Entanglement Entropy in the XY Model

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Outline

- Introduction: Von Neumann and Renyi Entropy as a measure of Entanglement
- Quantum Entropy of the XY model
- Ellipses of constant Entropy and the Essential Critical Point
- Modular properties of the entropy and of the partition function -Not enough time-
- Conclusions

Understanding Entanglement

- Consider a unique (pure) ground state
- Divide the system into two Subsystems: A & B
- If system wave-function is:

$$|\Psi^{A,B}\rangle = |\Psi^{A}\rangle \otimes |\Psi^{B}\rangle$$

$\rightarrow$ No Entanglement

(Measurements on B does not affect A state)

Understanding Entanglement (cont.)

- If the system wave-function is:

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \lambda_j |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

(with d > 1, $|\Psi_j^A\rangle$ & $|\Psi_j^B\rangle$ linearly independent):

→ Entangled (Measurements on B affect A state):

i.e. $\langle\Psi_i^B|\Psi^{A,B}\rangle = \lambda_j|\Psi_i^A\rangle$

How to measure Entanglement?

- Compute Density Matrix of subsystem:

$$\rho_A = tr_B \left(|\Psi^{A,B}\rangle \langle \Psi^{A,B}| \right)$$

- Entanglement for pure state as Quantum Entropy (Bennett, Bernstein, Popescu, Schumacher 1996):

$$S = -tr_A \left(\rho_A \ln \rho_A \right)$$

Von Neumann Entropy

More Entanglement Estimators

- Von Neumann Entropy: $S_A = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$

- Renyi Entropy: $S = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$

  (equal to Von Neumann for $\alpha \to 1$)

- Tsallis Entropy

- Concurrence (Two-Tangle)

- ...

Entropy of a subsystem

$$|\Psi^{A,B}\rangle = \sum \lambda_j |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}| = \sum \lambda_j^2 |\Psi^A\rangle \langle \Psi^A|$$

$$\rho_B = \mathrm{tr}_A |\Psi^{A,B}\rangle \langle \Psi^{A,B}| = \sum \lambda_j^2 |\Psi^B\rangle \langle \Psi^B|$$

$$S_A = -\mathrm{tr}\left(\rho_A \log \rho_A\right) = -\mathrm{tr}\left(\rho_B \log \rho_B\right) = S_B$$

- NB: $S_{AB} = 0 < S_A + S_B$ (Unlike thermodynamic entropy)

Entropy as a measure of entanglement

- Assume Bell State as unity of Entanglement:

$$|\mathrm{Bell}\rangle = \frac{|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\sqrt{2}}$$

- Von Neumann Entropy measures how many Bell-Pairs are contained in a given state $|\Psi^A\rangle$ (i.e. closeness of state to maximally entangled one)

Entanglement in a Spin Chain

- Consider the Ground state of a Hamiltonian:

$$H = \sum_{i=1}^{N} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - h \sum_i \sigma_i^z$$

- Block of spins in the space interval [1, n] is subsystem A

- The rest of the ground state is subsystem B.

→ Entanglement of a block of spins on a space interval [1, n] with the rest of the ground state <u>as a function of n</u>

Entropy as a Correlation Function

- We study the bi-partite entropy of the ground state of a system: $\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}|$

$$S(n) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$ Von Neumann

$$S_\alpha(n) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$ Renyi

- Multi-Point correlation function with contributions from all two-point correlators
- Highly non-trivial correlation function: new insights?

General Behavior

- We study the behavior for block size $n \to \infty$

(Double scaling limit: $0 << n << N$)

$$S(n) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

- For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)

$$S(n) \to \mathbf{Constant}$$

- For critical phases: (Calabrese, Cardy, 2004)

$$S(n) \to \frac{c}{3} \ln n + \ldots$$

The Anisotropic XY Model

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

- Jordan-Wigner followed by Bogoliubov transformation to diagonalize the Hamiltonian

$$H = \sum_q \varepsilon_q \left(\chi_q^\dagger \chi_q - 1/2 \right) \qquad \varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q}$$

- The XY Model is essentially Free Fermions
- Correlators for physical quantities involve inverting the transformation to FF: complications

The Phase Diagram of the XY Model

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

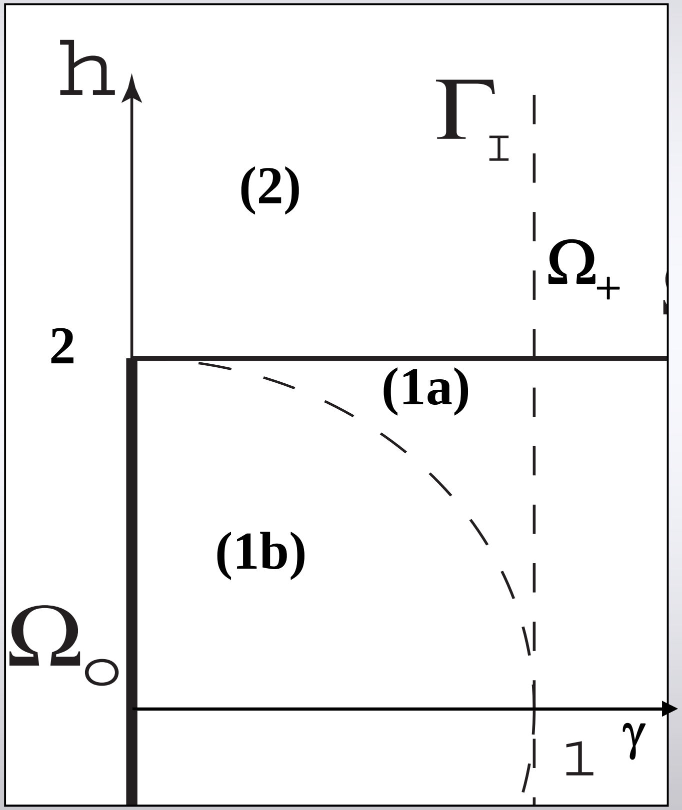
$$\varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q}$$

Phase Diagram:

- 3 non-critical regions (**2,1a,1b**)
- 2 critical phases:

  Ω_0: Isotropic XY

  Ω_+: Critical magnetic field



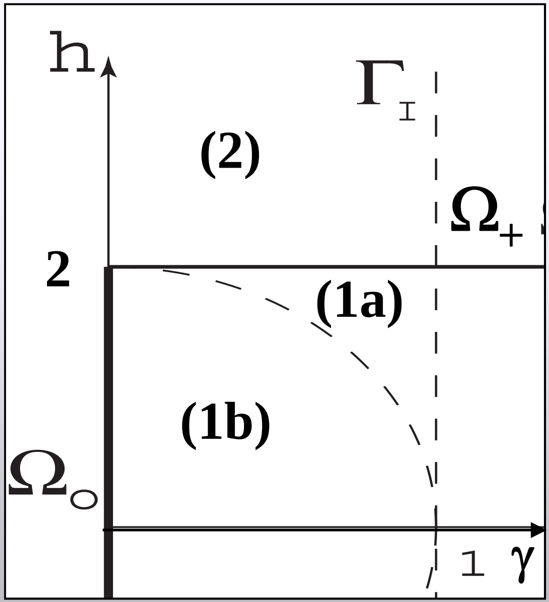
Phase Diagram of the XY Model
(only γ>0 shown)

Entropy on the gapped phases for |GS>

1. Case 1a: $2\sqrt{1-\gamma^2} < h < 2$. medium magnetic field
2. Case 1b: $0 \leq h < 2\sqrt{1-\gamma^2}$, small magnetic field
3. Case 2: $h > 2$, strong magnetic field

- We define an Elliptic Parameter:

$$k = \begin{cases} \gamma \,/\, \sqrt{(h/2)^2 + \gamma^2 - 1}, & \text{Case 2} \\ \sqrt{(h/2)^2 + \gamma^2 - 1} \,/\, \gamma, & \text{Case 1a} \\ \sqrt{1 - \gamma^2 - (h/2)^2} \,/\, \sqrt{1 - (h/2)^2}, & \text{Case 1b} \end{cases}$$



Asymptotic Entropy

$$S = -\mathrm{tr}\,(\rho_A \log \rho_A) \qquad S_R = \frac{1}{1-\alpha} \ln \mathrm{tr}\,(\rho_A^\alpha)$$

- For **h > 2** :

$$S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln (k\ k') - \frac{1}{3} \frac{1}{\alpha-1} \ln \left(\frac{\theta_2(0|q^\alpha)\ \theta_4(0|q^\alpha)}{\theta_3^2(0|q^\alpha)} \right) - \frac{1}{3} \ln 2$$

$$S(\rho_A) = \frac{1}{6} \left[\ln \frac{4}{k\ k'} + (k^2 - k'^2) \frac{2 I(k) I(k')}{\pi} \right]$$

$$q \equiv \mathrm{e}^{-\pi I(k')/I(k)}$$

$$k' = \sqrt{1-k^2}$$

- For **h < 2** :

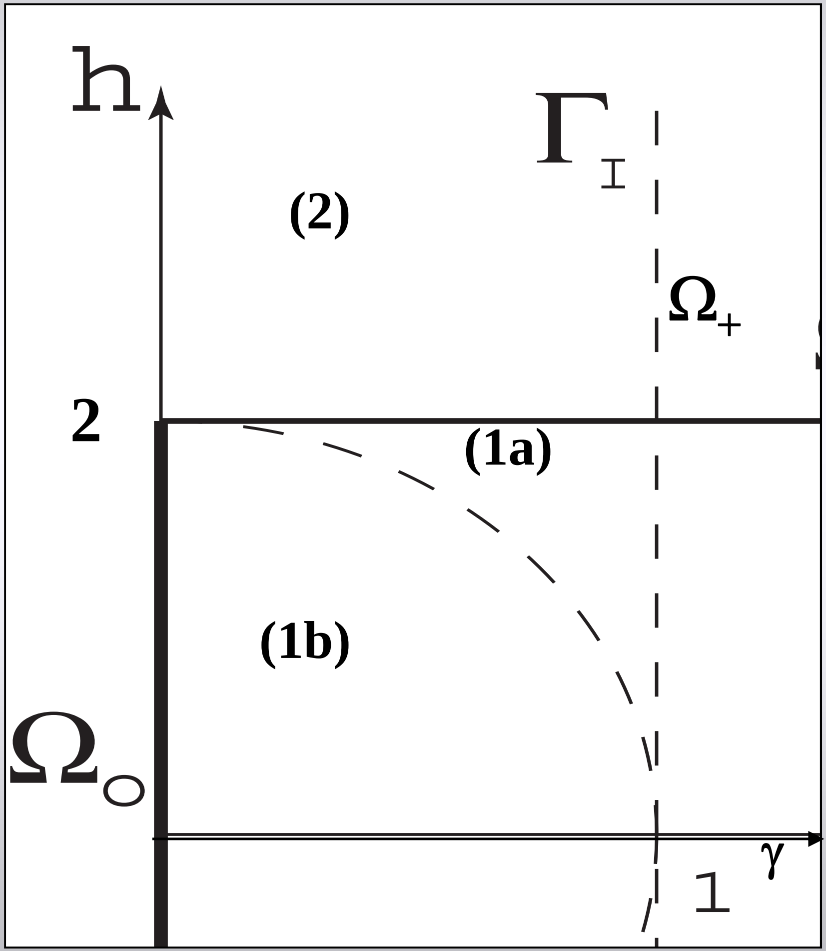
$$S_R = \frac{1}{6} \frac{\alpha}{\alpha - 1} \ln \left(\frac{k'}{k^2} \right) + \frac{1}{3} \frac{1}{\alpha-1} \ln \left(\frac{\theta_2^2(0|q^\alpha)}{\theta_3(0|q^\alpha)\ \theta_4(0|q^\alpha)} \right) - \frac{1}{3} \ln 2$$

$$S(\rho_A) = \frac{1}{6} \left[\ln \left(\frac{k^2}{16 k'} \right) + (2 - k^2) \frac{2 I(k) I(k')}{\pi} \right] + \ln\ 2$$

Asymptotic Entropy of the XY model

- We have a completely analytical expression for the asymptotic entropy

- Let's extract some physics out of it!!



Minima of the Entropy

- Absolute minimum at $\mathbf{h} \to \infty$ or $\gamma \to 0$ ($\mathbf{h} > 2$) : $\mathbf{S}_\infty \to \mathbf{0}$

  as the ground state becomes ferromagnetic ($\uparrow \ldots \uparrow$)

- Local minimum $\mathbf{S}_\infty = \ln 2$

  at the boundary between cases **1a** and **1b** ($h = 2\sqrt{1-\gamma^2}$)

- The ground states is factorized:

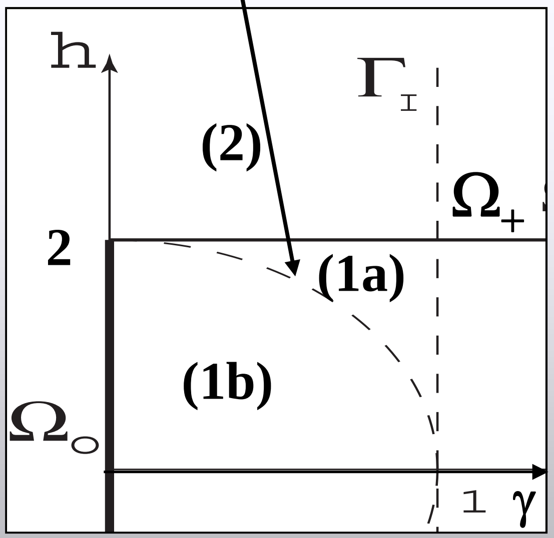
  (each state is factorized and has **no entropy**)

$$|GS_1\rangle = \prod_{n \in \text{lattice}} [\cos(\theta)|\uparrow_n\rangle + \sin(\theta)|\downarrow_n\rangle]$$

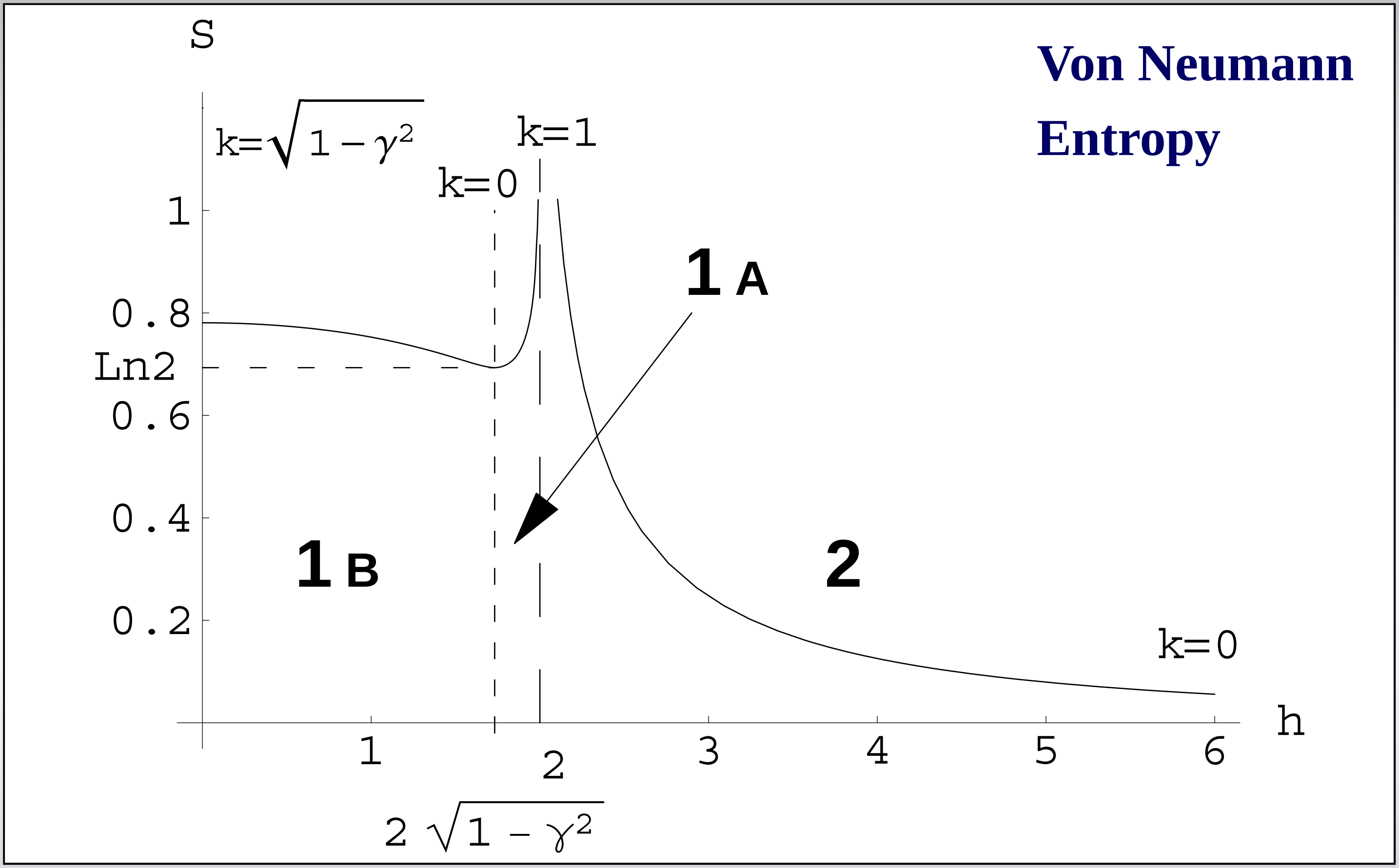
$$|GS_2\rangle = \prod_{n \in \text{lattice}} [\cos(\theta)|\uparrow_n\rangle - \sin(\theta)|\downarrow_n\rangle]$$

$$|GS\rangle_+ = |GS_1\rangle + |GS_2\rangle$$

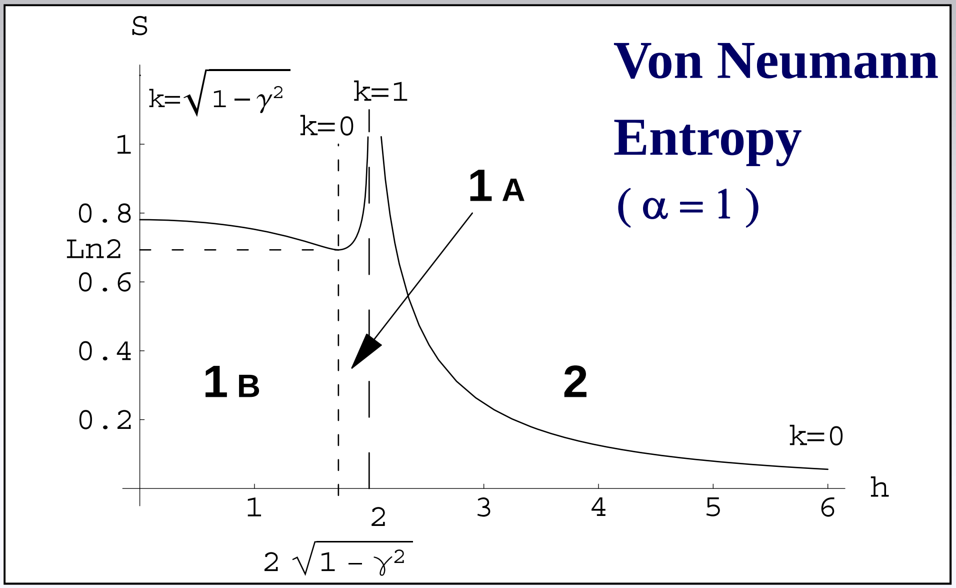
$$\cos^2(2\theta) = (1-\gamma)/(1+\gamma)$$

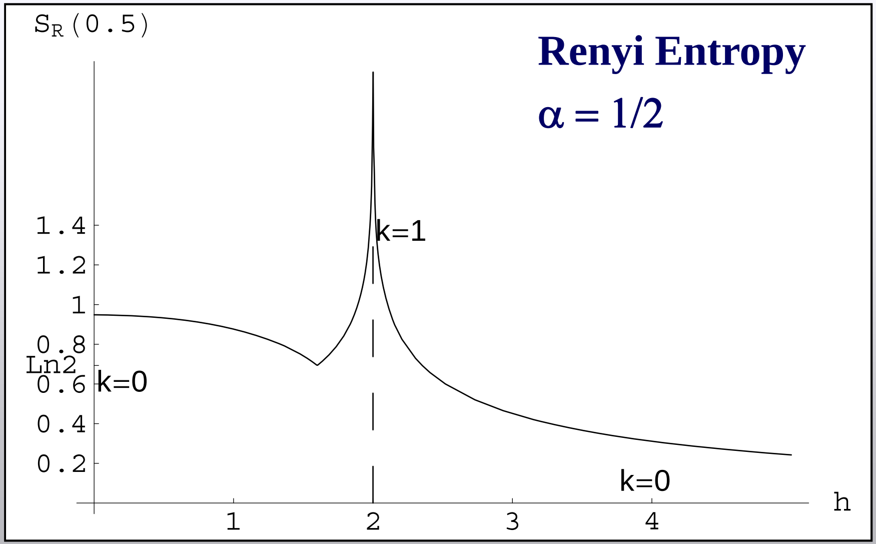


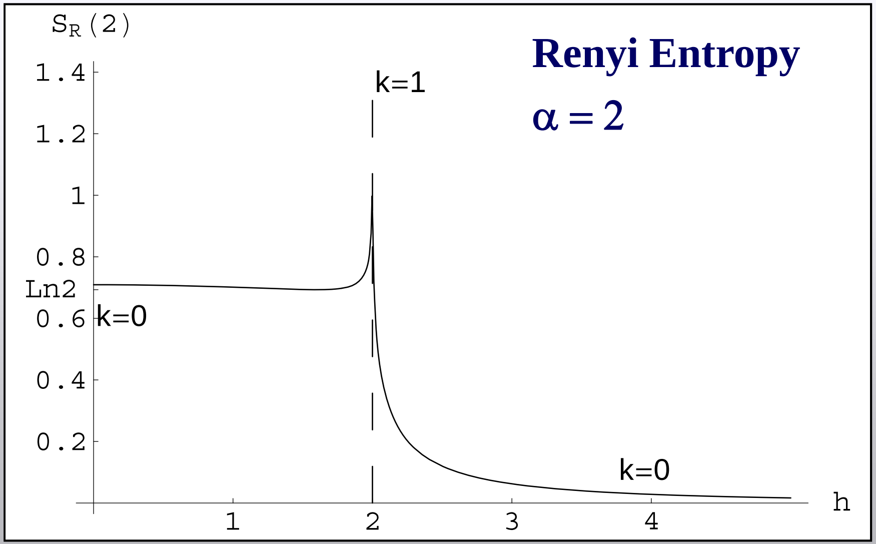
Entropy at fixed γ



Entropy at fixed γ

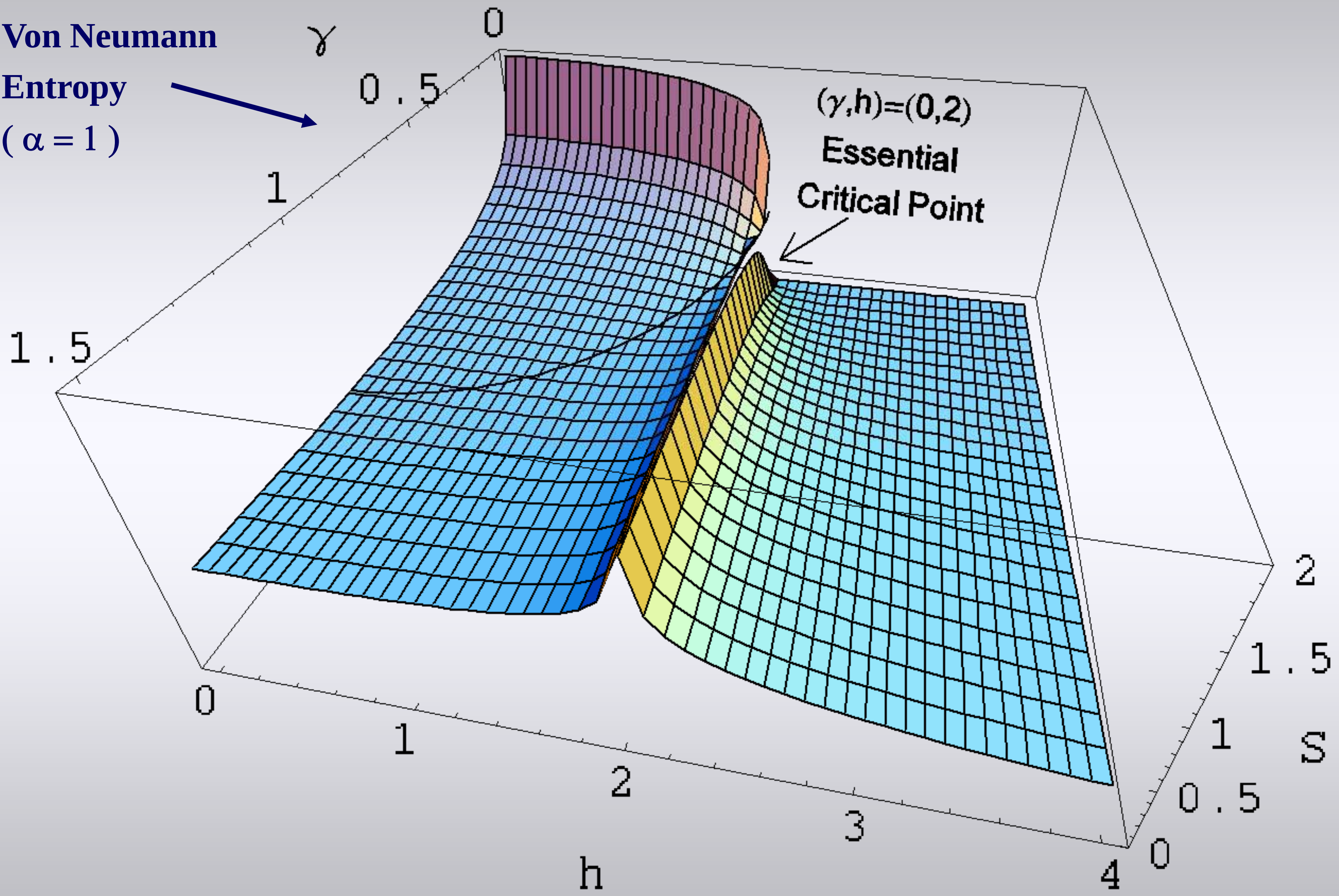






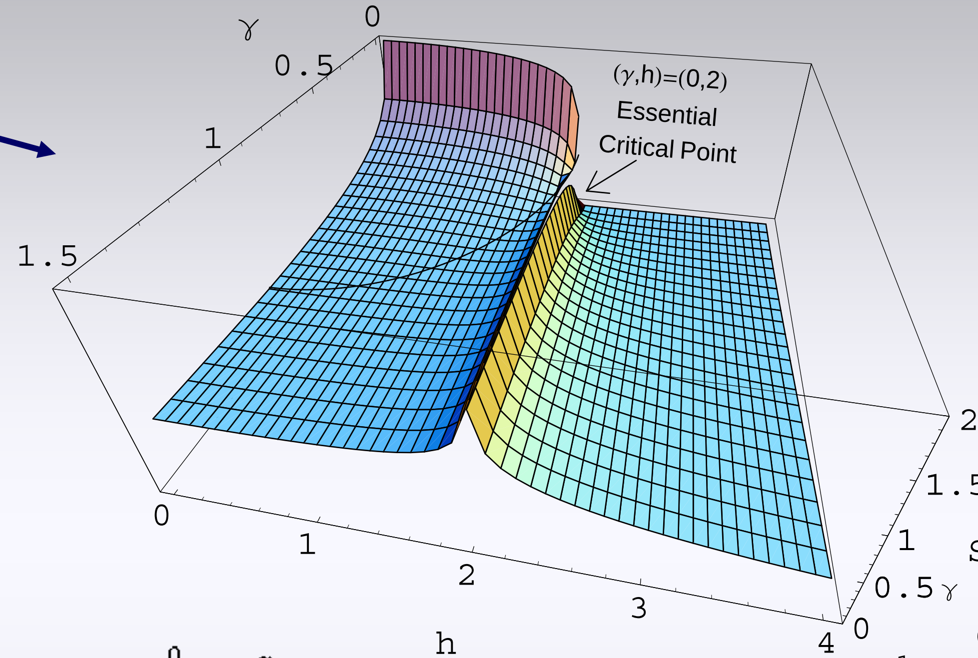
3-D plot of the Entropy

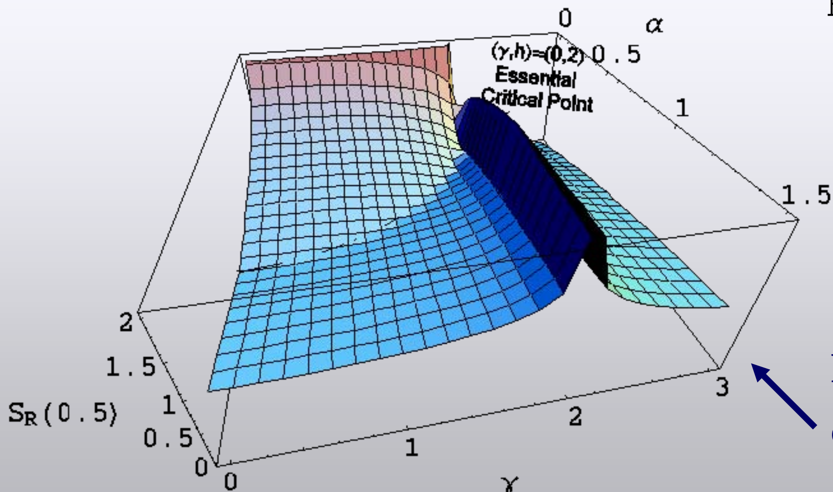
Von Neumann Entropy ($\alpha = 1$)



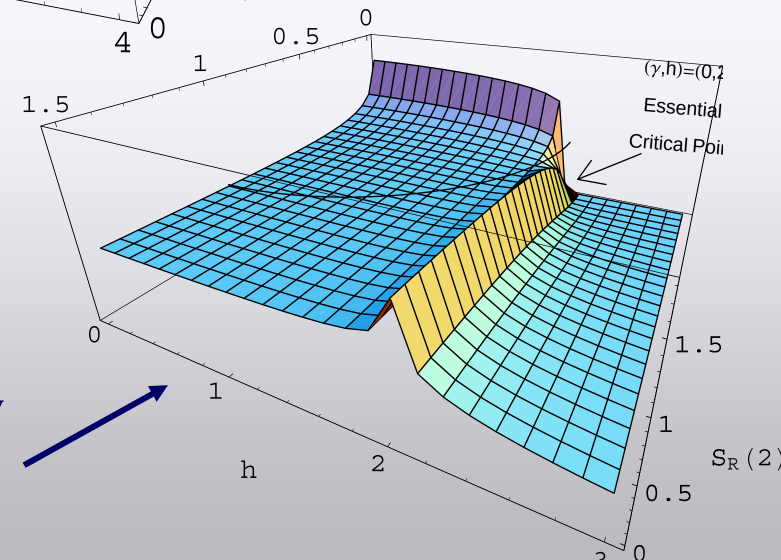
3-D plot of the Entropy

Von Neumann Entropy
(α = 1)



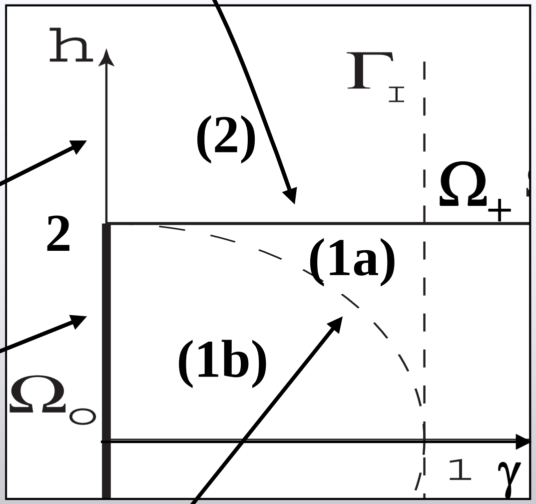


Renyi Entropy
α = 1/2   α=2



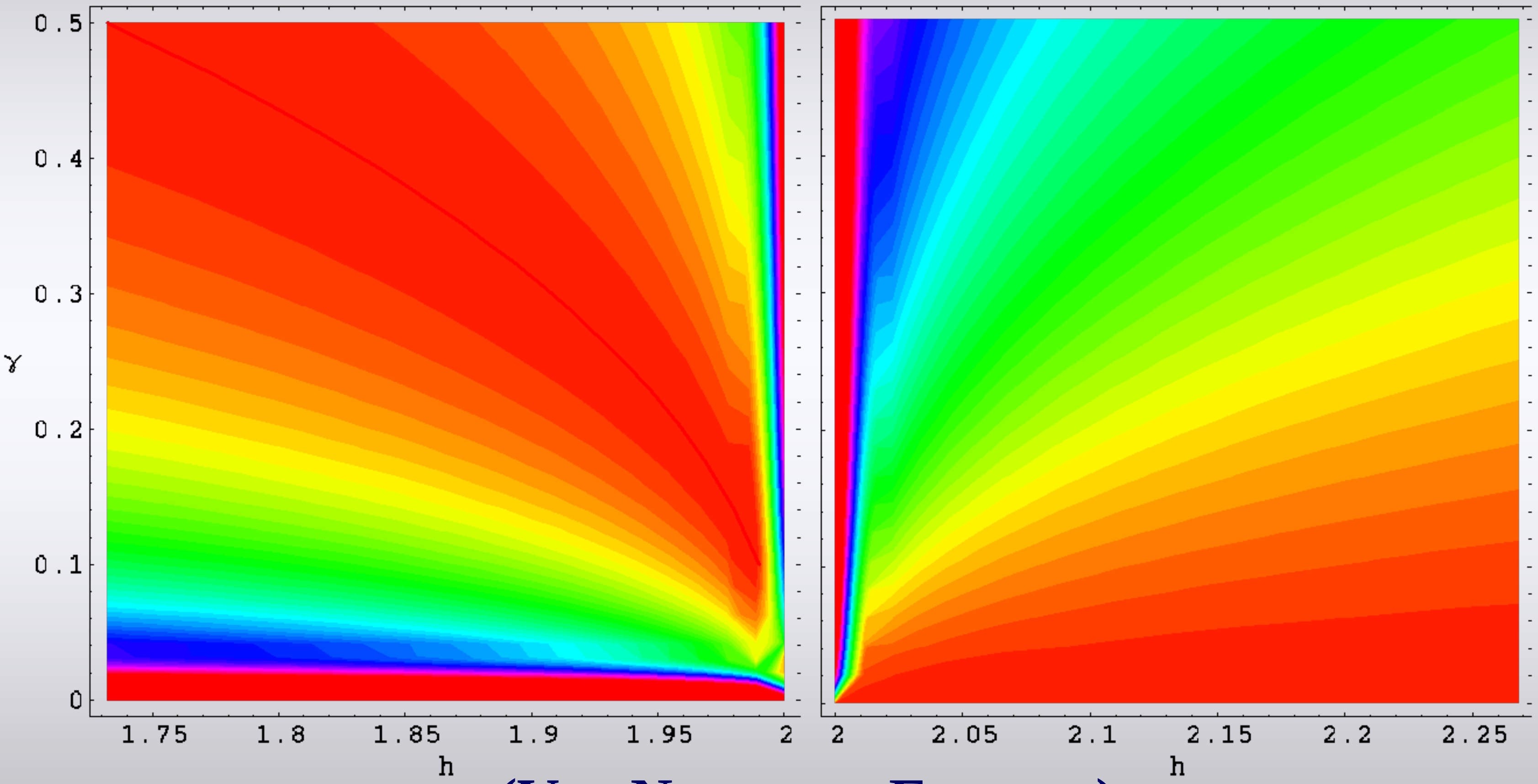
The Essential Critical Point (ECP)

- Point $(\gamma, h) = (0,2)$ is special:
- Theory is critical, but not CFT (quadratic spectrum)
- We can study the entropy close to this point:
  - –Approaching it along $h=2, \gamma>0$: $S_\infty = \infty$
  - –Approaching it along $\gamma=0, h>2$: $S_\infty = 0$
  - –Approaching it along $\gamma=0, h<2$: $S_\infty = \infty$
  - –Approaching it along $h = 2\sqrt{1-\gamma^2}$: $S_\infty = \ln 2$



Entropy around the ECP

$\alpha = 1$



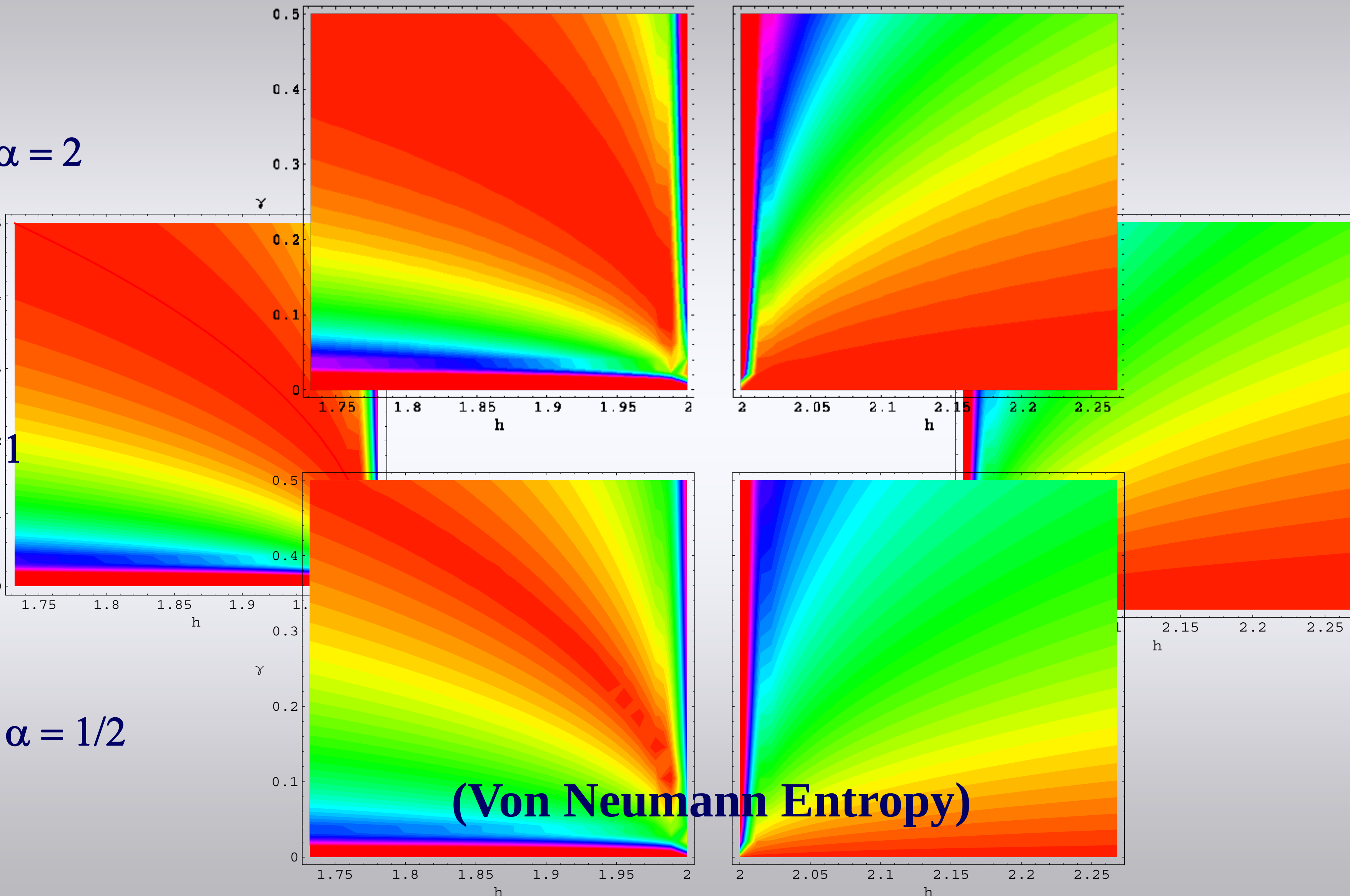
(Von Neumann Entropy)

Entropy around the ECP

$\alpha = 2$

$\alpha = 1$

$\alpha = 1/2$



(Von Neumann Entropy)

Recalling the formulae

- For **h > 2** :

$$S_R = \frac{1}{6}\,\frac{\alpha}{\alpha-1}\;\ln(k\;k') - \frac{1}{3}\,\frac{1}{\alpha-1}\;\ln\left(\frac{\theta_2(0|q^\alpha)\;\theta_4(0|q^\alpha)}{\theta_3^2(0|q^\alpha)}\right) - \frac{1}{3}\ln 2$$

$$S(\rho_A) = \frac{1}{6}\left[\ln\frac{4}{k\;k'} + (k^2 - k'^2)\frac{2I(k)I(k')}{\pi}\right]$$

$$q \equiv \mathrm{e}^{-\pi I(k')/I(k)}$$

$$\boldsymbol{k' = \sqrt{1-k^2}}$$

- For **h < 2** :

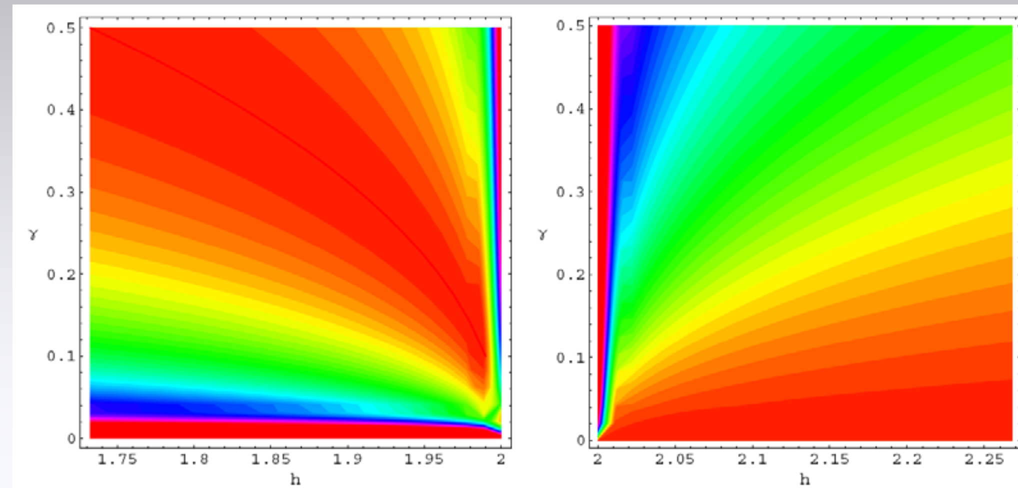
$$S_R = \frac{1}{6}\,\frac{\alpha}{\alpha-1}\;\ln\left(\frac{k'}{k^2}\right) + \frac{1}{3}\,\frac{1}{\alpha-1}\;\ln\left(\frac{\theta_2^2(0|q^\alpha)}{\theta_3(0|q^\alpha)\;\theta_4(0|q^\alpha)}\right) - \frac{1}{3}\ln 2$$

$$S(\rho_A) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + (2-k^2)\,\frac{2I(k)I(k')}{\pi}\right] + \ln\;2$$

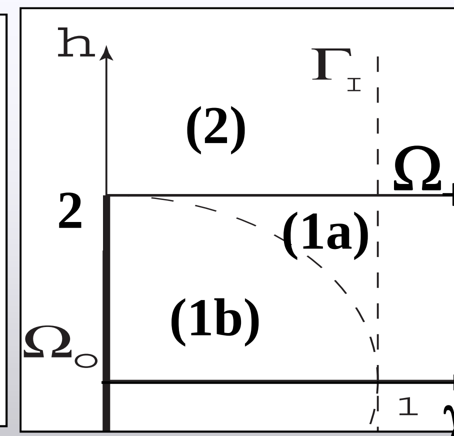
The entropy depends just on one parameter (k)

Curves of constant Entropy

- Curves of constant Entropy are curves of constant ***k***
- These curves are Hyperbolae and Ellipses:



Case 2 $(h > 2)$:	$\left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1,$	$0 \leq \kappa < \infty$
Case 1a $(2\sqrt{1-\gamma^2} < h < 2)$:	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1,$	$\kappa > 1$
Case 1b $(h < 2\sqrt{1-\gamma^2})$:	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1,$	$\kappa < 1.$

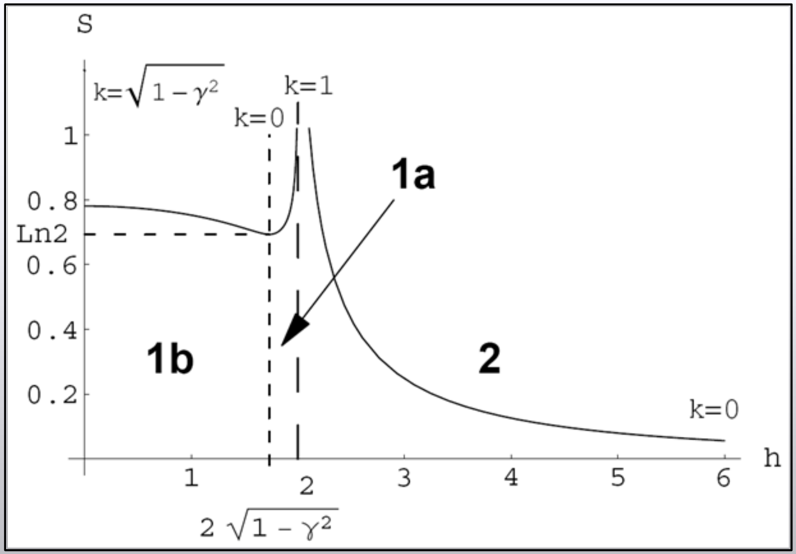


- All these curves <u>pass through</u> the Essential Critical Point!

Importance of the Essential Critical Point

- From any point in the phase diagram one reaches the ECP following a curve of constant Entropy
- The range of the Entropy in the phase diagram is the positive real axis

$\Rightarrow$ Near the ECP the Entropy reaches every positive value!



- Small variations in the parameters change the Entropy dramatically!
- ECP important for Quantum Control

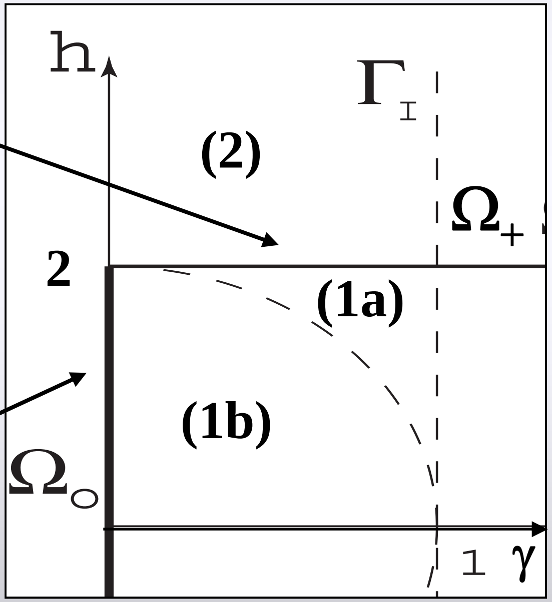
Entropy on the critical phases

Phase transitions: as the gap closes $S_\infty \longrightarrow +\infty$

$$S_\infty \to -\frac{1}{6}\ln|2-h| + \frac{1}{3}\ln 4\gamma + O(|2-h|\ln^2|2-h|)$$

$h \longrightarrow 2$ and $\gamma \neq 0$

Critical Magnetic Field:
(Calabrese, Cardy, 2004)



Isotropic XY model (XX Model):
(Jin, Korepin 2003)

$\gamma \longrightarrow 0$ and $0 < h < 2$

$$S_\infty \to -\frac{1}{3}\ln\gamma + \frac{1}{6}\ln(4-h^2) + \frac{1}{3}\ln 2 + O(\gamma\ln^2\gamma)$$

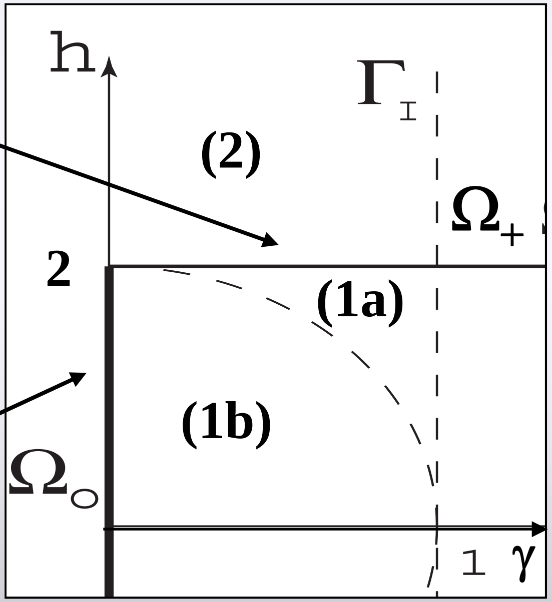
Entropy on the critical phases

Phase transitions: as the gap closes $S_\infty \to +\infty$

$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\Big(-\frac{1}{12}\ln|2-h| + \frac{1}{6}\ln 4\gamma\Big) + O(|h-2|\ln^2|h-2|)$$

$h \to 2$ and $\gamma \neq 0$

Critical Magnetic Field:



Conjecture:

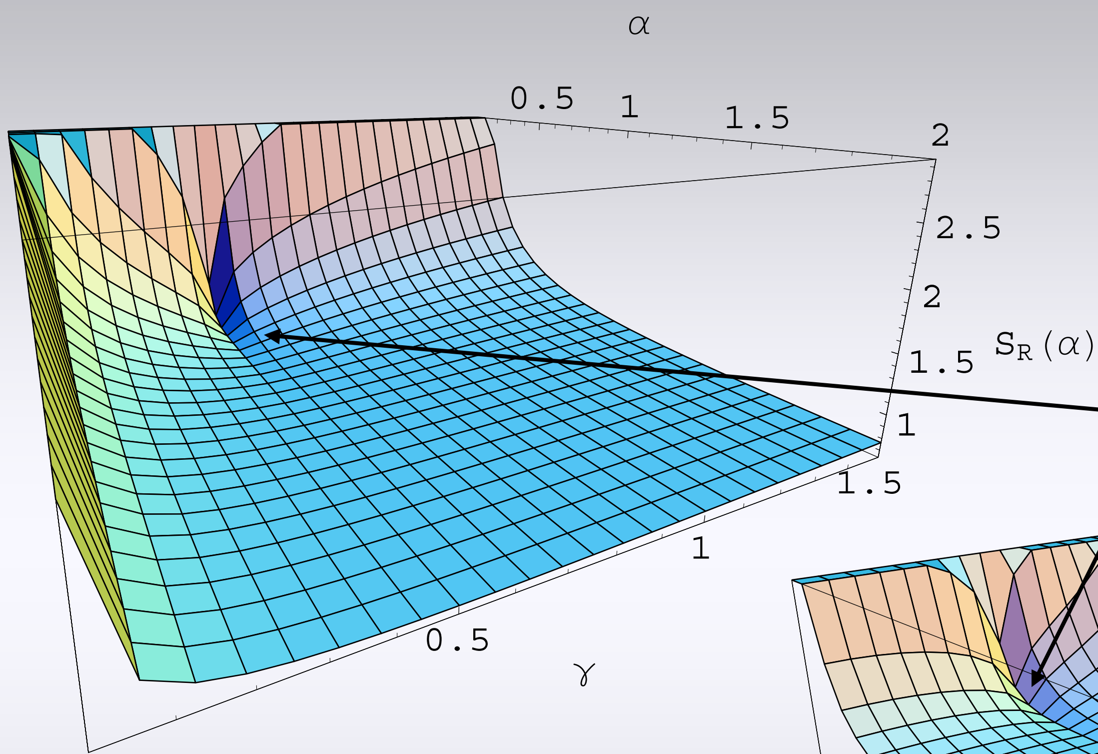
$$S_R(\alpha) = \frac{1+\alpha}{6\alpha}\, c\, \ln x + \dots$$

Isotropic XY model (XX Model):

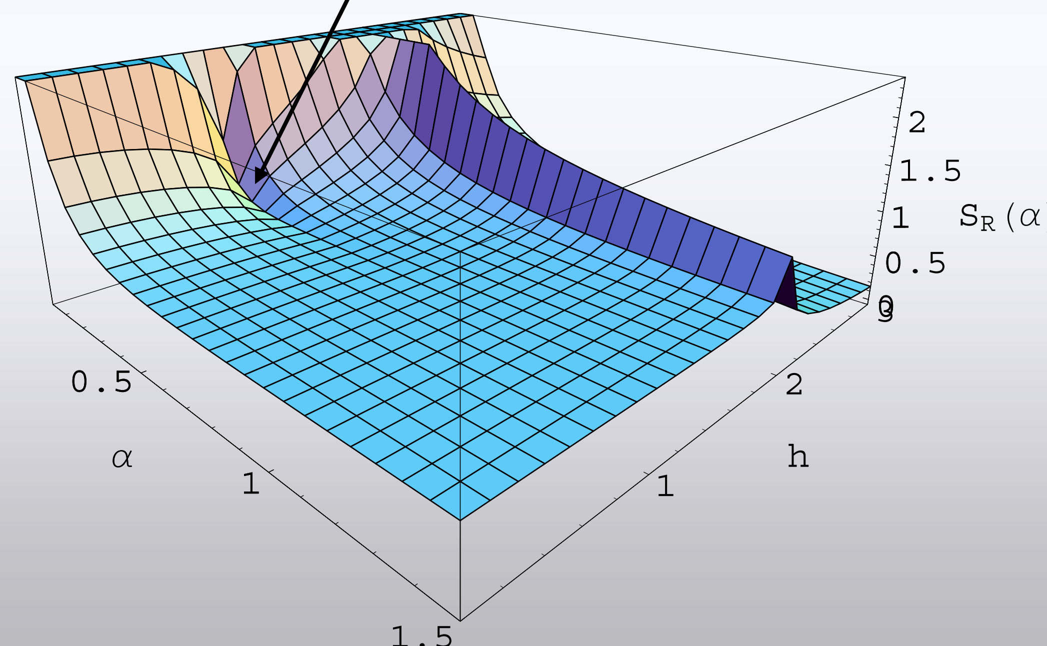
$\gamma \to 0$ and $0 < h < 2$

$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\Big(-\frac{1}{6}\ln\gamma + \frac{1}{12}\ln(4-h^2) + \frac{1}{6}\ln 2\Big) + O(\gamma\ln^2\gamma)$$

Entropy as a function of α



- Diverges for $\alpha \to 0$
- Except at the factorizing field ($h = 2\sqrt{1-\gamma^2}$): $S_R = \ln 2$



- Limit $\alpha \to \infty$ gives largest eigenvalue of density matrix (Single copy entanglement)

Conclusions

- We studied analytically the entropy (**Von Neumann** and **Renyi**) as a measure of bipartite entanglement in double scaling limit of the XY model
- Entropy diverges for critical phases, approaches a constant in gapped phases
- We achieved detail knowledge of the behavior of the entropy (also in α)
- Near Essential Critical Point, entropy reaches every positive value
- We can access the spectrum of the density matrix (we have the largest eigenvalue, we are working on the others)
- Entropy is sensitive to previously unnoticed modular properties of the model

Thank you!

Von Neumann Entropy of the XY model

Region	$S(\rho_A)$	Curves of Constant S	Range of Parameters
$\mathbf{2}:\quad h > 2$	$\frac{1}{6}\left[\ln\frac{4}{k\,k'} + \frac{2(k^2-k'^2)I(k)I(k')}{\pi}\right]$	$\left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 \leq k < 1$ $0 \leq \kappa < \infty$ $k = \sqrt{\frac{\kappa^2}{1+\kappa^2}}$
$\mathbf{1b}:\ 2\sqrt{1-\gamma^2} < h < 2$	$\frac{1}{6}\left[\ln\frac{k^2}{16k'} + \frac{2(2-k^2)I(k)I(k')}{\pi}\right] + \ln\ 2$	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 < k < 1$ $\kappa > 1$ $k = \sqrt{\frac{\kappa^2-1}{\kappa^2}}$
$\mathbf{1a}:\ h < 2\sqrt{1-\gamma^2}$	$\frac{1}{6}\left[\ln\frac{k^2}{16k'} + \frac{2(2-k^2)I(k)I(k')}{\pi}\right] + \ln\ 2$	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 < k < 1$ $\kappa < 1$ $k = \sqrt{1-\kappa^2}$
$h = 2\sqrt{1-\gamma^2}$	$\ln\ 2$	$\left(\frac{h}{2}\right)^2 + \gamma^2 = 1$	$k = 0$ $\kappa = 1$