Quantum Entropy of the XY Model

J. Phys. A: Math. Theor. 40 (2007) 8467, arXiv:0707.2534

Fabio Franchini*, A.R. Itz¶, B-Q. Jin† & V.E. Korepin‡

*The Abdus Salam ICTP, Strada Costiera 11, Trieste (TS), 34014, Italy, e-mail: **fabio@ictp.it** ;
¶Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, Indianapolis, IN, 46202, USA;
† College of Physics and Electronic Information, Wenzhou University, Wenzhou, Zhejiang, P.R. China;
‡ C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY, 11794, USA;

Abstract

Entanglement in the ground state of the **XY model** on the infinite chain can be measured by the **von Neumann entropy** or by the **Renyi entropy** of a block of neighboring spins. We study a double scaling limit: the size of the block is much larger than 1 but much smaller than the length of the whole chain. The entropy of the block has an asymptotic limit in the gapped regimes. We study this limiting entropy as a function of the anisotropy and of the magnetic field and identify its symmetry properties under the effect of **Modular Transformation** We identify the minima of the limiting entropy at product states and its divergences at the quantum phase transitions. We find that the curves of constant entropy are ellipses and hyperbolas and that they all meet at one point (*Essential Critical Point* –**ECP**-or *Multi-Critical Point* -**MCP**). Depending on the approach to this point, the entropy can take any value between 0 and ∞. In the vicinity of this point small changes in the parameters cause large change of the entropy.

Entropy and Entanglement

- **Entanglement** is a primary resource of **Quantum Information**
- There is **not** a unique way to quantify the entanglement of a system
- **Quantum entropy** is a good quantifier for the **bi-partite** entanglement
- Divide system into two subsystems: **A** & **B**
- Compute **Density Matrix** of subsystem:

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle\, \langle \Psi^{A,B}|$$

- **Von Neumann Entropy:**

$$S(\rho_A) = -\mathrm{tr}\,(\rho_A \log \rho_A)$$

- **Renyi Entropy:**

$$S_R(\rho_A, \alpha) = \frac{1}{1-\alpha} \ln \mathrm{tr}\,(\rho_A^\alpha)$$

- **N.B.** $\lim_{\alpha\to 1} S_R(\rho_A, \alpha) = S(\rho_A)$

Entanglement in a Spin Chain

- Consider the Ground State of a Hamiltonian:

$$H = -\sum_{j=-N}^{N} J_x S_j^x S_{j+1}^x + J_y S_j^y S_{j+1}^y + J_z S_j^z S_{j+1}^z + h S_j^z$$

- Block of spins in interval **[1,n]** is subsystem **A**
- The rest of the ground state is subsystem **B**

→ Entanglement of a block of spins on a space interval **[1, n]** with the rest of the ground state as a function of n

- **Consider** n→∞ (Double Scaling Limit: **1<<n<<N**)
- For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)

$$S(n) \to \text{Constant}$$

- For critical phases: (Calabrese, Cardy, 2004)

$$S(n) \to \frac{c}{3} \ln n + \ldots$$

The anisotropic XY Model

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

- **Jordan-Wigner Transf.:** spin degrees of freedom into spinless fermions

$$\begin{cases} \sigma_j^z = 2\psi_j^\dagger \psi_j - 1 \\ \sigma_j^+ = \psi_j^\dagger e^{i\pi \sum_{i<j} \psi_i^\dagger \psi_i} \end{cases} \qquad \sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i \sigma_j^y)$$

- In momentum space, the Hamiltonian becomes:

$$H = \sum_q 2(\cos q - h/2)\psi_q^\dagger \psi_q + i\gamma \sin q \left(\psi_q^\dagger \psi_{-q}^\dagger - \psi_{-q}\psi_q \right)$$

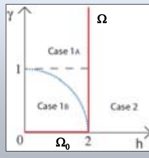
- A **Bogoliubov transf.** diagonalizas the Hamiltonian

$$H = \sum_q \varepsilon_q \left(\chi_q^\dagger \chi_q - 1/2 \right) \qquad \chi_q = \cos\frac{\vartheta_q}{2}\psi_q + i \sin\frac{\vartheta_q}{2}\psi_{-q}^\dagger$$

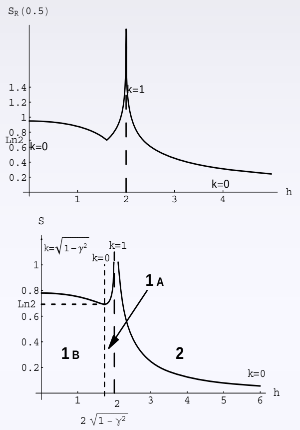
$$\varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q}$$

Phase Diagram:

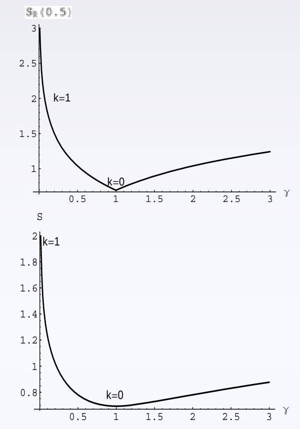
- 3 non-critical regions **(2,1a,1b)**
- 2 critical phases:
  - Ω_0: Isotropic XY
  - Ω: Critical magnetic field



Phase Diagram of the XY model



The limiting entropy as a function of the magnetic field at constant anisotropy **γ=0.5**. The entropy has a local minimum **S = ln 2** at $h = 2\sqrt{1-\gamma^2}$ and the absolute minimum for **h → ∞** where it vanishes. **S** is singular at the phase transition **h=2** where it diverges to **+∞**. The three cases are marked. Renyi entropy for **α = 0.5** on top, von Neumann entropy on the bottom.



The limiting entropy as a function of the anisotropy parameter at constant vanishing magnetic field **h = 0**. The entropy has a minimum **S = ln 2** at **γ=1** corresponding to the boundary between cases **1a** and **1b**. **S** diverges to **+∞** at the phase transition **γ=0**. Renyi entropy for **α = 0.5** on top, von Neumann entropy on the bottom.

Entropy of the XY Model

- Define the elliptic parameter in the three regions:

$$k = \begin{cases} \gamma \,/\, \sqrt{(h/2)^2 + \gamma^2 - 1}, & \text{Case 2} \\ \sqrt{(h/2)^2 + \gamma^2 - 1} \,/\, \gamma, & \text{Case 1a} \\ \sqrt{1 - \gamma^2 - (h/2)^2} \,/\, \sqrt{1 - (h/2)^2}, & \text{Case 1b} \end{cases}$$

- Representing the entropy as a Toeplitz determinant and employing Fredholm operators technique we represent it as a **series** which we can sum into:

$$S(\rho_A) = \frac{1}{6}\left[\ln \frac{4}{k\,k'} + (k^2 - k'^2)\frac{2I(k)I(k')}{\pi} \right], \qquad h > 2$$

$$S(\rho_A) = \frac{1}{6}\left[\ln \left(\frac{4k^2}{k'}\right) + (2 - k^2)\,\frac{2I(k)I(k')}{\pi} \right], \qquad h < 2$$

$$S_R(\rho_A, \alpha) = \frac{1}{6}\frac{\alpha}{1-\alpha} \ln(k\,k') - \frac{1}{3}\frac{1}{1-\alpha} \ln \left(\frac{\theta_2(0|\alpha i\tau)\,\theta_4(0|\alpha i\tau)}{\theta_3^2(0|\alpha i\tau)} \right) + \frac{1}{3}\ln 2\,, \qquad h > 2$$

$$S_R(\rho_A, \alpha) = \frac{1}{6}\frac{\alpha}{1-\alpha} \ln\left(\frac{k'}{k^2}\right) + \frac{1}{3}\frac{1}{1-\alpha} \ln \left(\frac{\theta_2^2(0|\alpha i\tau)}{\theta_3(0|\alpha i\tau)\,\theta_4(0|\alpha i\tau)} \right) + \frac{1}{3}\ln 2\,, \qquad h < 2$$

- Where we used the **complete elliptic integral of the 1st kind**:

$$I(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\,, \qquad k' = \sqrt{1-k^2}\,,$$

and the **elliptic theta-functions**:

$$\theta_2(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\left(n-\frac{1}{2}\right)^2\tau} e^{2iz\left(n-\frac{1}{2}\right)} \qquad \tau = I(k')/I(k)$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} e^{2izn} \qquad \theta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi n^2\tau} e^{2izn}\,.$$

Minima of the Entropy

- **Absolute minimum** at **h → ∞** or **γ → 0** (**h > 2**): $S_\infty \to 0$ (ferromagnetic ground state: ↑ . . . ↑)
- **Local minimum** S_∞ = **ln 2** at the boundary between cases **1a** and **1b** ($h = 2\sqrt{1-\gamma^2}$)
- The ground state is degenerate and factorized:

(each state is factorized and has **no entropy**)

$$|GS_1\rangle = \prod_{n \in \text{lattice}} \left[|\uparrow_n\rangle + \tan(\theta) |\downarrow_n\rangle \right], \qquad \cos^2(2\theta) = \frac{1-\gamma}{1+\gamma}$$

$$|GS_2\rangle = \prod_{n \in \text{lattice}} \left[|\uparrow_n\rangle - \tan(\theta) |\downarrow_n\rangle \right],$$

$$|GS\rangle = |GS_1\rangle + |GS_2\rangle$$

Curves of Constant Entropy & the Essential Critical Point

- The curves of constant entropy are Ellipses and Hyperbolae:

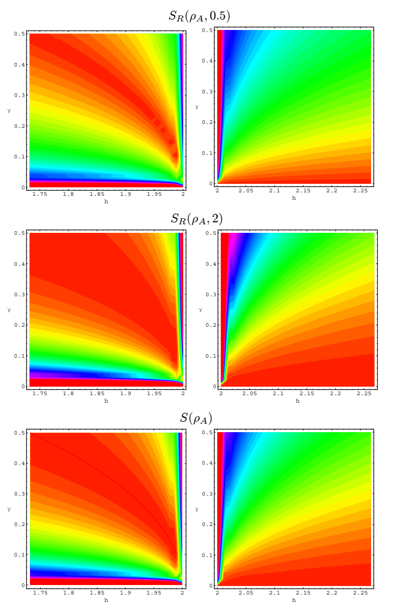
$$\text{Case 2} \quad \left\{ h > 2 : \right. \qquad \left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad 0 \le \kappa < \infty$$

$$\text{Case 1a} \quad \left\{ \begin{array}{l} h < 2, \\ \gamma > \sqrt{1-(h/2)^2} \end{array} \right. : \qquad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa > 1$$

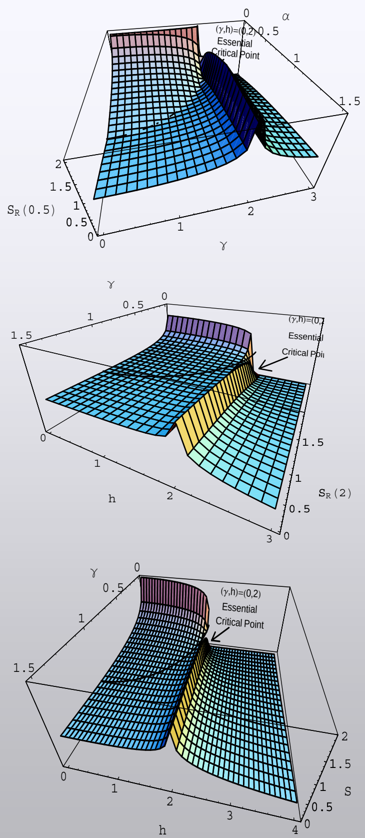
$$\text{Case 1b} \quad \left\{ \begin{array}{l} h < 2, \\ \gamma < \sqrt{1-(h/2)^2} \end{array} \right. : \qquad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa < 1.$$

- All these curves pass through at the Essential Critical Point: **(h,γ) = (2,0)**
- From any point of the phase diagram we can reach the ECP **along a curve of constant entropy**
- The range of the entropy is the positive real axis, **near the ECP the entropy assume any positive real value**
- Small variations in the parameters change the Entropy dramatically!

→ ECP is important for Quantum Control



Contour plot of the limiting entropy near the essential critical point **(h,γ) = (2,0)**. Regions of similar colors have similar entropy values and the lines where colors change are lines of constant entropy. **S** is diverging to **+∞** on the critical lines **h=2** and **h < 2**, **γ = 0**. One can see that near the essential critical point the lines of constant entropy grow denser. Renyi entropy for **α = 0.5** on top, **α = 2** in the middle and von Neumann entropy on the bottom.



Three-dimensional plot of the limiting entropy as a function of the anisotropy parameter **γ** and of the external magnetic field **h**. The local minimum **S = ln 2** at the boundary between cases **1a** and **1b** is visible and marked by a continuous line. **S** diverges to **+∞** at the phase transitions **h = 2** and **γ = 0, h < 2**. The entropy takes every positive value in the vicinity of the Essential Critical Point **(h,γ)=(2,0)**. Renyi entropy for **α = 0.5** on top, **α = 2** in the middle and von Neumann entropy on the bottom.

Entropy approaching the critical phases

- For γ → **0**:

$$S \sim -\frac{1}{3}\ln\gamma + \frac{1}{6}\ln\left[1 - (h/2)^2\right] + \ldots$$

$$S_R(\alpha) \sim \frac{1+\alpha}{\alpha}\left(-\frac{1}{6}\ln\gamma + \frac{1}{12}\ln\left[1-(h/2)^2\right] \right) + \ldots$$

- For h → **2**:

$$S \sim -\frac{1}{6}\ln\left|1-(h/2)^2\right| + \frac{1}{3}\ln\gamma + \ldots$$

$$S_R(\alpha) \sim \frac{1+\alpha}{\alpha}\left(-\frac{1}{12}\ln\left|1-(h/2)^2\right| + \frac{1}{6}\ln\gamma \right) + \ldots$$

- Von Neumann entropy results confirm conformal field theory results (Calabrese, Cardy, 2004)

Work in Progress: Modular Transformation Symmetry

- Starting from the definition of the entropy in one of the region, we can recover the definitions in the other regions by applying a modular transformation:

$$\tau_{(1a)} = 1 + \tau_{(1b)}\,, \qquad \tau_{(1a)} = \frac{\tau_{(2)}}{1 - \tau_{(2)}}$$

- Is this symmetry intrinsic in the XY model or just a feature of the entropy as a correlator? → **2**: