Doctoral Defense:

Hydrodynamic Correlations in Low-Dimensional Interacting Systems

1) The Emptiness Formation Probability

2) Spin-Charge (non-)Separation

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Outline

- Introduction & Motivations
- 1. The Emptiness Formation Probability in the XY model
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- Conclusions and direction for future research

Introduction

- 1-D systems, Zero-Temperature
- Correlators → Bosonization (linear approximation)
- Bosonization cannot describe large deviations:

  e.g. Emptiness Formation Probability or Spin-Charge Interaction
- Non-Linear Bosonization → Hydrodynamics
- Integrable model → Wave function (Bethe Ansatz)

Part 1: Emptiness Formation Probability

- EFP measures the probability that there are no particles in a region of length **n**
- One of the fundamental and simplest correlators in the theory of integrable models (Korepin et al.)
- Explicit expressions exist, but very complicated
- Interesting asymptotic behavior $\rightarrow$ Hydrodynamics

EFP for Spin Systems

- 1-d Spin Models: Probability of Formation of a Ferromagnetic String (PFFS) of length **n**:

$$\mathbf{P(n)} = \left\langle \prod_{i=1}^{n} \frac{1-\sigma_i^z}{2} \right\rangle$$

- Mapping to spinless fermion: PFFS becomes the Emptiness Formation Probability

$$\mathbf{P(n)} = \left\langle \prod_{i=1}^{n} \psi_i \psi_i^\dagger \right\rangle$$

- EFP measures the probability that there are no particles in a region of length **n**

The Anisotropic XY Model

$$H = \sum_i \left[\left(\frac{1+\gamma}{2} \right) \sigma_i^x \sigma_{i+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_i^y \sigma_{i+1}^y \right] - h \sum_i \sigma_i^z$$

- Jordan-Wigner transformation: spin degrees of freedom into spinless fermions

$$\begin{cases} \sigma_j^z = 2\psi_j^\dagger \psi_j - 1 \\ \sigma_j^+ = \psi_j^\dagger e^{i\pi \sum_{i<j} \psi_i^\dagger \psi_i} \end{cases}$$

$$\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$$

- In momentum space, the Hamiltonian becomes:

$$H = \sum_q 2(\cos q - h)\psi_q^\dagger \psi_q + i\gamma \sin q \left(\psi_q^\dagger \psi_{-q}^\dagger - \psi_{-q} \psi_q \right)$$

The Anisotropic XY Model (cont.)

- A Bogoliubov transformation diagonalizes the Hamiltonian

$$\chi_q = \cos\frac{\vartheta_q}{2}\psi_q + i\sin\frac{\vartheta_q}{2}\psi^\dagger_{-q}$$

$$H = \sum_q \varepsilon_q \left(\chi^\dagger_q \chi_q - 1/2\right) \qquad \varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}$$

- The XY Model is essentially Free Fermions

- Correlators for physical quantities involve inverting the transformation to FF: complications

The Phase Diagram of the XY Model

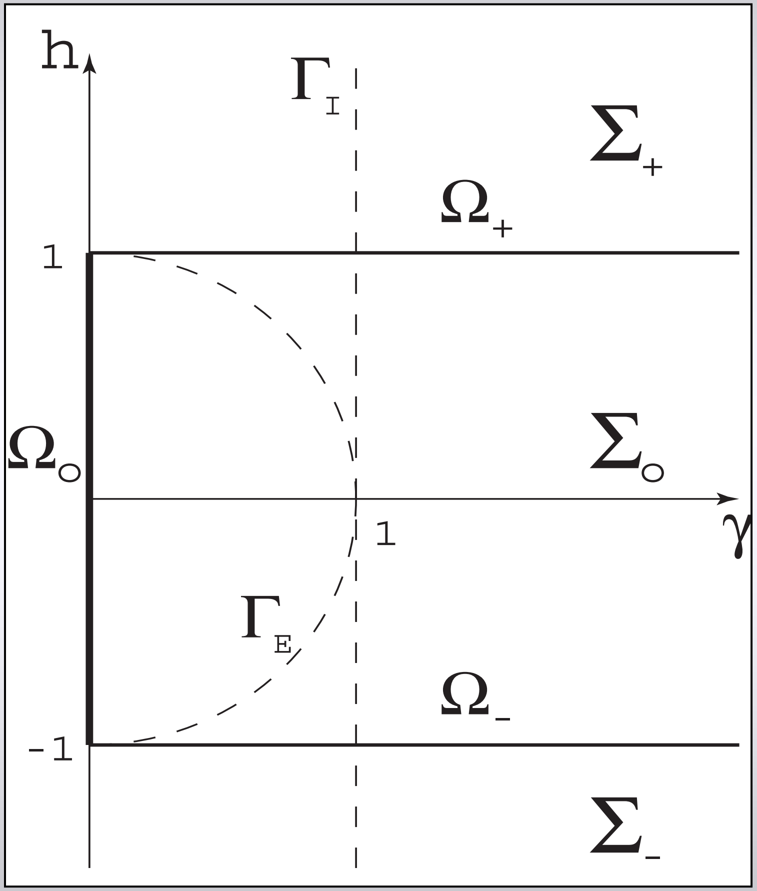
$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}$$

Phase Diagram:

- 3 non-critical regions (Σ_0, $\Sigma_\pm$)
- 3 critical phases:

Ω_0: Isotropic XY

$\Omega_\pm$: Critical magnetic field



Phase Diagram of the XY Model
(only $\gamma>0$ shown)

EFP for the XY Model

$$P(n) = \left\langle \prod_{i=1}^{n} \psi_i \psi_i^\dagger \right\rangle$$

- Wick Theorem and Pfaffian properties $\rightarrow$ EFP as the determinant of a **n** $\times$ **n** matrix: (Franchini & Abanov (2003))

$$P(n) = \left| \det(S_n) \right|$$

$$S_n = \left[\frac{1}{2} \int_{-\pi}^{\pi} \left(1 + \frac{\cos q - h + i\gamma \sin q}{\sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}} \right) e^{iq(j-k)} \frac{dq}{2\pi} \right]_{j,k=1}^{n}$$

Toeplitz Matrices

- Matrices like S_n are called Toeplitz: their elements depend only upon the difference of the indices

$$S_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & & \vdots \\ a_{-2} & a_{-1} & a_0 & & a_2 \\ \vdots & & & \ddots & a_1 \\ a_{-n+1} & \cdots & a_{-2} & a_{-1} & a_0 \end{pmatrix}$$

$$a_k = \int_{-\pi}^{\pi} \sigma(q) e^{iqk} \frac{dq}{2\pi}$$

- Asymptotic behavior of Det S_n: Szegö Theorem, Fisher-Hartwig conjecture and its generalization, Widom Theorem
- Det S_n depends on the "singularities" of the generating function

Toeplitz Matrices Techniques

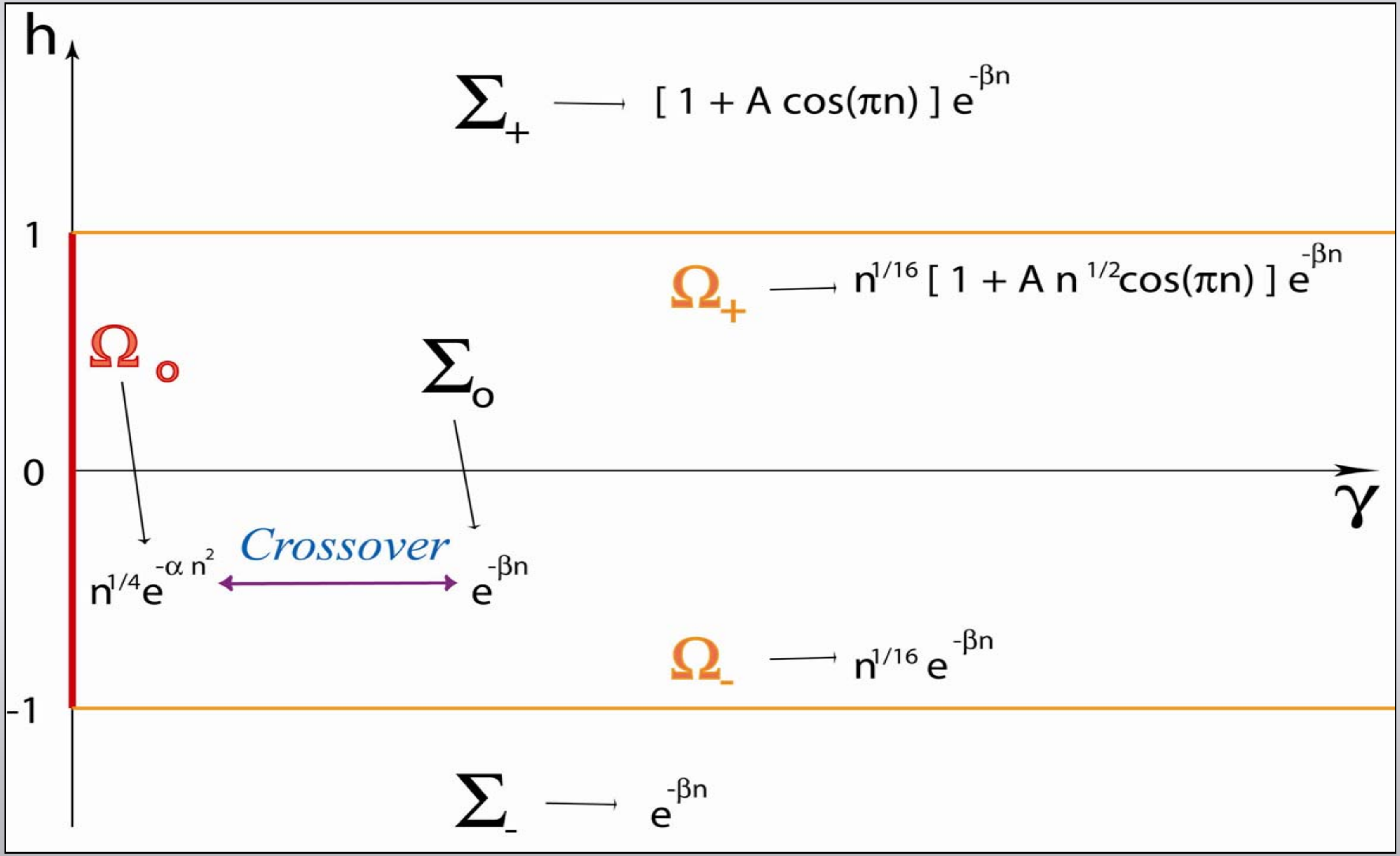
- Szegö Theorem: $\det S_n \overset{n\to\infty}{\sim} \exp\left[-n\int_0^{2\pi}\frac{\mathrm{d}q}{2\pi}\ln\sigma(q)\right]$

- McCoy et Al. (1970) used Toeplitz determinants to calculate 2-point correlators: $\rho^{\nu}_{lm} \equiv \langle 0\,|\sigma^{\nu}_{l}\;\sigma^{\nu}_{m}|\,0\rangle \qquad \nu = x, y, z$

$$\begin{aligned}\rho^{x}_{lm} &= \det|H(i-j)|^{j=l+1\ldots m}_{i=l\ldots m-1}\\ \rho^{y}_{lm} &= \det|H(i-j)|^{j=l\ldots m-1}_{i=l+1\ldots m}\end{aligned}$$

- We are the first to use the Generalized Fisher-Hartwig conjecture

The Phase Diagram of the XY Model and the Asymptotics of the EFP



Interpretation of these results

- Toeplitz determinant technique is exclusive for XY Model
- What is the physical meaning of the different behaviors?

  $\rightarrow$ Need for a more physical (general?) approach
- Collective description of the system $\rightarrow$ Bosonization
- EFP as probability of a collective configuration

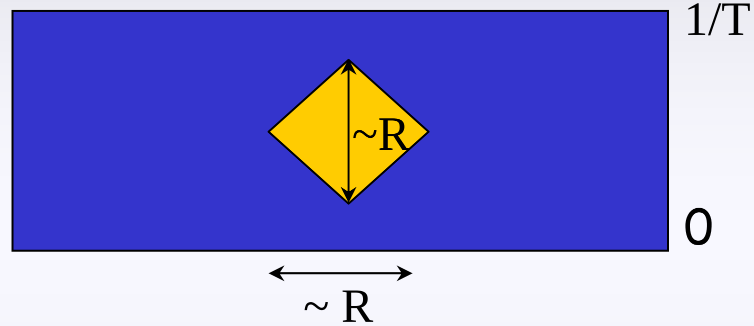
EFP as an Instanton Solution

- EFP as a rare fluctuation: $P(R) \sim e^{-S(R)}$

- Small String: $R \ll 1/T, \xi$

  $S(R) \sim R^2$

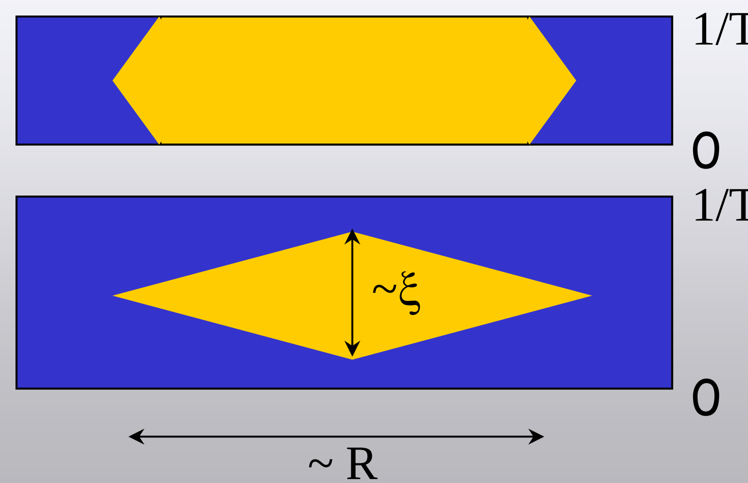
  (Gaussian)



- Large String: $R \gg 1/T$ or ξ

  $S(R) \sim R \times 1/T$ or ξ

  (exponential)



Limits of Bosonization

- Bosonization as a collective description
- Too large of a fluctuation: Bosonization cannot describe EFP (New problem: Depletion Formation Probability)
- Correct qualitative behavior:
  - Ω_0 (XX Model - Critical): Gaussian behavior (Abanov & Korepin, '02)
  - Crossover for small γ: Gaussian for $\mathbf{n} \ll 1/\sqrt{\gamma}$
    to Exponential for $\mathbf{n} \gg 1/\sqrt{\gamma}$ (Franchini & Abanov, '05)

Bosonization for the Isotropic XY Model

Korepin & Abanov 2002

- EFP as a rare fluctuation: $P(R) \sim e^{-S_0(R)}$

$$S = \int_{-\infty}^{+\infty} dx \int_0^{\beta} d\tau \, \frac{1}{2} (\partial_\mu \phi)^2$$

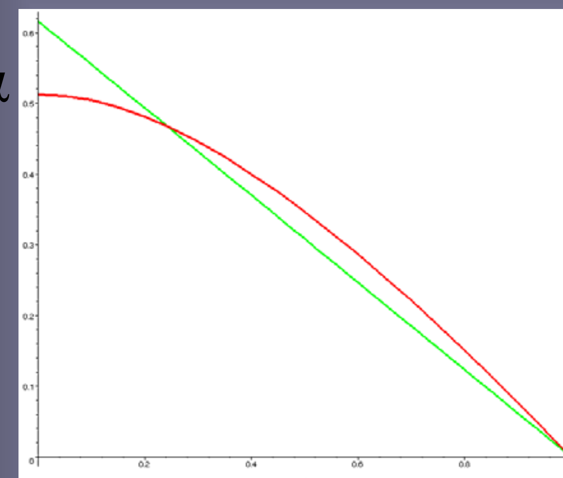
- From Bosonization:

$$S_0 = \pi^2/32 \, R^2 + \ldots \approx 0.34 \, R^2$$

- Exact result:

$$S_0 = ½ \, \mathrm{Ln} \, 2 \, R^2 + \ldots \approx 0.30 \, R^2$$

XXZ Model



Deift et Al. 1997
Kitanine et Al. 2002
Korepin et Al. 2002
Shiroishi et Al. 2002
Korepin & Abanov, 2002

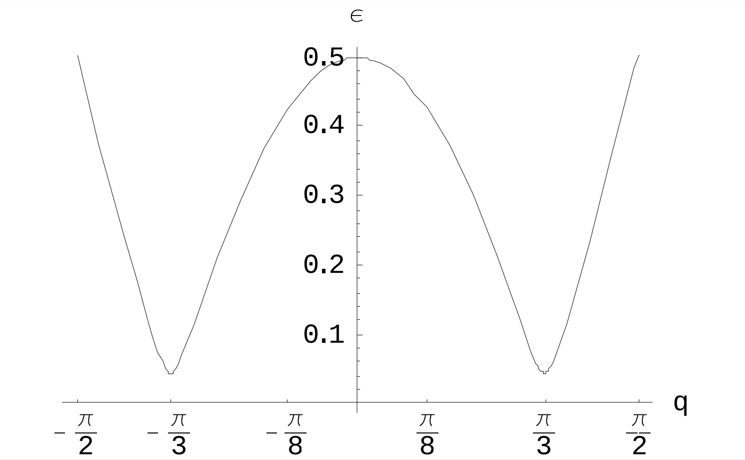
EFP Crossover from Bosonization

- We study the crossover for small γ: $n^{1/4} e^{-\alpha n^2} \xleftrightarrow{\text{Crossover}} e^{-\beta n}$

$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}$$

- Bosonized Lagrangian:

$$\mathcal{L} \simeq \left(\partial_\mu \theta\right)^2 + 2\gamma\theta^2$$



γ=0.05, h=0.5

m^2= 2γ: The anisotropy is an effective mass term that opens a gap

EFP Crossover from Bosonization

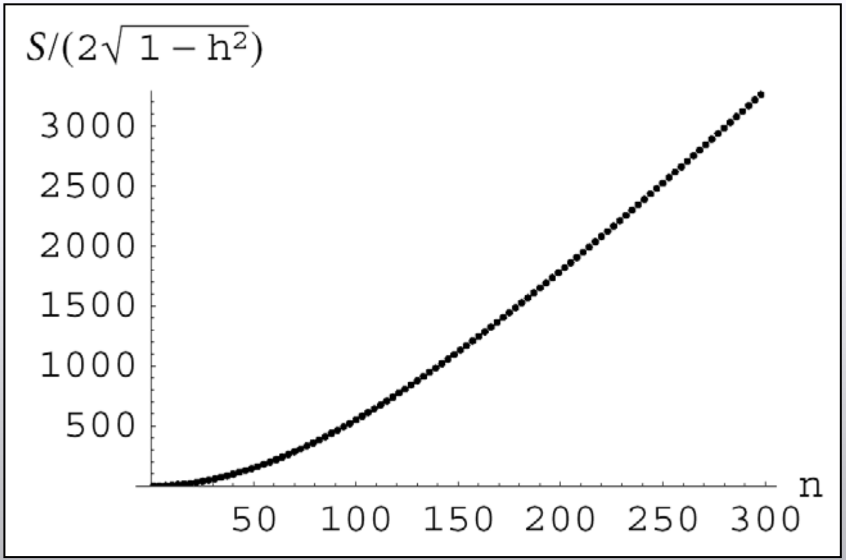
- We look for the saddle point solution of

$$(\partial_\mu \partial^\mu - m^2)\theta = 0$$ with EFP BC:

$$-\partial_t \theta(x,t)|_{t=0, 0<x<n} = 0$$

- The stationary action shows the expected crossover at n ~ **1**/√γ :

  P(n) ~ $e^{-S(n)}$



Part 1: Conclusions

- We express the EFP exactly as a determinant of a matrix
- We calculated the EFP in the whole phase-diagram of the XY model
- We provided the physical interpretation for the asymptotic behaviors
- We tested the limits of Bosonization

Part 2: Hydrodynamic Approach

- Collective field description: density ρ and velocity v
- From Galilean Invariance (Landau – 1941):

$$\mathcal{S}[\rho, v] = \int \mathrm{d}^2 x \left[\frac{\rho v^2}{2} - \rho \epsilon(\rho) + \ldots \right]$$

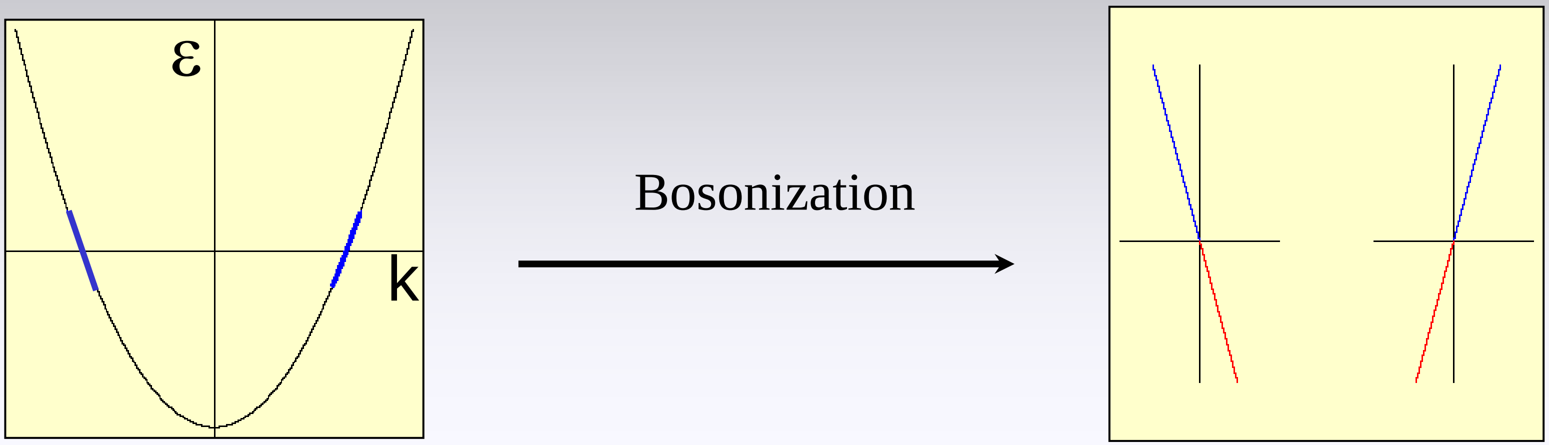
$$\partial_t \rho + \partial_x (\rho v) = 0$$

Continuity Equation

$$\partial_t v + v \partial_x v = -\partial_x \partial_\rho (\rho \epsilon)$$

Euler Equation

EFP & Hydrodynamics



- Hydrodynamics keeps non-linearity of the spectrum
- EFP as probability of Instanton configuration ($P(R) \sim e^{-S_0(R)}$) with

  BC: $\rho(t=0, |x|<R) = \bar{\rho} = 0$
- Bosonization valid for: $\frac{\rho_0 - \bar{\rho}}{\rho_0} \ll 1$ (ρ_0 : equilibrium density)

EFP results from Hydrodynamics

(Abanov 2005)

- Free Fermions: $$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}$$

$$\epsilon(\rho) = \frac{\pi^2}{6}\rho^2 \longrightarrow S_{EFP} = \frac{1}{2}\left[\pi\rho_0 R\right]^2 = \frac{1}{2}(k_F R)^2$$

- Calogero-Sutherland: $$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\sum_{1\le j<k\le N}\frac{\lambda(\lambda-1)}{(x_j - x_k)^2}$$

$$\epsilon(\rho) = \frac{\pi^2}{6}\lambda^2\rho^2 \longrightarrow S_{EFP} = \frac{\lambda}{2}\left[\pi\rho_0 R\right]^2$$

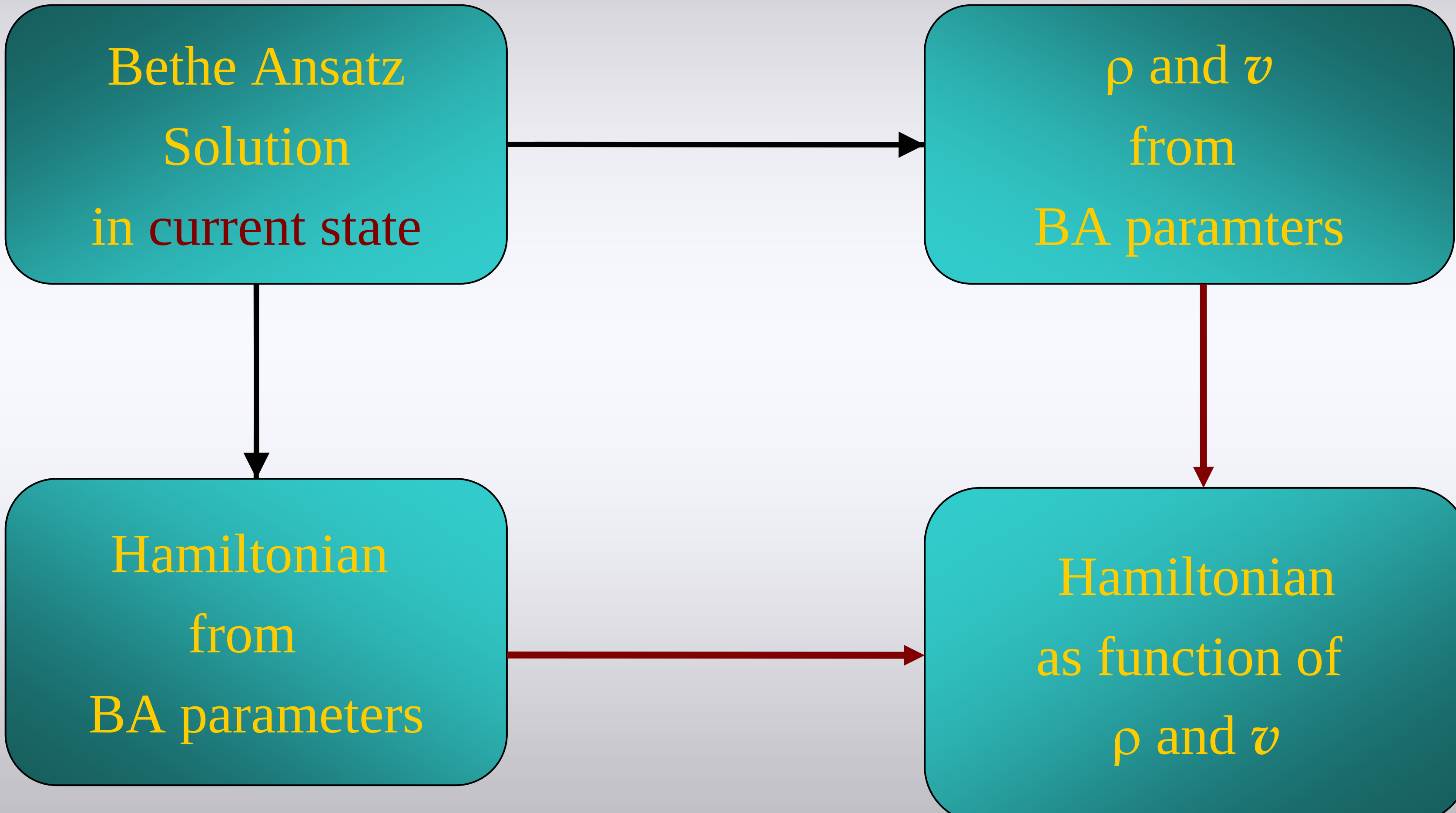
- Exact result known ($s \equiv \pi\,\rho_0\,R$) : $$-S = -\frac{\lambda}{2}s^2 - (1-\lambda)s + O(\ln s)$$

More EFP results from Hydrodynamics?

- Hydrodynamics correctly reproduce the leading behavior of EFP
- Work in progress: calculating EFP for lattice model

  (XY and XXZ model)
- Other problems → Construct the hydrodynamic description for more integrable systems

Hydrodynamics from Bethe Ansatz



Hydrodynamics Equations

- From microscopical description: $[\rho(x), v(y)] = -\mathrm{i}\hbar\delta'(x-y)$

$$\rho(x) \equiv \sum_{j=1}^{N} m\delta(x-x_j)$$

$$j(x) \equiv -\mathrm{i}\frac{\hbar}{2}\sum_{j=1}^{N}\left\{\frac{\partial}{\partial x_j}, \delta(x-x_j)\right\}, \qquad v \equiv \frac{1}{2}\left(\frac{1}{\rho}j + j\frac{1}{\rho}\right)$$

- Using the Hydrodynamic Hamiltonian: $\mathcal{H}(\rho, v) = \frac{\rho v^2}{2} + \rho\epsilon(\rho)$

$$\rho_t = \frac{\mathrm{i}}{\hbar}[H, \rho] = -\partial_x(\rho\, v),$$

$$v_t = \frac{\mathrm{i}}{\hbar}[H, v] = -\partial_x\left(\frac{v^2}{2} + (\rho\epsilon)_\rho\right)$$

Hydrodynamics for Free Fermions

- Exact Hydrodynamics ($\rho(k) = \frac{1}{2\pi}$)

$$\rho(x) = \int_{k_L}^{k_R} \rho(k) \mathrm{d}k = \frac{k_R - k_L}{2\pi}$$

$$J = \rho v = \int_{k_L}^{k_R} k \rho(k) \mathrm{d}k = \frac{k_R^2 - k_L^2}{4\pi}$$

- Same Hamiltonian as from non-linear Bosonization:

$$H = \int_{k_L}^{k_R} \frac{k^2}{2} \rho(k) \mathrm{d}k = \frac{k_R^3 - k_L^3}{12\pi} = \frac{\rho v^2}{2} + \frac{\pi^2}{6} \rho^3$$

$$[\rho(x), v(y)] = -\mathrm{i}\delta'(x - y)$$

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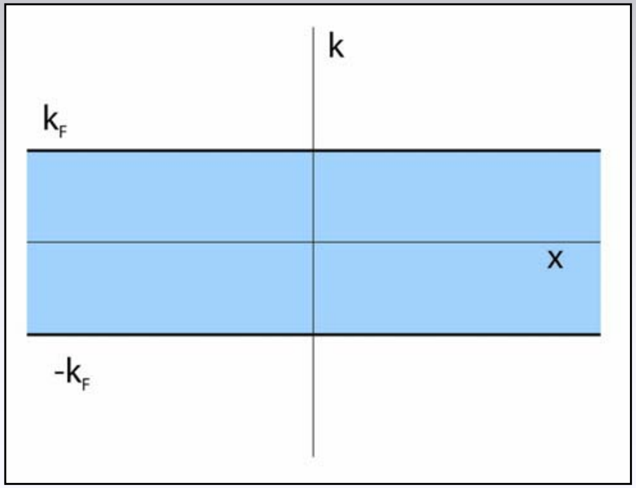
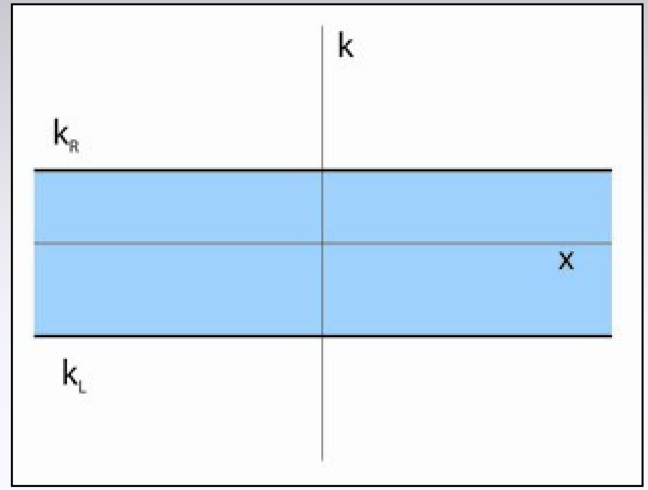
Phase-Space picture

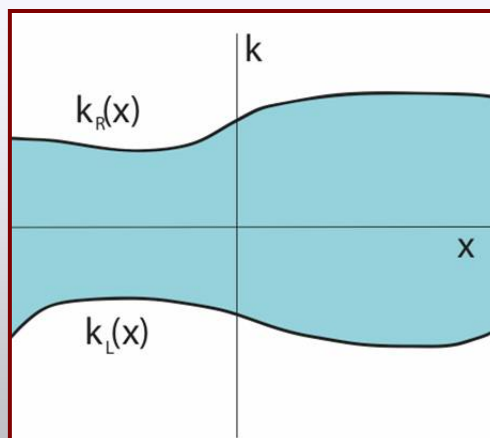
Ground State BA solution

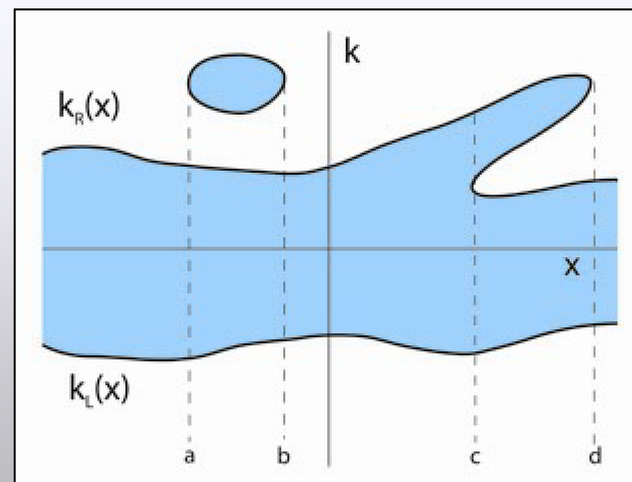




Current State BA solution

Perturbed (Hydrodynamic) Current State solution





Evolution of the solution beyond hydrodynamic description

Hydrodynamics for integrable models

- Bethe Ansatz solution: wave function as a superposition of two-particles scattering

- Essentially Free Fermions with corrected density

$$\rho(k) + \int_{q_L}^{q_R} K(k-p,c)\rho(p)\mathrm{d}p = \frac{1}{2\pi}$$

- Free Fermions: $K(k\text{-}p,c)=0$ → $\rho(k) = \frac{1}{2\pi}$

Part 2: Conclusions

- Hydrodynamics correctly reproduce the leading behavior of EFP
- We showed how to construct the hydrodynamic description of Integrable Models

Future developments

- Construct hydrodynamic description of spin models (XY and XXZ Model)
- Calculate EFP for these models

Part 3: Spin-Charge Separation

- Luttinger Liquid Spin-Charge separation comes from linear approximation of Bosonization (Haldane, 1979 & 1981, …):

$$H \sim v_c \left(\partial_x \phi_c\right)^2 + v_s \left(\partial_x \phi_s\right)^2 + \dots$$

$$\delta H = \frac{1}{k_F} \left(\partial_x \phi_c\right)^3 + \frac{3}{k_F} \left(\partial_x \phi_c\right) \left(\partial_x \phi_s\right)^2$$

- Perturbative calculations with spectrum curvature diverge
- Hydrodynamic approach takes into account the whole spectrum

Fermions with contact repulsion

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 4c \sum_{i<j} \delta(x_i - x_j)$$

- Complicated (Nested) Bethe Ansatz:

$$\begin{aligned} 2\pi\sigma(\Lambda) &= -\int_{B_L}^{B_R} \frac{4c\sigma(\Lambda')\mathrm{d}\Lambda'}{4c^2 + (\Lambda - \Lambda')^2} + \int_{Q_L}^{Q_R} \frac{2c\rho(k)\mathrm{d}k}{c^2 + (\Lambda - k)^2} \\ 2\pi\rho(k) &= 1 + \int_{B_L}^{B_R} \frac{4c\sigma(\Lambda)\mathrm{d}\Lambda}{c^2 + 4(k - \Lambda)^2} \end{aligned}$$

- The hydrodynamics variables are:

$$\rho = \int_{Q_L}^{Q_R} \rho(k)\mathrm{d}k, \qquad P = \int_{Q_L}^{Q_R} k\rho(k)\mathrm{d}k,$$

$$\rho_s = \int_{B_L}^{B_R} \sigma(\Lambda)\mathrm{d}\Lambda, \qquad P_s = \int_{B_L}^{B_R} p(\Lambda)\sigma(\Lambda)\mathrm{d}\Lambda$$

Spin-Charge Hydrodynamics

- Hydrodynamic Hamiltonian from the Bethe Ansatz:

$$H(Q_L, Q_R, B_L, B_R) = \int_{Q_L}^{Q_R} k^2 \rho(k) \mathrm{d}k = H(\rho, v, \rho_s, v_s)$$

by inverting the relations

$$\rho = \int_{Q_L}^{Q_R} \rho(k)\mathrm{d}k, \; P = \int_{Q_L}^{Q_R} k\rho(k)\mathrm{d}k, \; \rho_s = \int_{B_L}^{B_R} \sigma(\Lambda)\mathrm{d}\Lambda, \; P_s = \int_{B_L}^{B_R} p(\Lambda)\sigma(\Lambda)\mathrm{d}\Lambda$$

- The commutation relations complete the (implicit) hydrodynamic description:

$$[\rho(x), v(y)] = [\rho_s(x), v_s(y)] = -\mathrm{i}\delta'(x-y)$$

Part 3: Conclusions

- We constructed the Hydrodynamic description of Fermions with contact interaction in implicit form
- We can calculate the spin current carried by a charge disturbance $\rightarrow$ spin-charge coupling

Future directions

- Expand Hamiltonian to quadratic terms (bosonization) and beyond
- Find a close expression for the hydrodynamic Hamiltonian
- Apply this technique to spin Calogero-Sutherland Model

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Publications

- A.G. Abanov and F. Franchini; Phys. Lett. **A 316** (2003) 342-349,
  "*Emptiness Formation Probability for the Anisotropic XY Spin Chain in a Magnetic Field*"
  (also available on arXiv:cond-mat/0307001).
- F. Franchini and A.G. Abanov; J. Phys. **A 38** (2005) 5069-5096,
  "Asymptotics of Toeplitz Determinants and the Emptiness Formation Probability for the XY Spin Chain"
  (also available on arXiv:cond-mat/0502015).
- F. Franchini, A. R. Its, B.-Q. Jin and V. E. Korepin; to appear in "Proceedings of the
  26th International Colloquium on Group Theoretical Methods in Physics",
  "*Analysis of entropy of XY Spin Chain*
  (also available on arXiv:quant-ph/0606240)
- F. Franchini and A.G. Abanov; In progress,
  "*Coupling of Spin and Charge Degrees of Freedom in a Hydrodynamic Two-Fluid Approach*"
- F. Franchini and A.S. Goldhaber; In preparation,
  "*Aharonov-Bohm effect with many vortices*'"

Thank You!

In the Thesis, but not in this presentation

- Aharonov-Bohm effect in 2-D medium:

  Discrete spectrum;

  Exponential decay for a zero-energy particle;

  Topological trapping.

- Integrability of gradient-less hydrodynamic theories:

  Construction of the conserved currents.

Region	Critical	γ, h	P(n)	Zeros of $\sigma(q)$	Phase Jumps of $\sigma(q)$
Ω_0 (known)	Yes	$\gamma=0$, $-1<h<1$	$E\, n^{-1/4}\, e^{-\alpha n^2}$	$q \notin (-k_f, k_f)$	none
Σ_-	No	$h<-1$	$E\, e^{-\beta n}$	none	none
Ω_-	Yes	$h=-1$	$E\, n^{-1/16}\, [1+A n^{-1/2}]\, e^{-\beta n}$	none	π
Σ_0	No	$-1<h<1$	$E\, e^{-\beta n}$	π	π
Ω_+	Yes	$h=1$	$E\, n^{-1/16}\, [1+(-1)^n A n^{-1/2}]\, e^{-\beta n}$	π	$0, \pi$
Σ_+	No	$h>1$	$E\, [1+(-1)^n A]\, e^{-\beta n}$	$0, \pi$	$0, \pi$

EFP for the Anisotropic XY Model

Determinant Representation

- From Bethe Ansatz, correlators as complicated Fredholm integral operators (and minors of them)
- EFP is the simplest correlator, being expressed as the determinant of such an operator (Korepin et al.):

$$P(R) = \lim_{\alpha \to +\infty} \langle \Psi_G | \, e^{-\alpha \int_{-R}^{R} \rho(x) \, dx} \, | \Psi_G \rangle$$

$$= \frac{(0| \det[1 + \hat{V}] |0)}{\det[1 + \hat{K}]}$$

- With this formula it is possible to find the asymptotic behavior for large n, but only in very special cases

Multiple Integral Representation

- For the critical XXZ spin-½ Heisenberg chain ($\Delta=\cos\zeta$) (Kitanine et al. 2002):

$$\mathbf{P(n)} = \lim_{\xi_1 \dots \xi_n \to -i\frac{\zeta}{2}} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{Z_n\left(\{\lambda\},\{\xi\}\right)}{\prod_{a<b}^{n} \sinh(\xi_a - \xi_b)} \det_n \left(\frac{i}{2\zeta \sinh \frac{\pi}{\zeta}\left(\lambda_j - \xi_k\right)} \right) d^n\lambda$$

$$Z_n\left(\{\lambda\},\{\xi\}\right) = \prod_{a=1}^{n}\prod_{b=1}^{n} \frac{\sinh\left(\lambda_a - \xi_b\right)\sinh\left(\lambda_a - \xi_b - i\zeta\right)}{\sinh\left(\lambda_a - \lambda_b - i\zeta\right)} \cdot \frac{\det_n\left(\frac{-i\sin\zeta}{\sinh\left(\lambda_j - \xi_k\right)\sinh\left(\lambda_j - \xi_k - i\zeta\right)}\right)}{\prod_{a<b}^{n} \sinh(\xi_a - \xi_b)}$$

Hydrodynamics approach

- In general the Bethe Ansatz gives:

$$\rho(k) + \int_{q_L}^{q_R} K(k-p,c)\rho(p)\mathrm{d}p = \frac{1}{2\pi}$$

$$H = \int_{q_L}^{q_R} \frac{k^2}{2}\rho(k)\mathrm{d}k$$

- Using Galilean invariance we make a boost:

$$H = \frac{\rho v^2}{2} + \int_{-q}^{q} \frac{k^2}{2}\rho(k)\mathrm{d}k = \frac{\rho v^2}{2} + \rho\epsilon(\rho)$$

$$\rho(x) = \int_{k_L}^{k_R} \rho(k)\mathrm{d}k$$

Non-linear Bosonization

- We are interested in bilinears like:

$$: \psi^\dagger(x)\psi(x+\epsilon) := \frac{1}{2\pi} : \mathrm{e}^{\mathrm{i}\sqrt{4\pi}(\phi(x+\epsilon)-\phi(x))} : \mathrm{e}^{\mathrm{i}4\pi\langle\phi(x)\phi(x+\epsilon)\rangle}$$

$$= \frac{\mathrm{e}^{\mathrm{i}\sqrt{4\pi}\sum_{n=1}^{\infty}\frac{\epsilon^n}{n!}\phi^{(n)}(x)} - 1}{2\mathrm{i}\pi\epsilon}$$

- This is the generator for the currents:

$$\psi^\dagger(x)\psi(x+\epsilon) = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\psi^\dagger(x)\partial^n\psi(x) \equiv \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}J_n(x)$$

Non-linear Bosonization (cont.)

- The currents are:
  - Density: $$J_0 = \psi^\dagger(x)\psi(x) = \frac{1}{\sqrt{\pi}}\partial_x\phi(x)$$
  - Current Density: $$J_1 = \psi^\dagger(x)\partial_x\psi(x) = \mathrm{i}\left(\partial_x\phi(x)\right)^2 + \frac{1}{\sqrt{4\pi}}\partial_x^2\phi$$
  - Hamiltonian: $$J_2 = \psi^\dagger(x)\partial_x^2\psi(x)$$

$$= -\frac{\sqrt{4\pi}}{3}\left(\partial_x\phi(x)\right)^3 + \mathrm{i}\left(\partial_x\phi\right)\left(\partial_x^2\phi\right) + \frac{1}{3\sqrt{4\pi}}\partial_x^3\phi$$

Linear vs. Non-linear Bosonization

- One linearizes the spectrum around the Fermi Points:

$$H = -\psi^\dagger \partial_x^2 \psi \simeq -\sum_{L,R} \psi_{L,R}^\dagger \left(\partial_x \pm \mathrm{i} k_F\right)^2 \psi_{L,R}$$

- And after Bosonization:

$$H \sim k_F \left(\partial_x \phi_R\right)^2 + k_F \left(\partial_x \phi_L\right)^2 + \ldots$$

- While from the non-linear procedure we got:

$$H = \left(\partial_x \phi_R\right)^3 + \left(\partial_x \phi_L\right)^3$$

Spin-Charge Separation?

- One defines Spin and Charge degrees of freedom:

$$\phi_c = \phi_\uparrow + \phi_\downarrow \qquad \phi_s = \phi_\uparrow - \phi_\downarrow$$

- And the linear theory gives

$$H \sim k_F \left(\partial_x \phi_c\right)^2 + k_F \left(\partial_x \phi_s\right)^2 + \ldots$$

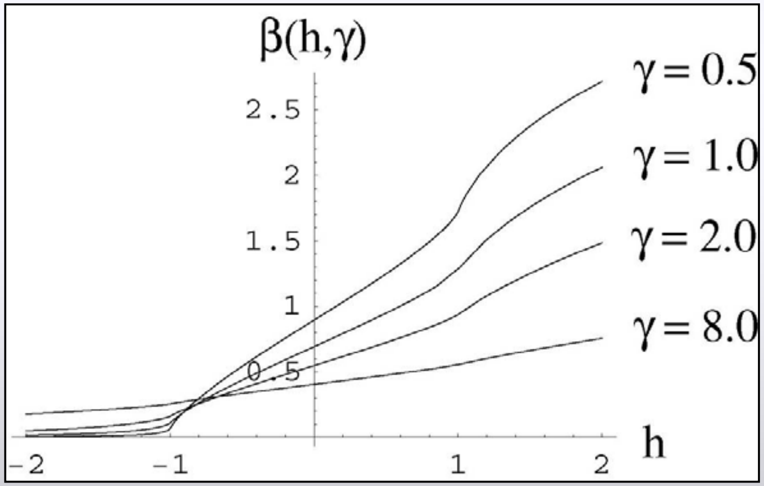
- While from the non-linear one gives:

$$H = \left(\partial_x \phi_c\right)^3 + 3 \left(\partial_x \phi_c\right) \left(\partial_x \phi_s\right)^2$$

EFP in the Non Critical Regions

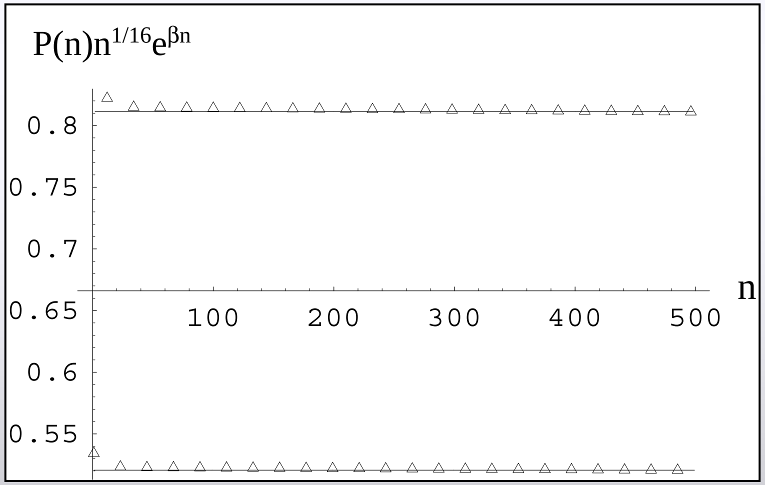
$$\Sigma_{-,0}: \quad P(n) \sim E_{-,0}(h,\gamma)\, e^{-\beta(h,\gamma)n}$$

$$\Sigma_{+}: \quad P(n) \sim E_{+}(h,\gamma)\left[1+(-1)^n A(h,\gamma)\right] e^{-\beta(h,\gamma)n}$$



$$\beta(h,\gamma) = -\int_{-\pi}^{\pi} \mathrm{Log}\left(\sigma(q)\right)\frac{dq}{2\pi}$$

(defined for $\gamma \neq 0$)



Numerical vs. Analytical results at $\gamma=1$, $h=1.5$

(Oscillatory behavior →

Z_2 symmetry breakdown at h=1)

Critical Phase: Ω_0

- Studied by Shiroshi et al. (2001) and in the '70s in the context of Unitary Random Matrices
- For $\gamma=0$, $\sigma(q)$ has only limited support
- Widom's Theorem $\rightarrow$

  the behavior is Gaussian with a power law pre-factor:

$$P(n) \sim 2^{\frac{5}{24}} e^{3\zeta'(-1)} (1-h)^{-\frac{1}{8}} n^{-\frac{1}{4}} \left(\frac{1+h}{2}\right)^{n^2/2} = E_0^c(h)\, n^{-\frac{1}{4}} e^{-\alpha(h) n^2}$$

EFP in the Critical Phases $\Omega_\pm$

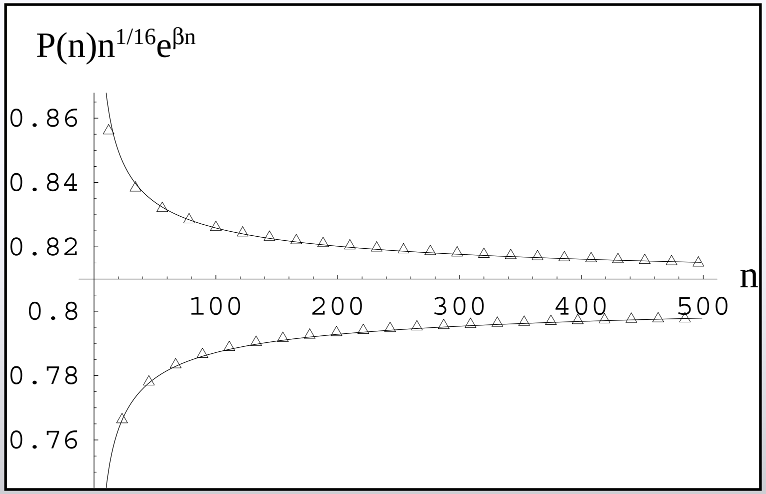
- Generalized Fisher-Hartwig conjecture: $P(n) \sim E(\gamma) n^{-\frac{1}{16}} e^{-\beta(h,\gamma)n}$
- Stretching the conjecture beyond its limits, we add a subleading term:

Ω_- :

$$P(n) \sim E^c_-(\gamma) n^{-\frac{1}{16}} \left[1 + A^c_-(\gamma) n^{-\frac{1}{2}} \right] e^{-\beta(-1,\gamma)n}$$

Ω_+ :

$$P(n) \sim E^c_+(\gamma) n^{-\frac{1}{16}} \left[1 + (-1)^n A^C_+(\gamma) n^{-\frac{1}{2}} \right] e^{-\beta(1,\gamma)n}$$



Numerical vs. Analytical results at $\gamma=1$, $h=1$

The Fisher-Hartwig Conjecture

- We parametrize the generating function as:

$$\sigma(q) = \tau(q) \prod_{r=1}^{R} e^{-\kappa_r \left(\pi - (q-\theta_r) \bmod 2\pi\right)} \left(2 - 2\cos(q - \theta_r)\right)^{\lambda_r}$$

where $\tau(q)$ is a smooth, non-zero function with winding number 0

- The asymptotic behavior of the determinant is:

$$\det(S_n) \underset{n\to\infty}{\sim} E[\tau,\kappa,\lambda]\, n^{\Sigma(\lambda_r^2 - \kappa_r^2)}\, e^{-\beta[\tau] n}$$

$$\beta[\tau] = -\int_{-\pi}^{\pi} \mathrm{Log}\left(\tau(q)\right) \frac{dq}{2\pi}$$

The generalized FH Conjecture

- When more than one parametrization exists:

$$\sigma(q) = \tau^a(q) \prod_{r=1}^{R} e^{-i\kappa_r^a \left(\pi - (q-\theta_r) \bmod 2\pi\right)} \left(2 - 2\cos(q-\theta_r)\right)^{\lambda_r^a}$$

the asymptotic behavior of the determinant is expressed as a sum of terms:

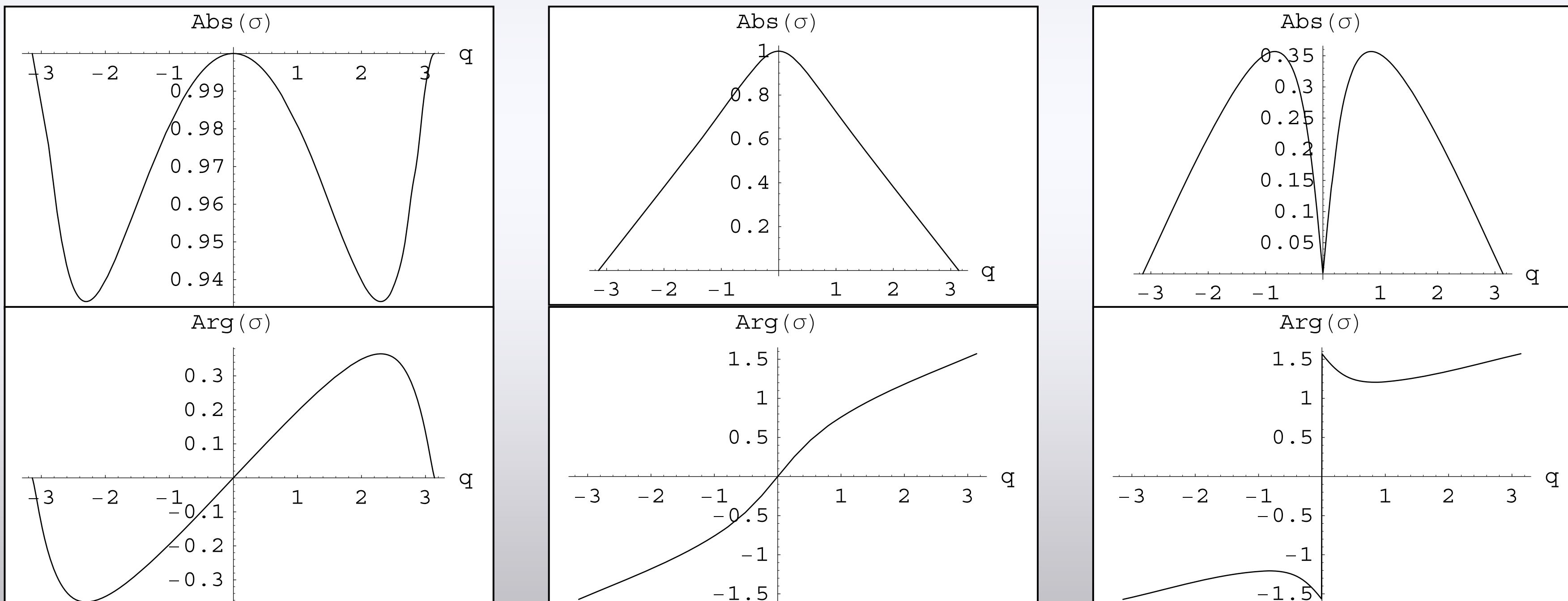
$$\det(S_n) \underset{n\to\infty}{\sim} \sum_{a\in T} E[\tau^a, \kappa^a, \lambda^a]\, n^{\Omega}\, e^{-\beta[\tau^a]n} \; ; \qquad \beta[\tau^a] = -\int_{-\pi}^{\pi} \mathrm{Log}\left(\tau^a(q)\right) \frac{dq}{2\pi}$$

$$T = \left\{ a; \Sigma\left((\lambda_r^a)^2 - (\kappa_r^a)^2 \right) = \max_j \Sigma\left((\lambda_r^j)^2 - (\kappa_r^j)^2 \right) = \Omega \right\}$$

Non Critical Regions: Σ_-, Σ_0 and Σ_+

- The generating function has different structures:

Σ_- Σ_0 Σ_+

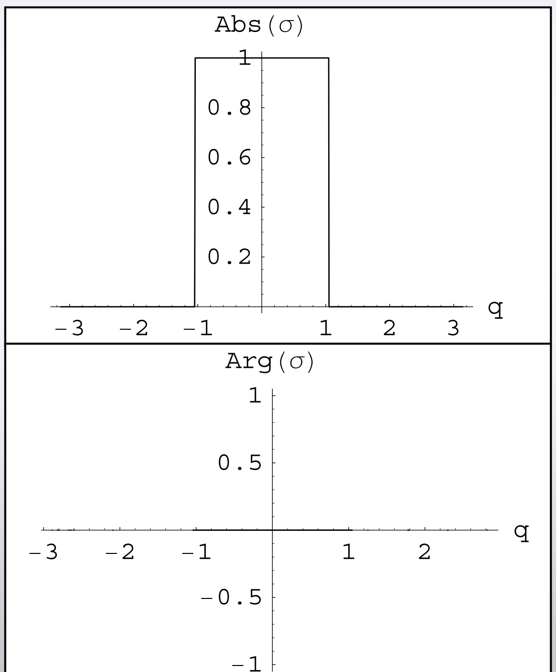


Abs(σ) and Arg(σ) for γ=1, h=-1.5; γ=1, h=0.5; γ=1, h=1.5

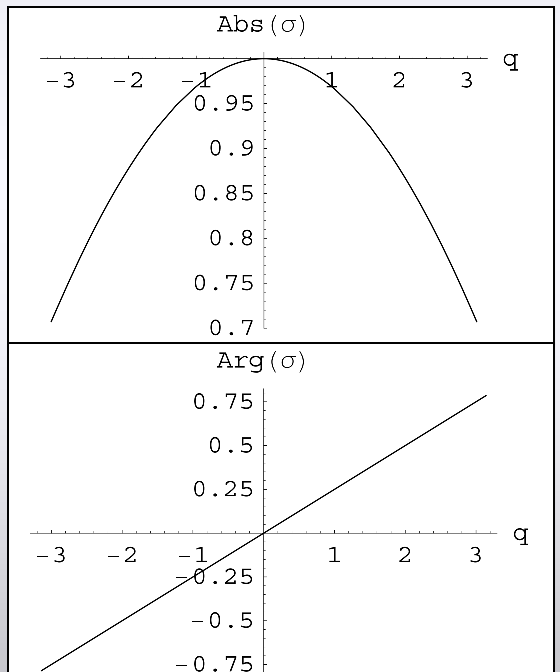
Critical Phases: Ω_0 , Ω_- , Ω_+

- The generating function presents the following behavior:

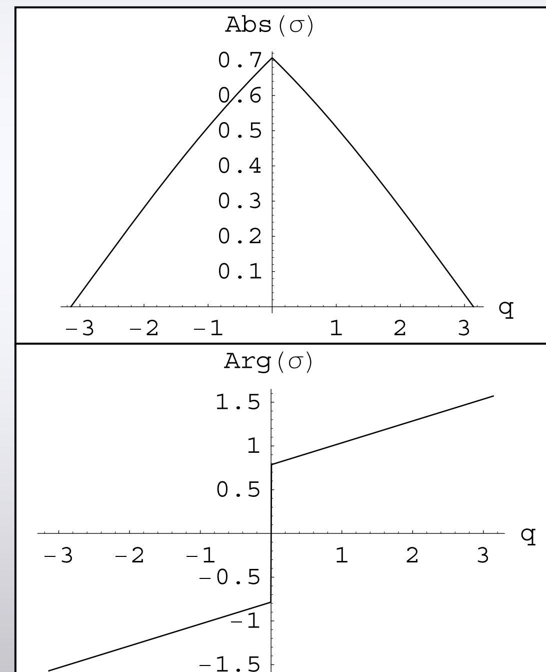
Ω_0 Ω_- Ω_+



Abs(σ) and Arg(σ) for γ=0, h=0.5



Abs(σ) and Arg(σ) for γ=1, h=-1



Abs(σ) and Arg(σ) for γ=1, h=1