial t_j} \log \tau_{n+1} - \frac{\partial}{\partial t_j} \log \tau_n, \qquad \forall\, j \geq 0. \tag{3.1.10}$$

Taking $\mathbf{t} = 0$ in (3.1.10) and recalling that $\gamma_n = \sum_{j\geq 0} \frac{(2j-1)!!}{\lambda^{2j+1}}\, {}_2F_1(-j, 1-n; 1; 2)$, and using the boundary condition $C_1(0;\lambda) = 0$ we obtain the expression (1.1.3) for one-point correlators.

The theorem is proved. □

3.2 An algorithm for computing connected GUE correlators. Examples

Based on Thm. 1.1.1 and formula (2.2.7), we give in this subsection a recursive procedure of calculating connected GUE correlators, which is very efficient in computation.

Definition 3.2.1 *Fix $\mathbf{b} = (b_1, b_2, b_3 \dots)$ an arbitrary sequence of positive integers. Define recursively a family of Laurent series $R^{\mathbf{b}}_K(n,\lambda) \in \mathrm{Mat}\left(2, \mathbb{Z}[n]((\lambda^{-1}))\right)$ with $K = \{k_1, \dots, k_m\}$ by*

$$\begin{aligned} R^{\mathbf{b}}_{\{\}}(n,\lambda) &:= \mathcal{R}_n(\lambda), \\ R^{\mathbf{b}}_K(n,\lambda) &:= \sum_{I \sqcup J = K - \{k_1\}} \left[R^{\mathbf{b}}_I(n,\lambda), \left(\lambda^{b_{k_1}}\, R^{\mathbf{b}}_J(n,\lambda)\right)_+ \right]. \end{aligned} \tag{3.2.1}$$

Here $k_1, \dots, k_m$ are distinct positive integers, $m = |K|$, and $\mathcal{R}_n(\lambda)$ is defined by eq. (1.1.4).

Lemma 3.2.2 *In the particular case that $b_1 = b_2 = b_3 = \dots = b$ we have*

$$R^{\mathbf{b}}_K(n,\lambda) = R^{\mathbf{b}}_{K'}(n,\lambda) =: R^b_{|K|}(n,\lambda), \qquad \textit{as long as } \ |K| = |K'|.$$

Moreover, the following formulae hold true for $R^b_m(n,\lambda)$, $m \geq 1$

$$R^b_m(n,\lambda) = \sum_{i=0}^{m-1} \binom{m-1}{i} \left[R^b_i(n,\lambda), \left(\lambda^b\, R^b_{m-1-i}(n,\lambda)\right)_+ \right].$$

Clearly, $R^b_0(n,\lambda) = \mathcal{R}_n(\lambda)$.

Proposition 3.2.3 *Let* $\mathbf{b} = (b_1, b_2, b_3 \dots)$ *be any sequence of positive integers, and* $K = \{k_1, \dots, k_m\}$ *any finite set of positive integers. The following formula holds true for connected GUE correlators*

$$\sum_{i,j\geq 1} \frac{\langle \operatorname{tr} M^{b_{k_1}} \cdots \operatorname{tr} M^{b_{k_m}} \operatorname{tr} M^i \operatorname{tr} M^j \rangle_c}{\lambda_1^{i+2} \lambda_2^{j+2}} = \sum_{I \sqcup J = K} \frac{\operatorname{tr} R_I^{\mathbf{b}}(N, \lambda_1)\, R_J^{\mathbf{b}}(N, \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{\delta_{m,0}}{(\lambda_1 - \lambda_2)^2}. \tag{3.2.2}$$

Here $m = |K|$*. In the particular case that* $b_1 = b_2 = \cdots = b$ *for some* $b \geq 1$*, we have* $\forall\, m \geq 0$

$$\sum_{i,j\geq 1} \frac{\left\langle \left(\operatorname{tr} M^b\right)^m \operatorname{tr} M^i \operatorname{tr} M^j \right\rangle_c}{\lambda_1^{i+1} \lambda_2^{j+1}} = \sum_{i=0}^{m} \binom{m}{i} \frac{\operatorname{tr} R_i^b(N, \lambda_1)\, R_{m-i}^b(N, \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{\delta_{m,0}}{(\lambda_1 - \lambda_2)^2}. \tag{3.2.3}$$

Proof. By using mathematical induction and by noticing that the term containing $Q_n(\mu)$ in r.h.s. of (2.2.7) does not contribute to generating series of correlators as it was proved in Thm. 1.2.5. □

Clearly, Def. 3.2.1, Lem. 3.2.2 and Prop. 3.2.3 give an algorithm of computing connected GUE correlators. Similar recursive formulation as (3.2.1)–(3.2.3) also works for logarithmic derivatives of tau-function of an arbitrary solution to the Toda lattice hierarchy.

Example 3.2.4 (1-point correlators) *We have* $\left\langle \operatorname{tr} M^{a+1} \right\rangle_c = 0$ *if* a *is an even integer then vanishes; otherwise,*

$$\left\langle \operatorname{tr} M^{a+1} \right\rangle_c = N\, a!! \left[{}_2F_1\left(-\frac{a+1}{2}, -N; 2; 2\right) - \frac{a+1}{2}\, {}_2F_1\left(-\frac{a-1}{2}, 1-N; 3; 2\right) \right], \quad a \geq 1. \tag{3.2.4}$$

For example,

$$\left\langle \operatorname{tr} M^2 \right\rangle_c = N^2, \qquad \left\langle \operatorname{tr} M^4 \right\rangle_c = N + 2N^3, \qquad \left\langle \operatorname{tr} M^6 \right\rangle_c = 10N^2 + 5N^4,$$

$$\left\langle \operatorname{tr} M^{20} \right\rangle_c = 16796N^{11} + 1385670N^9 + 31039008N^7 + 211083730N^5 + 351683046N^3 + 59520825N.$$

Example 3.2.5 (2-point correlators) $\left\langle \operatorname{tr} M^{a+1} \operatorname{tr} M^{b+1} \right\rangle_c$ *vanishes* $\forall\, a, b \geq 0$*, if* $a + b$ *is an odd integer; otherwise,*

$$\begin{aligned}
&\left\langle \operatorname{tr} M^{a+1} \operatorname{tr} M^{b+1} \right\rangle_c \\
&= N\,(a+b+1)!!\,(1+b)\, {}_2F_1\left(-\frac{a+b}{2}, 1-N; 2; 2\right) \\
&+2N^2 \sum_{\substack{0\leq j\leq b-2 \\ j\equiv b (\mathrm{mod} 2)}} (a+j+1)!!(b-j-1)!!\,(1+j)\, {}_2F_1\left(-\frac{a+j}{2}, 1-N; 2; 2\right) {}_2F_1\left(-\frac{b-j-2}{2}, 1-N; 2; 2\right) \\
&-N \sum_{\substack{0\leq j\leq b-1 \\ j\equiv b-1 (\mathrm{mod} 2)}} (a+j)!!(b-j-2)!!\,(1+j) \left[{}_2F_1\left(-\frac{a+1+j}{2}, -N; 1; 2\right) {}_2F_1\left(-\frac{b-j-1}{2}, 1-N; 1; 2\right) \right. \\
&\qquad \left. + {}_2F_1\left(-\frac{b-j-1}{2}, -N; 1; 2\right) {}_2F_1\left(-\frac{a+1+j}{2}, 1-N; 1; 2\right) \right].
\end{aligned} \tag{3.2.5}$$

For example, we have

$$\langle \operatorname{tr} M \operatorname{tr} M \rangle_c = N, \qquad \left\langle \operatorname{tr} M^2 \operatorname{tr} M^4 \right\rangle_c = 4N + 8N^3, \qquad \left\langle \operatorname{tr} M^4 \operatorname{tr} M^4 \right\rangle_c = 60N^2 + 36N^4,$$

$$\begin{aligned}&\left\langle \operatorname{tr} M^{18} \operatorname{tr} M^{20} \right\rangle_c \\ &= 4813380N\,(8840N^{18} + 3275220N^{16} + 478887552N^{14} + 34305326120N^{12} + 1259109855744N^{10} \\ &+23197400694000N^8 + 199375600144496N^6 + 689468897044260N^4 + 705221681016618N^2 \\ &+85187495274525).\end{aligned}$$

Example 3.2.6 (3-point correlators) *We have* $\forall\, i \geq 1$,

$$\begin{aligned}
\left\langle \left(\operatorname{tr} M^2\right)^2 \operatorname{tr} M^i \right\rangle_c &= N \operatorname{Coef}_{\lambda^{-i-1}} \left[\left(\lambda^2 - 1\right) \gamma_{N+1}(\lambda) - \left(\lambda^2 + 1\right) N \gamma_N(\lambda) \right] \\
\left\langle \operatorname{tr} M^2 \operatorname{tr} M^3 \operatorname{tr} M^i \right\rangle_c &= N \operatorname{Coef}_{\lambda^{-i-1}} \left[4\, \alpha_N(\lambda) - \lambda \left(\lambda^2 + 2N + 2\right) \gamma_N(\lambda) + \lambda \left(\lambda^2 + 2N - 2\right) \gamma_{N+1}(\lambda) + 2 \right] \\
\left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^i \right\rangle_c &= N \operatorname{Coef}_{\lambda^{-i-1}} \left[8\, \lambda\, \alpha_N(\lambda) - \left(\lambda^4 + 4N^2 + \lambda^2(4N+3) + 8N + 3\right) \gamma_N(\lambda) \right. \\
&\qquad \left. + \left(\lambda^4 + 4N^2 + \lambda^2(4N-3) - 8N + 3\right) \gamma_{N+1}(\lambda) + 4\lambda \right] \\
\left\langle \left(\operatorname{tr} M^4\right)^2 \operatorname{tr} M^i \right\rangle_c &= N \operatorname{Coef}_{\lambda^{-i-1}} \left[4\lambda \left(\lambda^2 + 6N\right) \left(2\alpha_N(\lambda) + 1\right) \right. \\
&\quad - \left(\lambda^6 + \lambda^4(4N+3) + \lambda^2(2N+1)(2N+9) + 36N(N+1) + 15\right) \gamma_N(\lambda) \\
&\quad \left. + \left(\lambda^6 + \lambda^4(4N-3) + \lambda^2(2N-1)(2N-9) - 36(N-1)N - 15\right) \gamma_{N+1}(\lambda) \right].
\end{aligned}$$

Here we recall

$$\alpha_n(\lambda) = n \sum_{j \geq 0} \frac{(2j+1)!!}{\lambda^{2j+2}}\, {}_2F_1(-j, 1-n; 2; 2), \quad \gamma_n(\lambda) = \sum_{j \geq 0} \frac{(2j-1)!!}{\lambda^{2j+1}}\, {}_2F_1(-j, 1-n, 1; 2).$$

For example,

$$\left\langle \left(\operatorname{tr} M^2\right)^3 \right\rangle_c = 8N^2, \qquad \left\langle \left(\operatorname{tr} M^2\right)^2 \operatorname{tr} M^4 \right\rangle_c = 24N + 48N^3,$$

$$\left\langle \left(\operatorname{tr} M^2\right)^2 \operatorname{tr} M^6 \right\rangle_c = 480N^2 + 240N^4, \qquad \left\langle \left(\operatorname{tr} M^2\right)^2 \operatorname{tr} M^8 \right\rangle_c = 1680N + 5600N^3 + 1120N^5,$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^2\right)^2 \operatorname{tr} M^{38} \right\rangle_c \;=\;& 1343120024400\, N^2 \left(2N^{18} + 1140N^{16} + 240312N^{14} + 24082880N^{12} \right. \\ &+1231558302N^{10} + 32196168420N^8 + 410364369452N^6 \\ &\left. +2294179050960N^4 + 4562960651307N^2 + 1979828515350\right).\end{aligned}$$

$$\left\langle \operatorname{tr} M^2\, \left(\operatorname{tr} M^3\right)^2 \right\rangle_c = 18N + 72N^3, \quad \langle \operatorname{tr} M^2 \operatorname{tr} M^3 \operatorname{tr} M^5 \rangle_c = 480N^2 + 360N^4,$$

$$\langle \operatorname{tr} M^2 \operatorname{tr} M^3 \operatorname{tr} M^7 \rangle_c = 480N^2 + 360N^4, \quad \langle \operatorname{tr} M^2 \operatorname{tr} M^3 \operatorname{tr} M^9 \rangle_c = 1470N + 6300N^3 + 1680N^5,$$

$$\begin{aligned}\langle \operatorname{tr} M^2 \operatorname{tr} M^3 \operatorname{tr} M^{39} \rangle_c \;=\;& 1033848966150N \left(16N^{20} + 10108N^{18} + 2401182N^{16} + 276911776N^{14} \right. \\ &+16743310948N^{12} + 536717003004N^{10} + 8831088179794N^8 \\ &+68958855149632N^6 + 219890931285060N^4 + 210352383917730N^2 \\ &\left. +24130040059125\right).\end{aligned}$$

$$\left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^4 \right\rangle_c = 468N^2 + 432N^4, \quad \left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^6 \right\rangle_c = 1350N + 6660N^3 + 2160N^5,$$

$$\left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^8 \right\rangle_c = 55440N^2 + 68040N^4 + 10080N^6,$$

$$\left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^{10} \right\rangle_c = 213570N + 1183140N^3 + 570780N^5 + 45360N^7,$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^3\right)^2 \operatorname{tr} M^{38} \right\rangle_c \;=\;& 1511010027450N \left(16N^{20} + 9628N^{18} + 2180250N^{16} + 239934736N^{14}\right.\\ &+13863233644N^{12} + 425408903244N^{10} + 6715474080598N^8\\ &+50449385602192N^6 + 155303372658492N^4 + 144060538320450N^2\\ &\left.+16119257529375\right).\end{aligned}$$

$$\left\langle \operatorname{tr} M^2 \left(\operatorname{tr} M^4\right)^2 \right\rangle_c = 480N^2 + 288N^4, \quad \left\langle \left(\operatorname{tr} M^4\right)^3 \right\rangle_c = 1728N^5 + 6336N^3 + 1440N,$$

$$\left\langle \left(\operatorname{tr} M^4\right)^2 \operatorname{tr} M^6 \right\rangle_c = 8640N^6 + 63360N^4 + 56160N^2,$$

$$\left\langle \left(\operatorname{tr} M^4\right)^2 \operatorname{tr} M^8 \right\rangle_c = 40320N^7 + 530880N^5 + 1162560N^3 + 221760N,$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^4\right)^2 \operatorname{tr} M^{38} \right\rangle_c \;=\;& 8058720146400N^2 \left(12N^{20} + 8408N^{18} + 2249790N^{16} + 297878352N^{14}\right.\\ &+21183159128N^{12} + 824144717136N^{10} + 17179527894426N^8\\ &+180912770249240N^6 + 860693336297694N^4 + 1496297650892364N^2\\ &\left.+582832451267325\right).\end{aligned}$$

Example 3.2.7 (4-point correlators) *We have* $\forall\, i \geq 1$,

$$\left\langle \left(\operatorname{tr} M^2\right)^3 \operatorname{tr} M^i \right\rangle_c = N \operatorname{Coef}_{\lambda^{-i-1}} \left[-4\lambda^3\alpha_N(\lambda) + \left(\lambda^4 - 3\right)\left(\gamma_N(\lambda) + \gamma_{N+1}(\lambda)\right) - 2\lambda^3\right]$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^3\right)^3 \operatorname{tr} M^i \right\rangle_c =\;& N \operatorname{Coef}_{\lambda^{-i-1}}\\ &\left[-2\left(\lambda^6 - 3\lambda^2 + 8N^3 + 12\lambda^2N^2 + 6\lambda^4N - 38N\right)\left(2\,\alpha_N(\lambda) + 1\right)\right.\\ &+ \left(\lambda^7 + 3\lambda^3\left(4N^2 - 2N - 1\right) + 3\lambda^5(2N+1) + 2\lambda(N-5)(2N+1)(2N+3)\right)\gamma_N(\lambda)\\ &\left.+ \left(\lambda^7 + 3\lambda^3\left(4N^2 + 2N - 1\right) + 3\lambda^5(2N-1) + 2\lambda(N+5)(2N-1)(2N-3)\right)\gamma_{N+1}(\lambda)\right]\end{aligned}$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^4\right)^3 \operatorname{tr} M^i \right\rangle_c =\;& N \operatorname{Coef}_{\lambda^{-i-1}}\\ &\left[\alpha_N(\lambda)\left(\lambda(828 + 2160N^2) + \lambda^3(152N - 32N^3) - 24\lambda^5(1 + 2N^2) - 24N\lambda^7 - 4\lambda^9\right)\right.\\ &+ \gamma_N(\lambda)\left(\lambda^{10} + 3\lambda^8(1 + 2N) + 6\lambda^6(1 + 2N + 2N^2) - 2\lambda^4(15 + 30N^2 - 4N^3)\right.\\ &\left.-(315 + 756N + 828N^2 + 144N^3)\lambda^2 - 1944N^3 - 2916N^2 - 56\lambda^4N - 2862N - 945\right)\\ &+ \gamma_{N+1}(\lambda)\left(\lambda^{10} - 3\lambda^8(1 - 2N) + 6\lambda^6(1 - 2N + 2N^2) + 2\lambda^4(15 + 30N^2 + 4N^3)\right.\\ &\left.-(315 - 756N + 828N^2 - 144N^3)\lambda^2 - 1944N^3 + 2916N^2 - 56\lambda^4N - 2862N + 945\right)\\ &\left.-2\lambda^9 - 12\lambda^7N - 12(1 + 2N^2)\lambda^5 + (76N - 16N^3)\lambda^3 + (414 + 1080N^2)\lambda\right].\end{aligned}$$

For example,

$$\left\langle \left(\operatorname{tr} M^2\right)^4 \right\rangle_c = 48N^2, \qquad \left\langle \left(\operatorname{tr} M^2\right)^3 \operatorname{tr} M^4 \right\rangle_c = 192N + 384N^3,$$

$$\left\langle \left(\operatorname{tr} M^2\right)^3 \ \operatorname{tr}\ M^{20} \right\rangle_c = 44341440N\left(4N^{10}+330N^8+7392N^6+50270N^4+83754N^2+14175\right),$$

$$\left\langle \left(\operatorname{tr} M^3\right)^4 \right\rangle_c = 4536N^2+5184N^4, \qquad \left\langle \left(\operatorname{tr} M^3\right)^3 \ \operatorname{tr} M^5 \right\rangle_c = 15390N+82620N^3+32400N^5,$$

$$\left\langle \left(\operatorname{tr} M^3\right)^3 \ \operatorname{tr} M^{17} \right\rangle_c = 149652360N^2\left(16N^{10}+1354N^8+33462N^6+282518N^4+730832N^2+374043\right),$$

$$\left\langle \left(\operatorname{tr} M^4\right)^3 \ \operatorname{tr} M^2 \right\rangle_c = 17280N+76032N^3+20736N^5, \qquad \left\langle \left(\operatorname{tr} M^4\right)^4 \right\rangle_c = 770688N^2+964224N^4+145152N^6,$$

$$\begin{aligned}\left\langle \left(\operatorname{tr} M^4\right)^3 \ \operatorname{tr} M^{20} \right\rangle_c &= 798145920N^2\left(60N^{12}+8674N^{10}+402650N^8+7343262N^6+51873380N^4\right.\\ &\left.+120454639N^2+57830535\right).\end{aligned}$$

Example 3.2.8 (5-point correlators) *We have* $\forall\, i \geq 1$,

$$\begin{aligned}&\left\langle \left(\operatorname{tr} M^2\right)^4 \operatorname{tr} M^i \right\rangle_c = N\, \mathrm{Coef}_{\lambda^{-i-1}}\\ &\left[\left(\lambda^6+\lambda^4-3\lambda^2-4\lambda^4N-15\right)\gamma_{N+1}(\lambda)-\left(\lambda^6-\lambda^4-3\lambda^2-4\lambda^4N+15\right)\gamma_N(\lambda)-8\lambda^3\alpha_N(\lambda)-4\lambda^3\right]\\ &\left\langle \left(\operatorname{tr} M^3\right)^4 \operatorname{tr} M^i \right\rangle_c = N\, \mathrm{Coef}_{\lambda^{-i-1}}\\ &\left[8\lambda\left(\lambda^6-64N^3-3\lambda^2\left(20N^2+7\right)-12\lambda^4N+304N\right)\alpha_N(\lambda)\right.\\ &+\left(\lambda^{10}+315\left(\lambda^2-3\right)-4N\left(2N\left(2N\left(4N^2+N-106\right)+463\right)-687\right)+\lambda^8(4N-3)\right.\\ &\left.-\lambda^6(8(N-2)N+9)+\lambda^4(45-8N(4N(2N-3)+13))+8\lambda^2N(2N(N(6-7N)+49)-93)\right)\gamma_{N+1}(\lambda)\\ &+\left(-\lambda^{10}-315\left(\lambda^2+3\right)-\lambda^8(4N+3)+\lambda^6(8N(N+2)+9)+\lambda^4(8N(4N(2N+3)+13)+45)\right.\\ &\left.+8\lambda^2N(2N(N(7N+6)-49)-93)+4N(2N(2N(N(4N-1)-106)-463)-687)\right)\gamma_N(\lambda)\\ &\left.+4\lambda\left(\lambda^6-64N^3-3\lambda^2\left(20N^2+7\right)-12\lambda^4N+304N\right)\right].\end{aligned}$$

For example,

$$\left\langle \left(\operatorname{tr} M^2\right)^5 \right\rangle_c = 384N^2, \qquad \left\langle \left(\operatorname{tr} M^2\right)^4 \ \operatorname{tr} M^4 \right\rangle_c = 1920N+3840N^3,$$

$$\left\langle \left(\operatorname{tr} M^2\right)^4 \ \operatorname{tr}\ M^{20} \right\rangle_c = 1152877440N\left(4N^{10}+330N^8+7392N^6+50270N^4+83754N^2+14175\right),$$

$$\left\langle \left(\operatorname{tr} M^3\right)^4 \ \operatorname{tr}\ M^2 \right\rangle_c = 62208N^4+54432N^2, \qquad \left\langle \left(\operatorname{tr} M^3\right)^4 \ \operatorname{tr} M^4 \right\rangle_c = 528768N^5+1181952N^3+204120N,$$

$$\left\langle \left(\operatorname{tr} M^3\right)^4 \ \operatorname{tr} M^{18} \right\rangle_c = 283551840N^2(5440827+11132606N^2+4554930N^4+576009N^6+25058N^8+320N^{10}).$$

The GUE correlators $\langle \operatorname{tr} M^{i_1} \ldots \operatorname{tr} M^{i_k}\rangle_c$ that we compute with $i_1+\cdots+i_k \leq 10$ agree with [1, 5, 22].

3.3 Polygon numbers on Riemann surfaces

In this subsection, we consider correlators of the form

$$\left\langle \left(\operatorname{tr} M^b\right)^k \right\rangle_c, \qquad b \geq 3,\ k \geq 1$$

which is exactly the case when $b_1 = b_2 = \cdots = b$ in the algorithm described by Def. 3.2.1–Prop. 3.2.3.

These correlators are polynomials in N whose coefficients are positive integers, called *polygon numbers* on Riemann surfaces. More precisely, we have

$$\left\langle \left(\operatorname{tr} M^b\right)^k \right\rangle_c = \sum_{0 \leq g \leq \frac{k}{4}(b-2)+\frac{1}{2}} n_{g,b,k}\, N^{2-2g+(\frac{b}{2}-1)k}$$

where $n_{g,b,k}$ counts the number of connected oriented labelled ribbon graphs of genus g with k vertices of valencies b. In other words, $n_{g,b,k}$ counts the number of labelled *maps* which are embedded into a connected oriented closed surface of genus g; see details for example in [5, 11], or Appendix A.

Using our algorithm we worked out a program computing the polygon numbers. Below are tables of first several polygon numbers with $b = 3, 4, 5, 6, 7, 8$.

k	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
2	12	3	0	0	0	0
4	5184	4536	0	0	0	0
6	9797760	19362240	3061800	0	0	0
8	45148078080	164367221760	89414357760	0	0	0
10	392212641300480	2332019568291840	2834113460935680	357485480352000	0	0
12	5 560 971 849 577 267 200	49 838 762 032 083 763 200	110 757 832 882 937 856 000	47 537 982 337 808 793 600	0	0

Table 1: Triangle numbers $n_{g,3,k}$.

The above numbers of triangulations agree with the ones computed by Fleming [14], and with Table 7 in Appendix A.

The numbers of quadrangulations (Table 2) for $k \leq 6$ agree with the ones computed by Pierce [23].

Most of the numbers in Tables 3–6 seem not to be computed in the literature.

k	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	2	1	0	0	0	0
2	36	60	0	0	0	0
3	1728	6336	1440	0	0	0
4	145152	964224	770688	0	0	0
5	17915904	192098304	348033024	58060800	0	0
6	2956124160	47357706240	158525890560	92253634560	0	0
7	614873825280	13922807316480	76300251955200	100275872071680	13948526592000	0
8	154928203970560	4755537360322 560	39364669475389 440	95431198231756 800	45881115652915 200	0
9	45 977 357 978 173 440	1 850 918 058 999 152 640	21 844 654 140 570 992 640	86 654 328 700 277 882 880	93 561 769 862 061 096 960	11 473 053 680 664 576 000

Table 2: Quadrangle numbers $n_{g,4,k}$.

k	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
2	180	600	165	0	0	0
4	6480000	93960000	332100000	219510000	0	0
6	1242216000000	45300060000000	546671268000000	2354983470000000	2843338018500000	389492853750000
8	6246072 00000000000	4472187552 0000000000	12283912451664 00000000	152254618488 00000000000	814486310134308 00000000	155872936216116 000000000
10	6135286523184 00000000000	748085330014848 00000000000	381919155214554 7200000000000	994894820469295 20000000000000	1345274552969624 982600000000000	886998667042254 5388000000000000

Table 3: Pentagon numbers $n_{g,5,k}$.

k	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	5	10	0	0	0	0
2	600	4800	4770	0	0	0
3	216000	4176000	17290800	12315600	0	0
4	142560000	5287680000	54015984000	161062992000	93360956400	0
5	141523200000	8805542400000	174855024000000	1291104489600000	3123016385040000	1565262377280000
6	190356480000000	1819210752 0000000	611671917312 000000	8806826030976 000000	527219331093504 00000	1096721661511872 00000
7	3255095808 00000000	448921046016 00000000	233492421222144 0000000	5707503816562944 0000000	666456378352813 440000000	34239929870156 77440000000

Table 4: Hexagon numbers $n_{g,6,k}$.

k	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
2	2800	34300	81340	16695	0	0
4	4609920000	270256560000	5470015824000	42516370176000	108544213999200	56597795793000
6	43505659008 000000	66630746448 00000000	4225455365922 00000000	134784066695700 244000000	219133289516560 146000000	16984808089605 44078400000
8	11023506707447 80800000000	3382196457804530 68800000000	4700014305292697 9005440000000	369200898096647 5731179520000000	17291108386034951 6876140800000000	48125851798487907 98122421760000000

Table 5: Heptagon numbers $n_{g,7,k}$.

k	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
1	14	70	21	0	0	0
2	9800	215600	1009400	781200	0	0
3	21952000	1218336000	20217792000	110898368000	158932166400	24309331200
4	92198400000	10058845440000	386873706240000	6319266481920000	42291774083328000	96422698084608000
5	588594585600000	1094166355968 00000	79095340412928 00000	277053418672128 000000	482663835053568 0000000	39425239788834 816000000

Table 6: Octagon numbers $n_{g,8,k}$.

4 Calculating the GUE correlators from the two-dimensional Frobenius manifold of the Toda lattice

In this section, we briefly outline following [9, 7, 8] an algorithm for computing the genus expansion of the GUE free energy. It is just a specification of the general algorithm of [9, 7] applied to the two-dimensional Frobenius manifold with the potential

$$F = \frac{1}{2} u\, v^2 + e^u.$$

Such a Frobenius manifold appears in the description of the structure of the long wave limit of the Toda lattice. The algorithm of [9] proves to be quite powerful for computation of the low genera multipoint correlators. We have used it for checking the explicit examples presented above.

The algorithm will be illustrated on computation of the weighted numbers of triangulations on surfaces of genus 0, 1 and 2. We will only describe the main steps of the algorithm referring the reader to [9] for details.

Let $v = v(x, s)$ be the solution to the cubic equation

$$v\left(1 - 9s\, v + 18 s^2 v^2\right) = 6s\, x \tag{4.0.6}$$

in the form of a power series in s vanishing at $s = 0$,

$$v = 6\, s\, x + 324\, s^3 x^2 + 31104\, s^5 x^3 + \dots. \tag{4.0.7}$$

Put

$$w = \frac{x}{1 - 6\, s\, v} = x\left(1 + 36\, s^2 x + 3240\, s^4 x^2 + \dots\right). \tag{4.0.8}$$

Introduce also the series

$$u = \log w = \log x + 36\, s^2 x + 2592\, s^4 x^2 + \dots. \tag{4.0.9}$$

Then the genus zero free energy is given by the series expansion of the following expression

$$\begin{aligned}\mathcal{F}_0 &= \frac{1}{2}\left(v^2 w + \frac{1}{2} w^2\right) - 6s\left(\frac{1}{2}v^3 w + v w^2\right) + 18\, s^2\left(\frac{1}{4}v^4 w + v^2 w^2 + \frac{1}{3}w^3\right)\\ &\quad - x\left(\frac{1}{2}v^2 + w\right) + 6\, s\, x\left(\frac{1}{6}v^3 + v\, w\right) + \frac{1}{2}u\, x^2\\ &= \frac{1}{2}x^2\left(\log x - \frac{3}{2}\right) + 6\, s^2 x^3 + 216\, s^4 x^4 + \dots.\end{aligned} \tag{4.0.10}$$

Recall that the coefficient of s^k at $x = 1$ is equal to the weighted number of planar triangulations with k triangles.

The genus one free energy is given by the formula

$$\begin{aligned}\mathcal{F}_1 &= \frac{1}{24}\log\left(v_x^2 - w\, u_x^2\right)\\ &= -\frac{1}{12}\log x + \frac{3}{2}s^2 x + 189\, s^4 x^2 + \dots.\end{aligned} \tag{4.0.11}$$

The genus two expression is more involved. Introduce two series

$$u_{1,2} = v \pm 2\sqrt{w} = 6\, s\, x + 324\, s^3 x^2 + 31104\, s^5 x^3 \pm 2\sqrt{x}\left(1 + 18\, s^2 x + 1458\, s^4 x^2 + \dots\right). \tag{4.0.12}$$

Then

$$\begin{aligned}24^2\,\mathcal{F}_2 &= \frac{4\, u_1''^3\, u_{12}}{5\, u_1'^4} - \frac{4\, u_2''^3\, u_{12}}{5\, u_2'^4} - \frac{u_1''\, u_2''}{4\, u_1'\, u_2'}\\ &+ \frac{3\, u_1''}{4\, u_1'^3}\left(\frac{1}{2}\, u_1''\, u_2' - \frac{7}{5}\, u_1'''\, u_{12}\right) + \frac{3\, u_2''}{4\, u_2'^3}\left(\frac{1}{2}\, u_2''\, u_1' + \frac{7}{5}\, u_2'''\, u_{12}\right)\\ &+ \frac{1}{4\, u_1'^2}\left(\frac{33}{10}\, u_1''^2 - \frac{9}{10}\, u_1'''\, u_2' + \frac{1}{10}\, u_1''\, u_2'' + u_1^{IV}\, u_{12}\right)\\ &+ \frac{1}{4\, u_2'^2}\left(\frac{33}{10}\, u_2''^2 - \frac{9}{10}\, u_2'''\, u_1' + \frac{1}{10}\, u_1''\, u_2'' - u_2^{IV}\, u_{12}\right)\\ &- \frac{1}{4\, u_1'}\left(\frac{17}{5}\, u_1''' + \frac{1}{2}\, u_2'''\right) - \frac{1}{4\, u_2'}\left(\frac{17}{5}\, u_2''' + \frac{1}{2}\, u_1'''\right)\\ &- \frac{1}{10\, u_{12}^2}\left(\frac{u_1'^3}{u_2'} + \frac{u_2'^3}{u_1'}\right) - \frac{1}{u_{12}^2}\left(u_1'^2 - \frac{11}{5}\, u_1'\, u_2' + u_2'^2\right)\\ &+ \frac{u_1'' - u_2''}{u_{12}}\left(\frac{u_2'}{5\, u_1'} + \frac{u_1'}{5\, u_2'} + 1\right)\\ &= -\frac{1}{240 x^2} + \frac{8505}{2}s^6 x + \dots\end{aligned} \tag{4.0.13}$$

Here $u_{12} = u_1 - u_2$, $u_{1,2}' = \partial_x u_{1,2}$, $u_{1,2}'' = \partial_x^2 u_{1,2}$ etc. Applying this algorithm one obtains the following table of the weighted numbers of triangulations of surfaces of genera 0, 1, 2 with $k \leq 20$ triangles (k is necessarily even); the computation takes less than a second.

k	$g=0$	$g=1$	$g=2$
2	6	3/2	0
4	216	189	0
6	13608	26892	8505/2
8	119744	4076568	2217618
10	540416448/5	3213210384/5	3905028468
12	11609505792	104047172352	231226436160
14	9425943686016/7	120228382104192/7	62004956093424
16	165505114570752	2877311706393600	15594280091334144
18	21285494650967040	487638320996544768	3749645355442763904
20	14195644503284514816/5	417102705028906942464/5	4360691488086816325632/5

Table 7: Weighted triangle numbers $a_g(3^k)$.

In a similar way one can easily compute the weighted numbers of quadrangulations etc. of surfaces of genus 0, 1, 2. In principle one can extend this algorithm to higher genera but the calculations become more involved.

We plan to do large g and large k asymptotics of connected GUE correlators based on the two algorithms described above in an upcoming publication.

A Appendix. GUE, Toda lattice and enumeration of ribbon graphs

A.1 GUE partition function and orthogonal polynomials

Consider the GUE partition function represented as an integral over the space $\mathcal{H}(N)$ of $N \times N$ Hermitean matrices $M = (M_{ij})$

$$Z_N(\mathbf{s};\epsilon) = \frac{(2\pi)^{-N}\epsilon^{-\frac{1}{12}}}{Vol(N)} \int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\mathrm{tr}\, V(M)} dM. \tag{A.1.1}$$

Here the formal series V depending on the parameters $\mathbf{s} = (s_3, s_4, \dots)$ has the form

$$V(M) = \frac{1}{2}M^2 - \sum_{j\geq 3} s_j M^j. \tag{A.1.2}$$

The integral with respect to the measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}\,(M_{ij})\, d\mathrm{Im}\,(M_{ij})$$

will be understood[2] as a formal asymptotic expansion[3] with respect to the small parameter $\epsilon \to +0$. The pre-factor $Vol(N)^{-1}$ corresponds to the volume, with respect to the Haar measure, of the quotient of the unitary group over the maximal torus $[U(1)]^N$

$$Vol(N) = Vol\left(U(N)/\left[U(1)\right]^N\right) = \frac{\pi^{\frac{N(N-1)}{2}}}{G(N+1)} \tag{A.1.3}$$

Here G is the Barnes G-function taking the value

$$G(N+1) = \prod_{n=1}^{N-1} n! \tag{A.1.4}$$

at positive integers. The formula (A.1.3) will be re-derived below.

Denote $\mathcal{D}_N$ the set of diagonal $N \times N$ matrices $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ with real ordered eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$. The map

$$\begin{aligned} &U(N)/\left[U(1)\right]^N \times \mathcal{D}_N \to \mathcal{H}(N) \\ &(U, \Lambda) \mapsto U\,\Lambda\,U^* \end{aligned} \tag{A.1.5}$$

is a local diffeomorphism away from a subset of codimension three in $\mathcal{H}(N)$. Because of invariance of the measure w.r.t. to the action of unitary group one obtains

$$\int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M)} dM = Vol\left(U(N)/\left[U(1)\right]^N\right) \int_{\mathcal{D}_N} \Delta^2(\lambda)\, e^{-\frac{1}{\epsilon}\sum_{k=1}^N V(\lambda_k)} d\lambda_1 \dots d\lambda_N.$$

Here

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

is the Vandermonde determinant. Due to symmetry of the integrand one can rewrite the last formula as

$$\int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M)} dM = \frac{1}{N!} Vol\left(U(N)/\left[U(1)\right]^N\right) \int_{\mathbb{R}^N} \Delta^2(\lambda)\, e^{-\frac{1}{\epsilon}\sum_{k=1}^N V(\lambda_k)} d\lambda_1 \dots d\lambda_N.$$

Denote

$$p_n(\lambda) = \lambda^n + a_{1n}\lambda^{n-1} + \dots + a_{nn}, \quad n = 0,\, 1, \dots \tag{A.1.6}$$

a system of monic polynomials orthogonal w.r.t. to the exponential weight

$$\int_{-\infty}^{\infty} p_n(\lambda)\, p_m(\lambda)\, e^{-\frac{1}{\epsilon}V(\lambda)} d\lambda = h_n\, \delta_{mn}. \tag{A.1.7}$$

[2]An alternative way is to consider the integral (A.1.1) at $\epsilon = 1$ as a formal expansion in the parameters s_k. Then one can extend $Z_N(\mathbf{s}; 1)$ also to the variables s_1 and s_2, $V(M) = \frac{1}{2}M^2 - \sum_{j\ge 1} s_j M^j$. In this way one obtains a generating series for correlators of all traces $\operatorname{tr} M^j$ for arbitrary $j \ge 1$.

[3]Under certain assumptions for the polynomial $V(M)$ it can be rigorously justified [11, 12] that the formal series considered below are asymptotic expansions of convergent matrix integrals.

Representing the Vandermonde as

$$\Delta(\lambda) = \det \begin{pmatrix} p_0(\lambda_1) & p_0(\lambda_2) & \dots & p_0(\lambda_N) \\ p_1(\lambda_1) & p_1(\lambda_2) & \dots & p_1(\lambda_N) \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ p_{N-1}(\lambda_1) & p_{N-1}(\lambda_2) & \dots & p_{N-1}(\lambda_N) \end{pmatrix}$$

one obtains an expression of the last integral via the normalizing factors of the orthogonal polynomials

$$\int_{\mathbb{R}^N} \Delta^2(\lambda) e^{-\frac{1}{\epsilon}\sum_{k=1}^N V(\lambda_k)} d\lambda_1 \dots d\lambda_N = N!\, h_0\, h_1 \dots h_{N-1}.$$

We conclude that

$$\int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M)} dM = Vol\left(U(N)/\left[U(1)\right]^N\right)\, h_0\, h_1 \dots h_{N-1}. \tag{A.1.8}$$

The formula (A.1.3) for the volume $Vol\left(U(N)/\left[U(1)\right]^N\right)$ can be easily derived from the last equation. Indeed, evaluating the lhs of eq. (A.1.8) at the Gaussian point $\mathbf{t} = 0$ one obtains

$$\int_{\mathcal{H}(N)} e^{-\frac{1}{2\epsilon}\operatorname{tr} M^2} dM = 2^{\frac{N}{2}} \left(\pi\, \epsilon\right)^{\frac{N^2}{2}}.$$

At $\mathbf{s} = 0$ the orthogonal polynomials (A.1.6)–(A.1.7) are expressed via Hermite polynomials

$$p_n(\lambda) = \epsilon^{\frac{n}{2}} \operatorname{He}_n(x), \quad \lambda = \epsilon^{\frac{1}{2}} x.$$

From

$$\int_{-\infty}^{\infty} \operatorname{He}_n(x) \operatorname{He}_m(x)\, e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}\, n!\, \delta_{mn}$$

it follows that

$$h_n(\mathbf{s} = 0) = \epsilon^{n+\frac{1}{2}} \sqrt{2\pi}\, n!$$

So, eq. (A.1.8) at $\mathbf{s} = 0$ takes the form

$$2^{\frac{N}{2}} \left(\pi\, \epsilon\right)^{\frac{N^2}{2}} = Vol\left(U(N)/\left[U(1)\right]^N\right) \cdot (2\pi)^{\frac{N}{2}}\, \epsilon^{\frac{N^2}{2}} \prod_{n=1}^{N-1} n!$$

This implies (A.1.3).

We conclude this Section with the following expression for the GUE partition function

$$Z_N(\mathbf{s}; \epsilon) = h_0\, h_1 \dots h_{N-1}. \tag{A.1.9}$$

Our nearest goal is to prove that this partition function is the tau-function of a particular solution of Toda hierarchy.

A.2 GUE and Toda

Denote v_n, w_n the coefficients of the three-term recursion relation for the orthogonal polynomials $p_n(\lambda)$

$$\lambda\, p_n(\lambda) = p_{n+1}(\lambda) + v_n\, p_n(\lambda) + w_n\, p_{n-1}(\lambda), \quad n \geq 0 \tag{A.2.1}$$

$p_{-1} = 0$. That is, the orthogonal polynomials are eigenvectors of the second order difference operator

$$(L\,\psi)_n = \psi_{n+1} + v_n\, \psi_n + w_n\, \psi_{n-1}. \tag{A.2.2}$$

The corresponding tri-diagonal matrix will also be denoted $L = (L_{ij})$.

Denote

$$(f, g) = \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, e^{-\frac{1}{\epsilon}V(\lambda)} d\lambda \tag{A.2.3}$$

an inner product on the space of polynomials. Recall that all integrals are understood as formal series in $\epsilon^{1/2}$ (actually, after division by $\sqrt{\epsilon}$ they contain only integer powers of ϵ). The symmetry

$$(\lambda\, p_n, p_m) = (p_n, \lambda\, p_m) \;\; \Leftrightarrow \;\; L_{mn} h_m = L_{nm} h_n$$

implies

$$w_n = \frac{h_n}{h_{n-1}} = \frac{Z_{n+1} Z_{n-1}}{Z_n^2}. \tag{A.2.4}$$

Here $h_n = (p_n, p_n)$ (see eq. (A.1.7) above).

For an arbitrary square matrix $X = (X_{ij})$ denote X_- and X_+ its upper- and lower-triangular parts

$$X_- = (X_{ij})_{i<j}\,, \quad X_+ = (X_{ij})_{i \geq j}\,, \quad X = X_+ + X_-.$$

Lemma A.2.1 *The orthogonal polynomials $p_n = p_n(\lambda)$ satisfy*

$$\epsilon\, \frac{\partial p_n}{\partial s_j} + (A_j\, p)_n = 0, \quad A_j = -\left(L^j\right)_-\,, \quad j \geq 3. \tag{A.2.5}$$

Proof. Write

$$\frac{\partial p_n(\lambda)}{\partial s_j} = \sum_{i=0}^{n-1} A^{(j)}_{i\,n}\, p_i(\lambda), \quad n \geq 1$$

for some coefficients $A^{(j)}_{in}$. Differentiating in s_j the equation $(p_n, p_m) = 0$ for $m < n$ we obtain

$$A^{(j)}_{mn} h_m + \frac{1}{\epsilon}\left(\lambda^j p_n, p_m\right) = 0.$$

Introduce matrices of multiplication by powers of λ

$$\lambda^j p_n(\lambda) = \sum_{i=0}^{n+j} \left(L^j\right)_{in} p_i(\lambda). \tag{A.2.6}$$

We have

$$\left(\lambda^j p_n, p_m\right) = \left(L^j\right)_{mn} h_m,$$

hence

$$\epsilon\, A^{(j)}_{mn} = -\left(L^j\right)_{mn}, \quad m < n \tag{A.2.7}$$

that is,

$$\epsilon\, A^{(j)} = -\left(L^j\right)_- .$$

□

Repeating a similar calculation for $m = n$ we obtain

$$\frac{\partial}{\partial s_j} \log \frac{Z_{n+1}}{Z_n} \equiv \frac{\partial}{\partial s_j} \log h_n = \left(L^j\right)_{nn}. \tag{A.2.8}$$

Corollary A.2.2 *The difference operator L satisfies*

$$\epsilon\, \frac{\partial L}{\partial s_j} = [A_j, L], \quad A_j = \left(L^j\right)_+ . \tag{A.2.9}$$

Proof. Differentiating equation

$$\lambda\, p_n = \sum_{i=0}^{n+1} L_{in}\, p_i$$

in s_j and using eq. (A.2.5) obtain

$$\epsilon\, \frac{\partial L}{\partial s_j} = \left[L, \left(L^j\right)_-\right].$$

Since the operators L and L^j commute we arrive at (A.2.9). □

Proposition A.2.3 *The GUE partition function Z is a tau-function, in the sense of Definition 1.2.4 of the Toda lattice hierarchy.*

Proof. Cor. A.2.2 tells that w_n, v_n is a particular solution to the Toda lattice hierarchy.

It then follows from (A.2.4) and (A.2.8) that the partition function Z_n satisfies eq. (1.2.9) and eq. (1.2.10). Let $t_j = s_{j+1},\ j = 0, 1, 2, \ldots$. Here s_1, s_2 are understood as in the footnote 2. Eq. (A.2.8) implies that

$$\frac{\partial}{\partial t_j} \log \frac{Z_{n+1}}{Z_n} = (j+1)\, h_{j-1}(n), \qquad j \geq 0$$

where $h_{j-1}(n) := \frac{1}{j+1}(L^{j+1})_{nn}$. Define

$$\gamma_n(\lambda) = \frac{1}{\lambda} + \sum_{j\geq 0} \frac{(j+1)\, h_{j-1}(n-1)}{\lambda^{j+2}}.$$

We have

$$\sum_{j\geq 0} \frac{1}{\lambda^{j+2}} \frac{\partial}{\partial t_j} \log \frac{Z_{n+1}}{Z_n} = \sum_{j\geq 0} \frac{1}{\lambda^{j+2}} (j+1)\, h_{j-1}(n) = \gamma_{n+1}(\lambda) - \frac{1}{\lambda}.$$

So

$$\sum_{i,j\geq 0} \frac{1}{\mu^{i+2}\lambda^{j+2}} \frac{\partial^2}{\partial t_j \partial t_i} \log \frac{Z_{n+1}}{Z_n} = \nabla(\mu)\, \gamma_{n+1}(\lambda).$$

Here, $\nabla(\mu)$ is defined in (2.2.6). Noting that

$$\begin{aligned}\nabla(\mu)\,\gamma_{n+1}(\lambda) &= \frac{\gamma_{n+1}(\lambda)(1+2\alpha_{n+1}(\mu))-\gamma_{n+1}(\mu)(1+2\alpha_{n+1}(\lambda))}{\lambda-\mu}+\gamma_{n+1}(\lambda)\gamma_{n+1}(\mu)\\ &= \frac{\operatorname{tr} R_{n+1}(\lambda)R_{n+1}(\mu)-\operatorname{tr} R_n(\lambda)R_n(\mu)}{(\lambda-\mu)^2}\end{aligned}$$

we obtain

$$\sum_{i,j\geq 0}\frac{1}{\mu^{i+2}\lambda^{j+2}}\frac{\partial^2}{\partial t_j\partial t_i}\log Z_n=\frac{\operatorname{tr} R_n(\lambda)R_n(\mu)-1}{(\lambda-\mu)^2}.$$

In the above formulae, R_n is the matrix resolvent of L. The proposition is proved. □

A.3 GUE and enumeration of ribbon graphs

After rescaling $M\mapsto \epsilon^{\frac{1}{2}}M$ and expansion in powers of the parameters one obtains

$$\int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M)}dM=\epsilon^{\frac{N^2}{2}}\int_{\mathcal{H}(N)} e^{-\frac{1}{2}\operatorname{tr} M^2+\sum_{j\geq 3}\epsilon^{\frac{j}{2}-1}s_j\operatorname{tr} M^j}dM$$

Expanding in a series in ϵ yields

$$\frac{Z_N(\mathbf{s};\epsilon)}{Z_N(0;\epsilon)}=\sum_{k\geq 0}\frac{1}{k!}\sum_{m\geq 0}\epsilon^m\sum_{i_1+\cdots+i_k=k+2m}s_{i_1}\dots s_{i_k}\left\langle \operatorname{tr} M^{i_1}\dots\operatorname{tr} M^{i_k}\right\rangle \tag{A.3.1}$$

where

$$\left\langle \operatorname{tr} M^{i_1}\dots\operatorname{tr} M^{i_k}\right\rangle:=\frac{\int \operatorname{tr} M^{i_1}\dots\operatorname{tr} M^{i_k}\,e^{-\frac{1}{2}\operatorname{tr} M^2}\,dM}{\int e^{-\frac{1}{2}\operatorname{tr} M^2}dM}. \tag{A.3.2}$$

The coefficients (A.3.2) of the perturbative expansion (A.3.1) are polynomials in N that can be computed by applying the Wick rule. E.g.,

$$\left\langle \operatorname{tr} M^4\right\rangle=2N^3+N,\quad \left\langle\left(\operatorname{tr} M^3\right)^2\right\rangle=12N^3+3N,\quad \left\langle \operatorname{tr} M^6\right\rangle=5N^4+10N^2$$

etc. Terms of the polynomial (A.3.2) correspond to oriented ribbon graphs with k vertices. Expansion of the logarithm of the partition function has a similar structure keeping connected graphs only:

$$\log\frac{Z_N(\mathbf{s};\epsilon)}{Z_N(0;\epsilon)}=\sum_{k\geq 0}\frac{1}{k!}\sum_{m\geq 0}\epsilon^m\sum_{i_1+\cdots+i_k=k+2m}s_{i_1}\dots s_{i_k}\left\langle \operatorname{tr} M^{i_1}\dots\operatorname{tr} M^{i_k}\right\rangle_c. \tag{A.3.3}$$

Introduce the 't Hooft coupling parameter

$$x=N\,\epsilon.$$

Re-expanding in ϵ the logarithm of the partition function we arrive at the main statement of this section, see [5].

Theorem A.3.1 *Logarithm of the tau-function of the solution to the Toda hierarchy given by the GUE partition function has the following expansion*

$$\log Z_N(\mathbf{s};\epsilon)|_{N=\frac{x}{\epsilon}} = \log Z_N(0;\epsilon) + \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(x; s_3, s_4, \dots)$$

$$\mathcal{F}_g(x; s_3, s_4, \dots) = \sum_{k\geq 0}\sum_{i_1,\dots,i_k} a_g(i_1,\dots,i_k)\, s_{i_1}\dots s_{i_k} x^h$$

$$a_g(i_1,\dots,i_k) = \frac{1}{k!}\,\#\{\textit{connected oriented labelled ribbon graphs of genus } g \textit{ with } k \textit{ vertices of valencies } i_1,\dots,i_k\}$$

$$= \frac{1}{k!}\sum_\Gamma \rho(\Gamma) = \sum_\Gamma \frac{1}{\#\operatorname{Sym}\Gamma} \tag{A.3.4}$$

$$h = 2-2g-\left(k-\frac{|i|}{2}\right),\ |i| = i_1+\dots+i_k,$$

where the two last summations are taken over all connected (unlabelled) ribbon graphs Γ *of genus* g *with* k *vertices of valencies* $i_1, \dots, i_k$*,* $\rho(\Gamma)$ *is the number of labelled ribbon graphs having the same topological shape* Γ*, and* $\#\operatorname{Sym}\Gamma$ *is the order of the symmetry group of* Γ*.*

In the particular case $i_1 = i_2 = \dots = i_k = 3$ the dual to the ribbon graph is a triangulation of the surface of genus g consisting of k triangles. Thus $a_g(3^k) := a_g(3,\dots,3)$ (k times) is equal to the weighted number of triangulations of genus g with k triangles. In a similar way, $a_g(4^k)$ is the weighted number of quadrangulations of a surface of genus g with k squares, etc.

Since

$$Z_N(0;\epsilon) = (2\pi)^{-\frac{N}{2}}\epsilon^{\frac{N^2}{2}-\frac{1}{12}}\,G(N+1)$$

we can also expand the first term in (A.3.4) with the help of the asymptotic expansion of the Barnes G-function

$$\log G(z+1) \sim \frac{z^2}{2}\left(\log z - \frac{3}{2}\right) + \frac{z}{2}\log 2\pi - \frac{1}{12}\log z + \zeta'(-1) + \sum_{\ell\geq 1}\frac{B_{2\ell+2}}{4\ell(\ell+1)z^{2\ell}},\quad z\to\infty.$$

This yields the following genus expansion of the logarithm of the tau-function of the interpolated Toda hierarchy where the shift operator $\psi_n \mapsto \psi_{n+1}$ acting on functions on a lattice is replaced with the translation $\psi(x)\mapsto\psi(x+\epsilon)$ acting on smooth functions on the real line,

$$\begin{aligned}\log\tau(x; s_3, s_4,\dots;\epsilon) &= \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \zeta'(-1) + \sum_{g\geq 2}\epsilon^{2g-2}\frac{B_{2g}}{4g(g-1)x^{2g-2}}\\ &\quad + \sum_{g\geq 0}\epsilon^{2g-2}\mathcal{F}_g(x; s_3, s_4,\dots).\end{aligned}\tag{A.3.5}$$

Remark A.3.2 *The genus expansion* (A.3.5) *is often written as* $1/N$ *expansion, setting* $x = 1$*, so* $\epsilon = 1/N$*,*

$$\log Z_N(\mathbf{s}; N^{-1}) = -\frac{N^2}{2}\log N - \frac{N}{2}\log 2\pi + \frac{1}{12}\log N + \log G(N+1) + \sum_{g\geq 0} N^{2-2g}\mathcal{F}_g(s_3, s_4,\dots)$$

$$\mathcal{F}_g(s_3, s_4,\dots) = \sum_{k\geq 0}\sum_{i_1,\dots,i_k} a_g(i_1,\dots,i_k)\, s_{i_1}\dots s_{i_k}.$$

The coefficients $a_g(i_1, \ldots, i_k)$ *are the same as in* (A.3.4).

Observe that the coefficients of the connected correlators as polynomials in N can be expressed via the numbers $a_g(i_1, \ldots, i_k)$ enumerating ribbon graphs. Namely, $\forall\, k \geq 1$ and $\forall\, i_1, \ldots, i_k$ such that $|i|$ is even, we have

$$\langle \operatorname{tr} M^{i_1} \operatorname{tr} M^{i_2} \, \ldots \, \operatorname{tr} M^{i_k} \rangle_c \;=\; k! \sum_{0 \leq g \leq \frac{|i|}{4} - \frac{k}{2} + \frac{1}{2}} a_g(i_1, \ldots, i_k)\, N^{2-2g-k+\frac{|i|}{2}}.$$

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