Entanglement Entropy in the XY Model

Fabio Franchini

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- work in progress...
Outline

• Introduction: Von Neumann and Renyi Entropy as a measure of Entanglement

• Quantum Entropy of the XY model

• Ellipses of constant Entropy and the Essential Critical Point

• Modular properties of the entropy and of the partition function - Not enough time -

• Conclusions
Understanding Entanglement

- Consider a unique (pure) ground state
- Divide the system into two Subsystems: A & B
- If system wave-function is:

  \[ |\Psi^{A,B} \rangle = |\Psi^A \rangle \otimes |\Psi^B \rangle \]

  \[ \rightarrow \quad \text{No Entanglement} \]

  (Measurements on B does not affect A state)
• If the system wave-function is:

\[ |\Psi^{A,B}\rangle = \sum_{j=1}^{d} \lambda_j |\Psi^A_j\rangle \otimes |\Psi^B_j\rangle \]

(with \( d > 1 \), \( |\Psi^A_j\rangle \& |\Psi^B_j\rangle \) linearly independent):

→ **Entangled** (Measurements on B affect A state):

i.e.  \[ \langle \Psi^B_i |\Psi^{A,B}\rangle = \lambda_j |\Psi^A_i\rangle \]
How to measure Entanglement?

• Compute Density Matrix of subsystem:

\[ \rho_A = tr_B \left( |\Psi^{A,B}\rangle \langle \Psi^{A,B}| \right) \]

• Entanglement for pure state as Quantum Entropy (Bennett, Bernstein, Popescu, Schumacher 1996):

\[ S = -tr_A \left( \rho_A \ln \rho_A \right) \]

_Von Neumann Entropy_
More Entanglement Estimators

- Von Neumann Entropy: \( S_A = -\text{tr} (\rho_A \log \rho_A) \)

- Renyi Entropy: \( S = \frac{1}{1-\alpha} \ln \text{tr} (\rho_A^\alpha) \)
  (equal to Von Neumann for \( \alpha \to 1 \))

- Tsallis Entropy

- Concurrence (Two-Tangle)

- ...
Entropy of a subsystem

\[ |\Psi^{A,B}\rangle = \sum \lambda_j |\Psi^A_j\rangle \otimes |\Psi^B_j\rangle \]

\[ \rho_A = \text{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}| = \sum \lambda_j^2 |\Psi^A\rangle \langle \Psi^A| \]

\[ \rho_B = \text{tr}_A |\Psi^{A,B}\rangle \langle \Psi^{A,B}| = \sum \lambda_j^2 |\Psi^B\rangle \langle \Psi^B| \]

\[ S_A = -\text{tr} (\rho_A \log \rho_A) = -\text{tr} (\rho_B \log \rho_B) = S_B \]

• NB: \( S_{AB} = 0 < S_A + S_B \) (Unlike thermodynamic entropy)
Entropy as a measure of entanglement

• Assume Bell State as unity of Entanglement:

$$|\text{Bell}\rangle = \frac{|\downarrow \downarrow\rangle + |\uparrow \uparrow\rangle}{\sqrt{2}}$$

• Von Neumann Entropy measures how many Bell-Pairs are contained in a given state $$|\Psi^A\rangle$$ (i.e. closeness of state to maximally entangled one)
Entanglement in a Spin Chain

• Consider the Ground state of a Hamiltonian:

\[
H = \sum_{i=1}^{N} \left[ J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - \hbar \sum_i \sigma_i^z
\]

• Block of spins in the space interval \([1, n]\) is subsystem A

• The rest of the ground state is subsystem B.

→ Entanglement of a block of spins on a space interval \([1, n]\) with the rest of the ground state as a function of \(n\)
We study the bi-partite entropy of the ground state of a system:

$$\rho_A = \text{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}|$$

$$S(n) = -\text{tr} (\rho_A \log \rho_A) \quad \text{Von Neumann}$$

$$S_\alpha(n) = \frac{1}{1 - \alpha} \ln \text{tr} (\rho_A^\alpha) \quad \text{Renyi}$$

- Multi-Point correlation function with contributions from all two-point correlators
- Highly non-trivial correlation function: new insights?
General Behavior

• We study the behavior for block size $n \to \infty$
  
  (Double scaling limit: $0 << n << N$)

  \[ S(n) = -\text{tr} \left( \rho_A \log \rho_A \right) \]

• For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)
  \[ S(n) \to \text{Constant} \]

• For critical phases: (Calabrese, Cardy, 2004)
  \[ S(n) \to \frac{c}{3} \ln n + \ldots \]
The Anisotropic XY Model

\[ H = -\sum_i \left[ (1 + \gamma) \sigma_i^x \sigma_{i+1}^x + (1 - \gamma) \sigma_i^y \sigma_{i+1}^y + h \sigma_i^z \right] \]

- Jordan-Wigner followed by Bogoliubov transformation to diagonalize the Hamiltonian

\[ H = \sum_q \varepsilon_q \left( \chi_q^\dagger \chi_q - 1/2 \right) \quad \varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q} \]

- The XY Model is essentially Free Fermions
- Correlators for physical quantities involve inverting the transformation to FF: complications
The Phase Diagram of the XY Model

\[ H = \sum_i \left[ (1 + \gamma) \sigma_i^x \sigma_{i+1}^x + (1 - \gamma) \sigma_i^y \sigma_{i+1}^y + h \sigma_i^z \right] \]

\[ \varepsilon_q = \sqrt{\left(\frac{h}{2} - \cos q\right)^2 + \gamma^2 \sin^2 q} \]

Phase Diagram:

- 3 non-critical regions (2, 1a, 1b)
- 2 critical phases:
  - \( \Omega_0 \): Isotropic XY
  - \( \Omega_+ \): Critical magnetic field

Phase Diagram of the XY Model
(only \( \gamma > 0 \) shown)
Entropy on the gapped phases for $|GS\rangle$

1. Case 1a: $2\sqrt{1 - \gamma^2} < h < 2$, medium magnetic field
2. Case 1b: $0 \leq h < 2\sqrt{1 - \gamma^2}$, small magnetic field
3. Case 2: $h > 2$, strong magnetic field

• We define an Elliptic Parameter:

\[
k = \begin{cases} 
\frac{\gamma}{\sqrt{(h/2)^2 + \gamma^2 - 1}}, & \text{Case 2} \\
\frac{\sqrt{(h/2)^2 + \gamma^2 - 1}}{\gamma}, & \text{Case 1A} \\
\frac{\sqrt{1 - \gamma^2 - (h/2)^2}}{\sqrt{1 - (h/2)^2}}, & \text{Case 1B}
\end{cases}
\]
Asymptotic Entropy

\[ S = -\text{tr} (\rho_A \log \rho_A) \quad S_R = \frac{1}{1 - \alpha} \ln \text{tr} (\rho_A^\alpha) \]

- For \( h > 2 \):

\[
S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln \left( k k' \right) - \frac{1}{3} \frac{1}{\alpha - 1} \ln \left( \frac{\theta_2(0|q^\alpha)}{\theta_3(0|q^\alpha)} \right) - \frac{1}{3} \ln 2
\]

\[
S(\rho_A) = \frac{1}{6} \left[ \ln \frac{4}{k k'} + (k^2 - k'^2) \frac{2I(k)I(k')}{\pi} \right]
\]

\[ q = e^{-\pi I(k')/I(k)} \quad k' = \sqrt{1 - k^2} \]

- For \( h < 2 \):

\[
S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln \left( \frac{k'}{k^2} \right) + \frac{1}{3} \frac{1}{\alpha - 1} \ln \left( \frac{\theta_2^2(0|q^\alpha)}{\theta_3(0|q^\alpha) \theta_4(0|q^\alpha)} \right) - \frac{1}{3} \ln 2
\]

\[
S(\rho_A) = \frac{1}{6} \left[ \ln \left( \frac{k^2}{16k'} \right) + (2k^2) \frac{2I(k)I(k')}{\pi} \right] \ln 2
\]
We have a completely analytical expression for the asymptotic entropy.

Let’s extract some physics out of it!!
Minima of the Entropy

- Absolute minimum at $h \to \infty$ or $\gamma \to 0$ ($h > 2$): $S_\infty \to 0$
  as the ground state becomes ferromagnetic ($\uparrow \ldots \uparrow$)

- Local minimum $S_\infty = \ln 2$
  at the boundary between cases 1a and 1b ($h = 2\sqrt{1 - \gamma^2}$)

- The ground states is factorized:
  (each state is factorized and has no entropy)

$$
|GS_1\rangle = \prod_{n \in \text{lattice}} \left[ \cos(\theta)|\uparrow_n\rangle + \sin(\theta)|\downarrow_n\rangle \right]
$$

$$
|GS_2\rangle = \prod_{n \in \text{lattice}} \left[ \cos(\theta)|\uparrow_n\rangle - \sin(\theta)|\downarrow_n\rangle \right]
$$

$$
|GS\rangle_+ = |GS_1\rangle + |GS_2\rangle
$$

$$
\cos^2(2\theta) = (1 - \gamma)/(1 + \gamma)
$$
Entropy at fixed $\gamma$

Von Neumann Entropy

Entanglement Entropy in the XY Model

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Entropy at fixed $\gamma$

Von Neumann Entropy
( $\alpha = 1$ )

Renyi Entropy
$\alpha = 1/2$

Renyi Entropy
$\alpha = 2$

Entanglement Entropy in the XY Model

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3-D plot of the Entropy

Von Neumann Entropy
\( \alpha = 1 \)

\( (\gamma, h) = (0, 2) \)

Essential Critical Point

Entanglement Entropy in the XY Model

n. 20

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3-D plot of the Entropy

Von Neumann Entropy
(\( \alpha = 1 \) )

Rényi Entropy
\( \alpha = 1/2 \quad \alpha = 2 \)

Essential Critical Point

\( (\gamma, h) = (0, 2) \)

Entanglement Entropy in the XY Model
The Essential Critical Point (ECP)

- Point \((\gamma, h) = (0, 2)\) is special:
  - Theory is critical, but not CFT (quadratic spectrum)
  - We can study the entropy close to this point:
    - Approaching it along \(h = 2, \gamma > 0\): \(S_\infty = \infty\)
    - Approaching it along \(\gamma = 0, h > 2\): \(S_\infty = 0\)
    - Approaching it along \(\gamma = 0, h < 2\): \(S_\infty = \infty\)
    - Approaching it along \(h = 2\sqrt{1 - \gamma^2}\): \(S_\infty = \ln 2\)
Entropy around the ECP

$\alpha = 1$

(Von Neumann Entropy)

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Entropy around the ECP

\[ \alpha = 2 \]

\[ \alpha = 1 \]

\[ \alpha = 1/2 \]

(Von Neumann Entropy)

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Recalling the formulae

For $h > 2$ :

\[
S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln \left( \frac{k}{k'} \right) - \frac{1}{3} \frac{1}{\alpha - 1} \ln \left( \frac{\theta_2(0|q^\alpha) \theta_4(0|q^\alpha)}{\theta_3^2(0|q^\alpha)} \right) - \frac{1}{3} \ln 2
\]

\[
S(\rho_A) = \frac{1}{6} \left[ \ln \frac{4}{k, k'} | (k^2, k'^2) \frac{2I(k)I(k')}{\pi} \right]
\]

For $h < 2$ :

\[
S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln \left( \frac{k'}{k^2} \right) + \frac{1}{3} \frac{1}{\alpha - 1} \ln \left( \frac{\theta_2^2(0|q^\alpha)}{\theta_3(0|q^\alpha) \theta_4(0|q^\alpha)} \right) - \frac{1}{3} \ln 2
\]

\[
S(\rho_A) = \frac{1}{6} \left[ \ln \left( \frac{k^2}{16 k'} \right) \mid (2, k^2) \frac{2I(k)I(k')}{\pi} \right] \mid \ln 2
\]

The entropy depends just on one parameter ($k$)
Curves of constant Entropy

- Curves of constant Entropy are curves of constant $k$
- These curves are Hyperbolae and Ellipses:

\[
\begin{align*}
\text{Case 2 } (h > 2) & : \quad \left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1, \quad 0 \leq \kappa < \infty \\
\text{Case 1a } (2\sqrt{1-\gamma^2} < h < 2) & : \quad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \quad \kappa > 1 \\
\text{Case 1b } (h < 2\sqrt{1-\gamma^2}) & : \quad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \quad \kappa < 1.
\end{align*}
\]

- All these curves pass through the Essential Critical Point!
Importance of the Essential Critical Point

- From any point in the phase diagram one reaches the ECP following a curve of constant Entropy
- The range of the Entropy in the phase diagram is the positive real axis

Near the ECP the Entropy reaches every positive value!

- Small variations in the parameters change the Entropy dramatically!
- ECP important for Quantum Control
**Entropy on the critical phases**

**Phase transitions:** as the gap closes $S_\infty \to +\infty$

$$S_\infty \to -\frac{1}{6} \ln |2 - h| + \frac{1}{3} \ln 4 \gamma + O(|2 - h| \ln^2 |2 - h|)$$

$h \to 2$ and $\gamma \neq 0$

**Critical Magnetic Field:**
(Calabrese, Cardy, 2004)

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**Isotropic XY model (XX Model):**
(Jin, Korepin 2003)

$\gamma \to 0$ and $0 < h < 2$

$$S_\infty \to -\frac{1}{3} \ln \gamma + \frac{1}{6} \ln (4 - h^2) + \frac{1}{3} \ln 2 + O(\gamma \ln^2 \gamma)$$
**Critical Magnetic Field:**

\[ S_R(\alpha) = \frac{1 + \alpha}{\alpha} \left( -\frac{1}{12} \ln |2 - h| + \frac{1}{6} \ln 4\gamma \right) + O(|h - 2| \ln^2 |h - 2|) \]

\( h \to 2 \) and \( \gamma \neq 0 \)

**Isotropic XY model (XX Model):**

\[ S_R(\alpha) = \frac{1 + \alpha}{6\alpha} c \ln x + \ldots \]

\( \gamma \to 0 \) and \( 0 < h < 2 \)

\[ S_R(\alpha) = \frac{1 + \alpha}{\alpha} \left( -\frac{1}{6} \ln \gamma + \frac{1}{12} \ln (4 - h^2) + \frac{1}{6} \ln 2 \right) + O(\gamma \ln^2 \gamma) \]
• Diverges for $\alpha \to 0$

• Except at the factorizing field ($h = 2\sqrt{1 - \gamma^2}$):
  $S_R = \ln 2$

• Limit $\alpha \to \infty$ gives largest eigenvalue of density matrix (Single copy entanglement)
Conclusions

• We studied analytically the entropy (Von Neumann and Renyi) as a measure of bipartite entanglement in double scaling limit of the XY model.

• Entropy diverges for critical phases, approaches a constant in gapped phases.

• We achieved detail knowledge of the behavior of the entropy (also in $\alpha$).

• Near Essential Critical Point, entropy reaches every positive value.

• We can access the spectrum of the density matrix (we have the largest eigenvalue, we are working on the others).

• Entropy is sensitive to previously unnoticed modular properties of the model.

Thank you!
# Von Neumann Entropy of the XY model

<table>
<thead>
<tr>
<th>Region</th>
<th>$S(\rho_A)$</th>
<th>Curves of Constant $S$</th>
<th>Range of Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>2: $h &gt; 2$</td>
<td>$\frac{1}{6} \left[ \ln \frac{4}{k k'} + \frac{2(k^2 - k'^2)I(k)I(k')}{\pi} \right]$</td>
<td>$\left(\frac{h}{2}\right)^2 - \left(\frac{\kappa}{\kappa'}\right)^2 = 1$</td>
<td>$0 \leq k &lt; 1$ $0 \leq \kappa &lt; \infty$ $k = \sqrt{\frac{\kappa^2}{1+\kappa^2}}$</td>
</tr>
<tr>
<td>1b: $2\sqrt{1 - \gamma^2} &lt; h &lt; 2$</td>
<td>$\frac{1}{6} \left[ \ln \frac{k^2}{16k'} + \frac{2(2 - k^2)I(k)I(k')}{\pi} \right] + \ln 2$</td>
<td>$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$</td>
<td>$0 &lt; k &lt; 1$ $\kappa &gt; 1$ $k = \sqrt{\frac{\kappa^2}{k^2}}$</td>
</tr>
<tr>
<td>1a: $h &lt; 2\sqrt{1 - \gamma^2}$</td>
<td>$\frac{1}{6} \left[ \ln \frac{k'^2}{16k} + \frac{2(2 - k'^2)I(k)I(k')}{\pi} \right] + \ln 2$</td>
<td>$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$</td>
<td>$0 &lt; k &lt; 1$ $\kappa &lt; 1$ $k = \sqrt{1 - \kappa^2}$</td>
</tr>
<tr>
<td>$h = 2\sqrt{1 - \gamma^2}$</td>
<td>$\ln 2$</td>
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