Doctoral Defense:

Hydrodynamic Correlations in Low-Dimensional Interacting Systems

1) The Emptiness Formation Probability
2) Spin-Charge (non-)Separation

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Ph.D. Defense (7-21-2006)
Outline

• Introduction & Motivations

• 1. The Emptiness Formation Probability in the XY model

• 2. Hydrodynamic Approach to the EFP

• 3. Hydrodynamics for Spin-Charge degrees of freedom

• Conclusions and direction for future research
Introduction

• 1-D systems, Zero-Temperature

• Correlators $\rightarrow$ Bosonization (linear approximation)

• Bosonization cannot describe large deviations:
  e.g. Emptiness Formation Probability or Spin-Charge Interaction

• Non-Linear Bosonization $\rightarrow$ Hydrodynamics

• Integrable model $\rightarrow$ Wave function (Bethe Ansatz)
Part 1: Emptiness Formation Probability

- EFP measures the probability that there are no particles in a region of length \( n \).
- One of the fundamental and simplest correlators in the theory of integrable models (Korepin et al.).
- Explicit expressions exist, but very complicated.
- Interesting asymptotic behavior → Hydrodynamics.
EFP for Spin Systems

• 1-d Spin Models: Probability of Formation of a Ferromagnetic String (PFFS) of length \( n \):

\[
P(n) = \left\langle \prod_{i=1}^{n} \frac{1 - \sigma_i^z}{2} \right\rangle
\]

• Mapping to spinless fermion: PFFS becomes the Emptiness Formation Probability

\[
P(n) = \left\langle \prod_{i=1}^{n} \psi_i \psi_i^\dagger \right\rangle
\]

• EFP measures the probability that there are no particles in a region of length \( n \)
The Anisotropic XY Model

\[ H = \sum_i \left[ \left( \frac{1+\gamma}{2} \right) \sigma_i^x \sigma_{i+1}^x + \left( \frac{1-\gamma}{2} \right) \sigma_i^y \sigma_{i+1}^y \right] - \hbar \sum_i \sigma_i^z \]

- Jordan-Wigner transformation: 

  \[ \begin{cases} 
  \sigma_j^z = 2 \psi_j^\dagger \psi_j - 1 \\
  \sigma_j^+ = \psi_j^\dagger \exp \left( i \pi \sum_{i<j} \psi_i^\dagger \psi_i \right) \\
  \sigma_j^- = \frac{1}{2} \left( \sigma_j^x \pm i \sigma_j^y \right) 
  \end{cases} \]

- In momentum space, the Hamiltonian becomes:

\[ H = \sum_q \left( 2 \cos q - \hbar \right) \psi_q^\dagger \psi_q + i \gamma \sin q \left( \psi_q^\dagger \psi_{-q}^\dagger - \psi_{-q} \psi_q \right) \]
The Anisotropic XY Model (cont.)

- A Bogoliubov transformation diagonalizes the Hamiltonian

$$\chi_q = \cos \frac{\vartheta_q}{2} \psi_q + i \sin \frac{\vartheta_q}{2} \psi_{-q}$$

$$H = \sum_q \varepsilon_q \left( \chi_q^\dagger \chi_q - 1/2 \right)$$

$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}$$

- The XY Model is essentially Free Fermions

- Correlators for physical quantities involve inverting the transformation to FF: complications
The Phase Diagram of the XY Model

\[ \varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q} \]

Phase Diagram:

- 3 non-critical regions \((\Sigma_0, \Sigma_{\pm})\)
- 3 critical phases:
  - \(\Omega_0\): Isotropic XY
  - \(\Omega_{\pm}\): Critical magnetic field

Hydrodynamic Description of Correlators  n. 9  Ph.D. Defense  (7-21-2006)
EFP for the XY Model

\[ P(n) = \left\langle \prod_{i=1}^{n} \psi_i \psi_i^\dagger \right\rangle \]

- Wick Theorem and Pfaffian properties →

EFP as the determinant of a \( n \times n \) matrix:

(\text{Franchini & Abanov (2003)})

\[ P(n) = \left| \det(S_n) \right| \]

\[ S_n = \left[ \frac{1}{2} \int_{-\pi}^{\pi} \left( 1 + \frac{\cos q - h + i \gamma \sin q}{\sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}} \right) e^{iq(j-k)} \frac{dq}{2\pi} \right]^{n} \left\{ \begin{array}{ll} j,k=1 & \end{array} \right. \]
Toeplitz Matrices

• Matrices like $S_n$ are called Toeplitz: their elements depend only upon the difference of the indices.

$$S_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & \vdots \\ a_{-2} & a_{-1} & a_0 & \ddots & a_2 \\ \vdots & \vdots & \ddots & \ddots & a_1 \\ a_{-n+1} & \cdots & a_{-2} & a_{-1} & a_0 \end{pmatrix}$$

$$a_k = \int_{-\pi}^{\pi} \sigma(q)e^{ikq} \frac{dq}{2\pi}$$


• $\text{Det } S_n$ depends on the “singularities” of the generating function.
Toeplitz Matrices Techniques

- Szegö Theorem:
  \[ \det S_n \xrightarrow{n \to \infty} \exp \left[ -n \int_0^{2\pi} \frac{dq}{2\pi} \ln \sigma(q) \right] \]

- McCoy et Al. (1970) used Toeplitz determinants to calculate 2-point correlators:
  \[ \rho_{lm}^\nu = \langle 0 | \sigma_i^\nu \sigma_m^\nu | 0 \rangle \quad \nu = x, y, z \]
  \[
  \rho_{lm}^x = \det |H(i - j)|_{i = l \ldots m-1}^{j = l+1 \ldots m}
  \]
  \[
  \rho_{lm}^y = \det |H(i - j)|_{i = l+1 \ldots m}^{j = l \ldots m-1}
  \]
  \[
  \rho_{lm}^z = \det |H(i - j)|_{i = l \ldots m}^{j = l+1 \ldots m}
  \]

- We are the first to use the Generalized Fisher-Hartwig conjecture
The Phase Diagram of the XY Model and the Asymptotics of the EFP

\[ \sum_+ = [1 + A \cos(\pi n)] e^{-\beta n} \]

\[ \Omega_+ = \frac{n^{1/16}}{} [1 + A n^{1/2} \cos(\pi n)] e^{-\beta n} \]

\[ \sum_0 = \frac{n^{1/4}}{e^{-\alpha n^2}} e^{-\beta n} \text{ Crossover} \]

\[ \Omega_o = \frac{n^{1/16}}{} e^{-\beta n} \]

\[ \sum_- = e^{-\beta n} \]
Interpretation of these results

- Toeplitz determinant technique is exclusive for XY Model
- What is the physical meaning of the different behaviors?  
  → Need for a more physical (general?) approach
- Collective description of the system → Bosonization
- EFP as probability of a collective configuration
**EFP as an Instanton Solution**

- **EFP as a rare fluctuation:** \( P(R) \sim e^{-S(R)} \)

- **Small String:** \( R << 1/T, \xi \)
  \[ S(R) \sim R^2 \]
  (Gaussian)

- **Large String:** \( R >> 1/T \) or \( \xi \)
  \[ S(R) \sim R \times 1/T \) or \( \xi \]
  (exponential)
Limits of Bosonization

- Bosonization as a collective description

- Too large of a fluctuation: Bosonization cannot describe EFP

  (New problem: Depletion Formation Probability)

- Correct qualitative behavior:
  
  - $\Omega_0$ (XX Model - Critical): Gaussian behavior (Abanov & Korepin, ‘02)
  
  - Crossover for small $\gamma$: Gaussian for $n << 1/\sqrt{\gamma}$

    to Exponential for $n >> 1/\sqrt{\gamma}$ (Franchini & Abanov, ‘05)
Bosonization for the Isotropic XY Model
Korepin & Abanov 2002

- EFP as a rare fluctuation: \( P(R) \sim e^{-S_0(R)} \)

\[
S = \int_{-\infty}^{+\infty} dx \int_{0}^{\beta} d\tau \frac{1}{2} (\partial_{\mu} \phi)^2
\]

- From Bosonization:
  \( S_0 = \frac{\pi^2}{32} R^2 + \ldots \approx 0.34 R^2 \)

- Exact result:
  \( S_0 = \frac{1}{2} \ln 2 R^2 + \ldots \approx 0.30 R^2 \)
EFP Crossover from Bosonization

• We study the crossover for small $\gamma$:

$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}$$

• Bosonized Lagrangian:

$$\mathcal{L} \simeq \left( \partial_\mu \theta \right)^2 + 2\gamma \theta^2$$

$m^2 = 2\gamma$: The anisotropy is an effective mass term that opens a gap
EFP Crossover from Bosonization

- We look for the saddle point solution of

\[(\partial_{\mu} \partial^{\mu} - m^2)\theta = 0\] with EFP BC:

\[-\partial_t \theta(x, t)\big|_{t=0, 0 < x < n} = 0\]

- The stationary action shows the expected crossover

at \(n \sim \frac{1}{\sqrt{\gamma}}\):

\(P(n) \sim e^{-S(n)}\)
Part 1: Conclusions

• We express the EFP exactly as a determinant of a matrix

• We calculated the EFP in the whole phase-diagram of the XY model

• We provided the physical interpretation for the asymptotic behaviors

• We tested the limits of Bosonization
Part 2: Hydrodynamic Approach

- Collective field description: density $\rho$ and velocity $v$

- From Galilean Invariance (Landau – 1941):

$$S[\rho, v] = \int d^2 x \left[ \frac{\rho v^2}{2} - \rho \epsilon(\rho) + \ldots \right]$$

**Continuity Equation**

$$\partial_t \rho + \partial_x (\rho v) = 0$$

**Euler Equation**

$$\partial_t v + v \partial_x v = -\partial_x \partial_\rho (\rho \epsilon)$$
• Hydrodynamics keeps non-linearity of the spectrum

• EFP as probability of Instanton configuration \( (P(R) \sim e^{-S_0(R)}) \) with

BC: \( \rho(t = 0, |x| < R) = \bar{\rho} = 0 \)

• Bosonization valid for: \( \frac{\rho_0 - \bar{\rho}}{\rho_0} \ll 1 \) \( (\rho_0 : \text{equilibrium density}) \)
EFP results from Hydrodynamics

(Abanov 2005)

- **Free Fermions:**
  \[ H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \]
  \[ \epsilon(\rho) = \frac{\pi^2}{6} \rho^2 \]
  \[ S_{EFP} = \frac{1}{2} [\pi \rho_0 R]^2 = \frac{1}{2} (k_F R)^2 \]

- **Calogero-Sutherland:**
  \[ H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \sum_{1 \leq j < k \leq N} \frac{\lambda(\lambda - 1)}{(x_j - x_k)^2} \]
  \[ \epsilon(\rho) = \frac{\pi^2}{6} \lambda^2 \rho^2 \]
  \[ S_{EFP} = \frac{\lambda}{2} [\pi \rho_0 R]^2 \]

- **Exact result known** \( (s \equiv \pi \rho_0 R) \):
  \[ -S = -\frac{\lambda}{2} s^2 - (1 - \lambda) s + O(\ln s) \]
More EFP results from Hydrodynamics?

• Hydrodynamics correctly reproduce the leading behavior of EFP

• Work in progress: calculating EFP for lattice model
  (XY and XXZ model)

• Other problems → Construct the hydrodynamic description for
  more integrable systems
Hydrodynamics from Bethe Ansatz

- Bethe Ansatz Solution in current state
- Hamiltonian from BA parameters
- $\rho$ and $\nu$ from BA parameters
- Hamiltonian as function of $\rho$ and $\nu$
Hydrodynamics Equations

• From microscopical description:

\[ \left[ \rho(x), v(y) \right] = -i\hbar\delta'(x - y) \]

\[ \rho(x) \equiv \sum_{j=1}^{N} m\delta(x - x_j) \]

\[ j(x) \equiv -i\frac{\hbar}{2} \sum_{j=1}^{N} \left\{ \frac{\partial}{\partial x_j}, \delta(x - x_j) \right\}, \quad \nu \equiv \frac{1}{2} \left( \frac{1}{\rho} \frac{j}{\rho} + \frac{j}{\rho} \right) \]

• Using the Hydrodynamic Hamiltonian:

\[ \mathcal{H}(\rho, \nu) = \frac{\rho v^2}{2} + \rho\varepsilon(\rho) \]

\[ \rho_t = \frac{i}{\hbar} [H, \rho] = -\partial_x (\rho v), \]

\[ v_t = \frac{i}{\hbar} [H, v] = -\partial_x \left( \frac{v^2}{2} + (\rho\varepsilon)_\rho \right) \]
Hydrodynamics for Free Fermions

• Exact Hydrodynamics \( \rho(k) = \frac{1}{2\pi} \)

\[
\rho(x) = \int_{k_L}^{k_R} \rho(k) dk = \frac{k_R - k_L}{2\pi}
\]

\[
J = \rho v = \int_{k_L}^{k_R} k \rho(k) dk = \frac{k_R^2 - k_L^2}{4\pi}
\]

• Same Hamiltonian as from non-linear Bosonization:

\[
H = \int_{k_L}^{k_R} \frac{k^2}{2} \rho(k) dk = \frac{k_R^3 - k_L^3}{12\pi} = \frac{\rho v^2}{2} + \frac{\pi^2}{6} \rho^3
\]

\[
[\rho(x), v(y)] = -i\delta'(x - y)
\]
Phase-Space picture

Ground State BA solution

Current State BA solution

Perturbed (Hydrodynamic) Current State solution

Evolution of the solution beyond hydrodynamic description

Hydrodynamic Description of Correlators n. 28 Ph.D. Defense (7-21-2006)
Hydrodynamics for integrable models

- **Bethe Ansatz** solution: wave function as a superposition of two-particles scattering

- Essentially Free Fermions with corrected density

\[
\rho(k) + \int_{q_L}^{q_R} K(k - p, c) \rho(p) dp = \frac{1}{2\pi}
\]

- Free Fermions: \( K(k-p, c) = 0 \) \( \rightarrow \) \[
\rho(k) = \frac{1}{2\pi}
\]
Part 2: Conclusions

- Hydrodynamics correctly reproduce the leading behavior of EFP
- We showed how to construct the hydrodynamic description of Integrable Models

Future developments

- Construct hydrodynamic description of spin models
  (XY and XXZ Model)
- Calculate EFP for these models
Part 3: Spin-Charge Separation

- Luttinger Liquid Spin-Charge separation comes from linear approximation of Bosonization (Haldane, 1979 & 1981, …):

\[ H \sim v_c (\partial_x \phi_c)^2 + v_s (\partial_x \phi_s)^2 + \ldots \]

\[ \delta H = \frac{1}{k_F} (\partial_x \phi_c)^3 + \frac{3}{k_F} (\partial_x \phi_c) (\partial_x \phi_s)^2 \]

- Perturbative calculations with spectrum curvature diverge

- Hydrodynamic approach takes into account the whole spectrum
Fermions with contact repulsion

\[ H = - \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 4c \sum_{i<j} \delta(x_i - x_j) \]

• Complicated (Nested) Bethe Ansatz:

\[
2\pi \sigma(\Lambda) = - \int_{B_L}^{BR} \frac{4c \sigma(\Lambda')d\Lambda'}{4c^2 + (\Lambda - \Lambda')^2} + \int_{QL}^{QR} \frac{2c \rho(k)dk}{c^2 + (\Lambda - k)^2}
\]

\[
2\pi \rho(k) = 1 + \int_{B_L}^{BR} \frac{4c \sigma(\Lambda)d\Lambda}{c^2 + 4(k - \Lambda)^2}
\]

• The hydrodynamics variables are:

\[
\rho = \int_{QL}^{QR} \rho(k)dk, \quad P = \int_{QL}^{QR} k\rho(k)dk, \\
\rho_s = \int_{B_L}^{BR} \sigma(\Lambda)d\Lambda, \quad P_s = \int_{B_L}^{BR} p(\Lambda)\sigma(\Lambda)d\Lambda
\]
Spin-Charge Hydrodynamics

- Hydrodynamic Hamiltonian from the Bethe Ansatz:

\[ H(Q_L, Q_R, B_L, B_R) = \int_{Q_L}^{Q_R} k^2 \rho(k)dk = H(\rho, v, \rho_s, v_s) \]

by inverting the relations

\[
\rho = \int_{Q_L}^{Q_R} \rho(k)dk, \quad P = \int_{Q_L}^{Q_R} k\rho(k)dk, \quad \rho_s = \int_{B_L}^{B_R} \sigma(\Lambda)d\Lambda, \quad P_s = \int_{B_L}^{B_R} p(\Lambda)\sigma(\Lambda)d\Lambda
\]

- The commutation relations complete the (implicit) hydrodynamic description:

\[
[\rho(x), v(y)] = [\rho_s(x), v_s(y)] = -i\delta'(x - y)
\]
Part 3: Conclusions

• We constructed the Hydrodynamic description of Fermions with contact interaction in implicit form

• We can calculate the spin current carried by a charge disturbance $\rightarrow$ spin-charge coupling

Future directions

• Expand Hamiltonian to quadratic terms (bosonization) and beyond

• Find a close expression for the hydrodynamic Hamiltonian

• Apply this technique to spin Calogero-Sutherland Model
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F. Franchini and A.G. Abanov; In progress, “Coupling of Spin and Charge Degrees of Freedom in a Hydrodynamic Two-Fluid Approach”

F. Franchini and A.S. Goldhaber; In preparation, “Aharonov-Bohm effect with many vortices”
In the Thesis, but not in this presentation

- Aharonov-Bohm effect in 2-D medium:
  - Discrete spectrum;
  - Exponential decay for a zero-energy particle;
  - Topological trapping.

- Integrability of gradient-less hydrodynamic theories:
  - Construction of the conserved currents.
<table>
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<tr>
<th>Region</th>
<th>Critical</th>
<th>$\gamma$, $h$</th>
<th>$P(n)$</th>
<th>Zeros of $\sigma(q)$</th>
<th>Phase Jumps of $\sigma(q)$</th>
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<tbody>
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<td>$\Omega_0$ (known)</td>
<td>Yes</td>
<td>$\gamma = 0$, $-1 &lt; h &lt; 1$</td>
<td>$E n^{-1/4} e^{-\alpha n^2}$</td>
<td>$q \notin (-k_f, k_f)$</td>
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<tr>
<td>$\Sigma_-$</td>
<td>No</td>
<td>$h &lt; -1$</td>
<td>$E e^{-\beta n}$</td>
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<td>Yes</td>
<td>$h = -1$</td>
<td>$E n^{-1/16} [1 + A n^{-1/2}] e^{-\beta n}$</td>
<td>none</td>
<td>$\pi$</td>
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<tr>
<td>$\Sigma_0$</td>
<td>No</td>
<td>$-1 &lt; h &lt; 1$</td>
<td>$E e^{-\beta n}$</td>
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<td>$\Sigma_+$</td>
<td>No</td>
<td>$h &gt; 1$</td>
<td>$E [1 + (-1)^n A] e^{-\beta n}$</td>
<td>0, $\pi$</td>
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</table>

EFP for the Anisotropic XY Model
Determinant Representation

- From **Bethe Ansatz**, correlators as complicated Fredholm integral operators (and minors of them)

- EFP is the simplest correlator, being expressed as the determinant of such an operator (Korepin et al.):

\[
P(R) = \lim_{\alpha \to +\infty} \langle \Psi_G | e^{-\alpha \int_{-R}^{R} \rho(x) dx} | \Psi_G \rangle
\]

\[
= \frac{(0 | \det[1 + \hat{V}] | 0)}{\det[1 + \hat{K}]}
\]

- With this formula it is possible to find the asymptotic behavior for large \( n \), but only in very special cases
Multiple Integral Representation

For the critical XXZ spin-½ Heisenberg chain (Δ=cosζ) (Kitanine et al. 2002):

\[ P(n) = \lim_{\xi_1 \cdots \xi_n \to -i \frac{\zeta}{2}} n! \int_{-\infty}^{\infty} \prod_{a \neq b} \sinh(\xi_a - \xi_b) \frac{Z_n(\{\lambda\}, \{\xi\})}{\prod_{a \neq b} \sinh(\xi_a - \xi_b)} \det_n \begin{pmatrix} i \\ 2\zeta \sinh \frac{\pi}{\zeta} (\lambda_j - \xi_k) \end{pmatrix} d^n\lambda \]

\[ Z_n(\{\lambda\}, \{\xi\}) = \prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\sinh(\lambda_a - \xi_b) \sinh(\lambda_a - \xi_b + i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta) \sinh(\lambda_j - \xi_k) \sinh(\lambda_j - \xi_k + i\zeta)} \cdot \frac{\det_n \begin{pmatrix} -i \sin \zeta \\ \sinh(\lambda_j - \xi_k) \sinh(\lambda_j - \xi_k + i\zeta) \end{pmatrix}}{\prod_{a \neq b} \sinh(\xi_a - \xi_b)} \]
Hydrodynamics approach

- In general the Bethe Ansatz gives:

\[
\rho(k) + \int_{q_L}^{q_R} K(k - p, c) \rho(p) dp = \frac{1}{2\pi} \quad H = \int_{q_L}^{q_R} \frac{k^2}{2} \rho(k) dk
\]

- Using Galilean invariance we make a boost:

\[
H = \frac{\rho v^2}{2} + \int_{q}^{-q} \frac{k^2}{2} \rho(k) dk = \frac{\rho v^2}{2} + \rho \epsilon(\rho)
\]

\[
\rho(x) = \int_{k_L}^{k_R} \rho(k) dk
\]
Non-linear Bosonization

• We are interested in bilinears like:

\[ : \psi^\dagger(x) \psi(x + \epsilon) : = \frac{1}{2\pi} : e^{i\sqrt{4\pi}(\phi(x+\epsilon)-\phi(x))} : e^{i4\pi \langle \phi(x) \phi(x+\epsilon) \rangle} \]

\[ = e^{i\sqrt{4\pi} \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \phi^{(n)}(x)} - 1 \]

\[ \frac{2i\pi \epsilon}{2i\pi \epsilon} \]

• This is the generator for the currents:

\[ \psi^\dagger(x) \psi(x + \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \psi^\dagger(x) \partial^n \psi(x) \equiv \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} J_n(x) \]
• The currents are:

– Density:

\[ J_0 = \psi^\dagger(x) \psi(x) = \frac{1}{\sqrt{\pi}} \partial_x \phi(x) \]

– Current Density:

\[ J_1 = \psi^\dagger(x) \partial_x \psi(x) = i (\partial_x \phi(x))^2 + \frac{1}{\sqrt{4\pi}} \partial_x^2 \phi \]

– Hamiltonian:

\[ J_2 = \psi^\dagger(x) \partial_x^2 \psi(x) \]

\[ = -\frac{\sqrt{4\pi}}{3} (\partial_x \phi(x))^3 + i (\partial_x \phi) (\partial_x^2 \phi) + \frac{1}{3\sqrt{4\pi}} \partial_x^3 \phi \]
Linear vs. Non-linear Bosonization

• One linearizes the spectrum around the Fermi Points:

$$H = -\psi^\dagger \partial_x^2 \psi \approx - \sum_{L,R} \psi^\dagger_{L,R} (\partial_x \pm \imath k_F)^2 \psi_{L,R}$$

• And after Bosonization:

$$H \approx k_F (\partial_x \phi_R)^2 + k_F (\partial_x \phi_L)^2 + \ldots$$

• While from the non-linear procedure we got:

$$H = (\partial_x \phi_R)^3 + (\partial_x \phi_L)^3$$
Spin-Charge Separation?

- One defines Spin and Charge degrees of freedom:

\[ \phi_c = \phi_\uparrow + \phi_\downarrow \quad \phi_s = \phi_\uparrow - \phi_\downarrow \]

- And the linear theory gives:

\[ H \sim k_F (\partial_x \phi_c)^2 + k_F (\partial_x \phi_s)^2 + \ldots \]

- While from the non-linear one gives:

\[ H = (\partial_x \phi_c)^3 + 3(\partial_x \phi_c)(\partial_x \phi_s)^2 \]
EFP in the Non Critical Regions

$$\Sigma_{-,0} : \quad P(n) \sim E_{-,0}(h, \gamma) e^{-\beta(h,\gamma)n}$$

$$\Sigma_{+} : \quad P(n) \sim E_{+}(h, \gamma) \left[1 + (-1)^n A(h, \gamma)\right] e^{-\beta(h,\gamma)n}$$

$$\beta(h, \gamma) = -\int_{-\pi}^{\pi} \log(\sigma(q)) \frac{dq}{2\pi}$$

(defined for $\gamma \neq 0$)

Numerical vs. Analytical results at $\gamma=1, h=1.5$
(Oscillatory behavior $\rightarrow$ $Z_2$ symmetry breakdown at $h=1$)
Critical Phase: $\Omega_0$

- Studied by Shiroshi et al. (2001) and in the ‘70s in the context of Unitary Random Matrices
- For $\gamma=0$, $\sigma(q)$ has only limited support
- Widom’s Theorem $\rightarrow$
  
  the behavior is **Gaussian** with a **power law** pre-factor:

\[
P(n) \sim 2^{24} e^{3\zeta'(-1)} (1 - h)^{-\frac{1}{8}} n^{-\frac{1}{4}} \left(\frac{1 + h}{2}\right)^{n^2/2} = E_0^c(h) n^{-\frac{1}{4}} e^{-\alpha(h) n^2}
\]
EFP in the Critical Phases $\Omega_{\pm}$

- Generalized Fisher-Hartwig conjecture: $P(n) \sim E(\gamma)n^{-\frac{1}{16}}e^{-\beta(h,\gamma)n}$

- Stretching the conjecture beyond its limits, we add a subleading term:

  $\Omega_{-}$:

  \[ P(n) \sim E_{-}(\gamma)n^{-\frac{1}{16}}\left[1 + A_{-}(\gamma)n^{-\frac{1}{2}}\right]e^{-\beta(-1,\gamma)n} \]

  $\Omega_{+}$:

  \[ P(n) \sim E_{+}(\gamma)n^{-\frac{1}{16}}\left[1 + (-1)^n A_{+}(\gamma)n^{-\frac{1}{2}}\right]e^{-\beta(1,\gamma)n} \]

Numerical vs. Analytical results at $\gamma=1$, $h=1$
The Fisher-Hartwig Conjecture

• We parametrize the generating function as:

\[
\sigma(q) = \tau(q) \prod_{r=1}^{R} e^{-\kappa_r (\pi - (q - \theta_r) \mod 2\pi)} (2 - 2\cos(q - \theta_r))^\lambda_r
\]

where \(\tau(q)\) is a smooth, non-zero function with winding number 0

• The asymptotic behavior of the determinant is:

\[
\det(S_n) \sim E[\tau, \kappa, \lambda] n^{\Sigma(\lambda_r^2 - \kappa_r^2)} e^{-\beta[\tau]n}
\]

\[
\beta[\tau] = -\int_{-\pi}^{\pi} \frac{\log(\tau(q))}{2\pi} dq
\]
The generalized FH Conjecture

- When more than one parametrization exists:

\[ \sigma(q) = \tau^a(q) \prod_{r=1}^{R} e^{-i\kappa^a_r \left( \pi - (q - \theta_r \text{mod} 2\pi) \right)} \left( 2 - 2 \cos(q - \theta_r) \right)^{\lambda_r^a} \]

the asymptotic behavior of the determinant is expressed as a sum of terms:

\[ \text{det}(S_n) \sim \sum_{a \in T} E[\tau^a, \kappa^a, \lambda^a] n^\Omega e^{-\beta[\tau^a] n} ; \quad \beta[\tau^a] = -\int_{-\pi}^{\pi} \log(\tau^a(q)) \frac{dq}{2\pi} \]

\[ T = \left\{ a; \Sigma \left( (\lambda^a_r)^2 - (\kappa^a_r)^2 \right) = \max_j \Sigma \left( (\lambda^j_r)^2 - (\kappa^j_r)^2 \right) = \Omega \right\} \]
Non Critical Regions: $\Sigma_-$, $\Sigma_0$ and $\Sigma_+$

- The generating function has different structures:

\[ \Sigma_- \]

\[ \Sigma_0 \]

\[ \Sigma_+ \]

Abs$(\sigma)$ and Arg$(\sigma)$ for $\gamma=1$, $h=-1.5$; $\gamma=1$, $h=0.5$; $\gamma=1$, $h=1.5$
Critical Phases: $\Omega_0$, $\Omega_-$, $\Omega_+$

- The generating function presents the following behavior:

$\Omega_0$

Abs($\sigma$) and Arg($\sigma$) for $\gamma=0$, $h=0.5$

$\Omega_-$

Abs($\sigma$) and Arg($\sigma$) for $\gamma=1$, $h=-1$

$\Omega_+$

Abs($\sigma$) and Arg($\sigma$) for $\gamma=1$, $h=1$

Hydrodynamic Description of Correlators