Nonlinear dynamics of spin and charge in the spin-Calogero model

by

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Outline

- Motivation
- Hydrodynamics from Bethe Ansatz
  (Free Fermions example)
- Spin Calogero-Sutherland model
  (and its gradient-less hydrodynamics)
- Connection to the Haldane-Shastry model
- Spin-Charge dynamics
- Emptiness Formation Probability
- Conclusions

Universality in 1-D systems

- In 1-D: no Fermi Liquid, but Luttinger Liquid
- Low-Energy approximation
- Linear dispersion relation (Lorentz invariance)
- Excitations are sound-waves
- Linear spectrum implies spin-charge separation: particles decouple into spinon and holon
- Curvature couples spin and charge dynamics

Non-linear effects

- Realization of 1-D systems

  $\rightarrow$ experimentally relevant

  (Quantum quenches; Non-equilibrium dynamics …)

- Several theoretical approaches toward non-linear effects (Universality?)

- So far, not much effort toward spin-charge dynamics

Hydrodynamic description

- We look at an integrable model:

  spin-Calogero Model (sCM)

- Collective field theory description: $\rho_{c,s}(x,t)$, $v_{c,s}(x,t)$

  $\Rightarrow$ Hydrodynamics approach

- Spin-less case well understood; spin-ful not so much

- Our is a heuristic construction based on Bethe Ansatz Solution ($\rightarrow$ valid for small gradients!)

The Bethe-Ansatz solution

System of N particles

↓ To specify the state

N integer quantum numbers: κ_α

↓ Using the Bethe Equations

$$k_\alpha L = 2\pi\kappa_\alpha - \sum_\beta^N \theta(k_\alpha - k_\beta)$$

N Quasi-Momenta: k_α

↓ Thermodynamic Limit: $N,L \to \infty$

$$2\pi\tau(k) = 1 + \int_{k_L}^{k_R} K(k-k')\tau(k')\mathrm{d}k'$$

Distribution of quasi-Momenta: $\tau(k)$

Hydrodynamics construction

	Integrable system	Free Fermions
	$\tau(k)$	$\tau(k) = \frac{1}{2\pi}$
Particle density:	$\rho = \int_{k_L}^{k_R} \tau(k) \mathrm{d}k$	$\rho = \int_{k_L}^{k_R} \frac{\mathrm{d}k}{2\pi} = \frac{k_R - k_L}{2\pi}$
Momentum density:	$j = \int_{k_L}^{k_R} k\tau(k) \mathrm{d}k$	$j = \int_{k_L}^{k_R} \frac{\mathrm{d}k}{2\pi} k = \frac{k_R^2 - k_L^2}{4\pi}$
Energy density:	$\frac{E}{L} = \int_{k_L}^{k_R} \frac{k^2}{2}\tau(k) \mathrm{d}k$	$\frac{E}{L} = \int_{k_L}^{k_R} \frac{\mathrm{d}k}{2\pi} \frac{k^2}{2} = \frac{k_R^3 - k_L^3}{12\pi}$

Hydrodynamics construction

Free Fermions

$$\tau(k) = \frac{1}{2\pi}$$

Particle density:

$$\rho = \int_{k_L}^{k_R} \frac{dk}{2\pi} = \frac{k_R - k_L}{2\pi}$$

$$v = \frac{k_R + k_L}{2}$$

Momentum density:

$$j = \int_{k_L}^{k_R} \frac{dk}{2\pi}\, k = \frac{k_R^2 - k_L^2}{4\pi} = \rho v$$

Energy density:

$$\frac{E}{L} = \int_{k_L}^{k_R} \frac{dk}{2\pi} \frac{k^2}{2} = \frac{k_R^3 - k_L^3}{12\pi} = \frac{\rho v^2}{2} + \frac{\pi^2}{6}\rho^3$$

Hydrodynamics construction

Free Fermions

$$\tau(k) = \frac{1}{2\pi}$$

We let the parameters have a slow space-dependence!

Particle density:

$$\rho(x) = \int_{k_L(x)}^{k_R(x)} \frac{\mathrm{d}k}{2\pi} = \frac{k_R(x) - k_L(x)}{2\pi}$$

Momentum density:

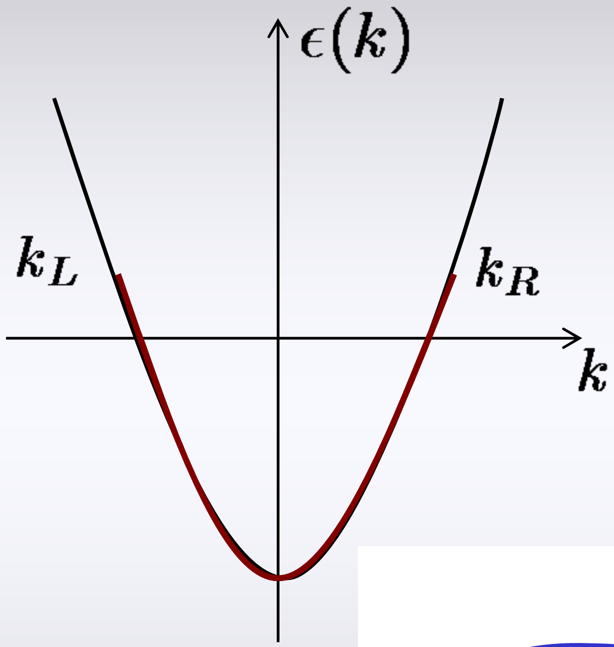
$$j(x) = \int_{k_L(x)}^{k_R(x)} \frac{\mathrm{d}k}{2\pi}\, k = \frac{k_R^2(x) - k_L^2(x)}{4\pi} = \rho(x) v(x)$$

Energy density:

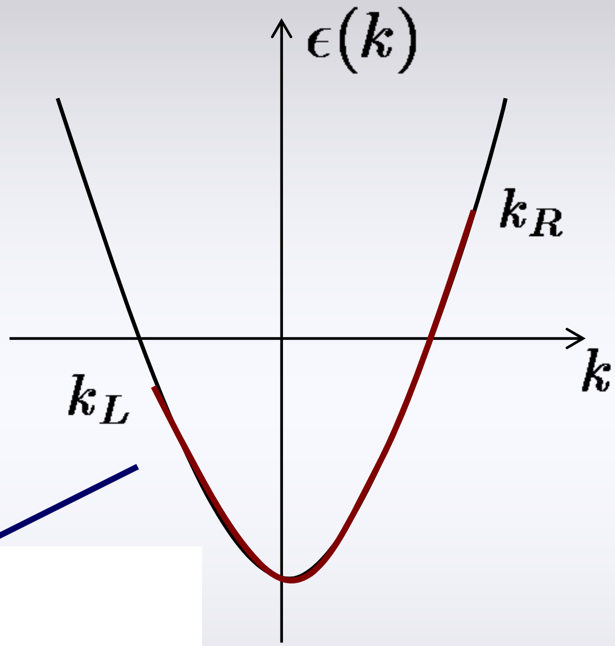
$$\frac{E}{L} = \int_{k_L(x)}^{k_R(x)} \frac{\mathrm{d}k}{2\pi} \frac{k^2}{2} = \frac{\rho(x)\, v^2(x)}{2} + \frac{\pi^2}{6}\rho^3(x) = \mathcal{H}(x)$$

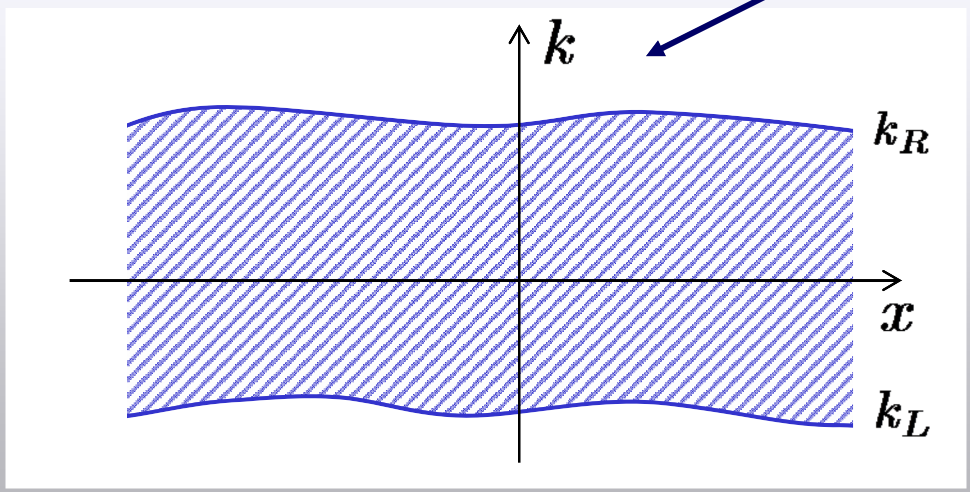
Recapping…

Equilibrium
Free Fermions



Non-Equilibrium
Free Fermions





Space-
Dependent
Configuration

Free Fermions Hydrodynamics

$$H = \int \mathrm{d}x \, \mathcal{H}(x) = \int \mathrm{d}x \left[\frac{\rho(x) \, v^2(x)}{2} + \frac{\hbar^2 \pi^2}{6} \rho^3(x) \right]$$

Dynamics from commutation relations:
(from microscopical analysis)

$$[\rho(x), v(y)] = -\mathrm{i}\hbar \delta'(x - y)$$

Continuity Equation:

$$\dot{\rho} = [H, \rho] = -\partial_x \, (\rho \, v)$$

Euler Equation:

$$\dot{v} = [H, v] = -\partial_x \left(\frac{v^2}{2} + \frac{\hbar^2 \pi^2}{2} \rho^2 \right)$$

Free Fermions Hydrodynamics

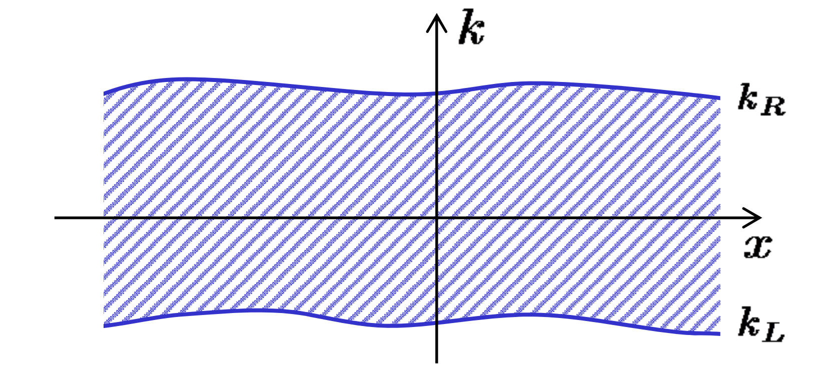
$$H = \int \mathrm{d}x \left[\frac{\rho(x)\, v^2(x)}{2} + \frac{\hbar^2 \pi^2}{6} \rho^3(x) \right] = \int \mathrm{d}x\, \hbar^2 \frac{k_R^3(x) - k_L^3(x)}{12\pi}$$

Note that: $[\rho(x), v(y)] = -i\hbar \delta'(x-y)$

↓

$$[k_L(x), k_L(y)] = -[k_R(x), k_R(y)] = 2\pi i \delta'(x-y)$$

Riemann-Hopf Equation: $\dot{k}_{R,L} + \hbar\, k_{R,L}\, \partial_x k_{R,L} = 0$



Left and Right Fermi points evolve independently!

Free Fermions Hydrodynamics

Two remarks:

1. In principle, gradient corrections from interaction.

   For Free Fermions, this is exact!

$$H = \int \mathrm{d}x \left[\frac{\rho\, v^2}{2} + \frac{\hbar^2 \pi^2}{6} \rho^3 \right] = \frac{\hbar^2}{12\pi} \int \mathrm{d}x \left[k_R^3 - k_L^3 \right]$$

2. Same result can be derived by conventional bosonization without linearization:

$$\Psi_{L,R}(x) =: \mathrm{e}^{\mathrm{i}\sqrt{4\pi}\phi_{L,R}(x)}$$

$$H = \frac{\hbar^2}{12\pi} \int \mathrm{d}x \; \left[(\partial_x \phi_R)^3 - (\partial_x \phi_L)^3 \right]$$

Free Fermions with Spin

- Just add the theory for each species:

$$H = \int \mathrm{d}x \left\{ \frac{1}{2}\rho_\uparrow v_\uparrow^2 + \frac{1}{2}\rho_\downarrow v_\downarrow^2 + \frac{\pi^2\hbar^2}{6}\left(\rho_\uparrow^3 + \rho_\downarrow^3\right) \right\}$$

- Expanding around ($\rho_0 = k_F/\pi$, $v_0 = 0$):

$$\begin{aligned} H &\approx \frac{\rho_0}{2} \int \mathrm{d}x \left(v_\uparrow^2 + \pi^2\hbar^2\delta\rho_\uparrow^2 + v_\downarrow^2 + \pi^2\hbar^2\delta\rho_\downarrow^2 \right) \\ &\approx \frac{\rho_0}{4}\hbar^2 \sum_{\alpha=\uparrow,\downarrow} \int \mathrm{d}x \left[(\partial_x\phi_{R,\alpha})^2 + (\partial_x\phi_{L,\alpha})^2 \right] \end{aligned}$$

$\Rightarrow$ traditional bosonization!

Bosonization of spinful Free Fermions

$$H \approx \frac{\rho_0}{4}\hbar^2 \sum_{\alpha=c,s} \int \mathrm{d}x \left[(\partial_x \phi_{R,\alpha})^2 + (\partial_x \phi_{L,\alpha})^2 \right]$$

- No true spin-charge separation (all excitations have same velocity)
- Peculiarity of FF (and of Calogero-Sutherland systems)
- Rieman-Hopf equation $\dot{k}_{\alpha,\chi} + \hbar k_{\alpha,\chi}\, \partial_x k_{\alpha,\chi} = 0$

  $k_{\alpha,\chi} \to \pi\rho_0 + k_{\alpha,\chi}$

  becomes wave equation $\dot{k}_{\alpha,\chi} + \hbar\pi\rho_0\, \partial_x k_{\alpha,\chi} = 0$

  (α=c,s; χ = R,L)

Semi-Classical Limit

$$H = \int \mathrm{d}x \left\{ \frac{1}{2}\rho_\uparrow v_\uparrow^2 + \frac{1}{2}\rho_\downarrow v_\downarrow^2 + \frac{\pi^2 \hbar^2}{6}\left(\rho_\uparrow^3 + \rho_\downarrow^3\right) \right\}$$

- In the classical limit, only velocity terms survive
- Semiclassical limit: $\rho \sim v/\hbar$

$\Rightarrow \; t \to t/\hbar$ and $v \to \hbar v$

$$H = \int \mathrm{d}x \left\{ \frac{1}{2}\rho_\uparrow v_\uparrow^2 + \frac{1}{2}\rho_\downarrow v_\downarrow^2 + \frac{\pi^2}{6}\left(\rho_\uparrow^3 + \rho_\downarrow^3\right) \right\}$$

- Commutation relations $\to$ Poisson Brackets

$$\{\rho_\alpha(x), v_\beta(y)\} = \delta_{\alpha\beta}\delta'(x-y)$$

Spin Calogero-Sutherland model

$$H \equiv -\frac{\hbar^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{2}\sum_{j\neq l}\frac{\lambda(\lambda \pm \mathbf{P}_{jl})}{(x_j - x_l)^2}$$

- P_{jl} particle-exchange operator
- SU(2) version of the traditional CS model ($P_{jl} = \pm 1$ for a ferromagnetic state)
- $\lambda \to \infty$: Haldane-Shastry spin chain

$$\text{Bosons} \longrightarrow \begin{cases} + & \Rightarrow \quad \text{Anti-ferromagnetic}\,, \\ - & \Rightarrow \quad \text{Ferromagnetic}\,, \end{cases}$$

$$\text{Fermions} \longrightarrow \begin{cases} + & \Rightarrow \quad \text{Ferromagnetic}\,, \\ - & \Rightarrow \quad \text{Anti-ferromagnetic}\,. \end{cases}$$

Anti-Ferromagnetic Fermions

$$H = -\frac{\hbar^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{2}\left(\frac{\pi}{L}\right)^2 \sum_{j\neq l}\frac{\lambda(\lambda - \mathbf{P}_{jl})}{\sin^2\frac{\pi}{L}(x_j - x_l)}$$

Periodic boundary conditions

- AF Ground state
- We chose a fermionic Hilbert space:

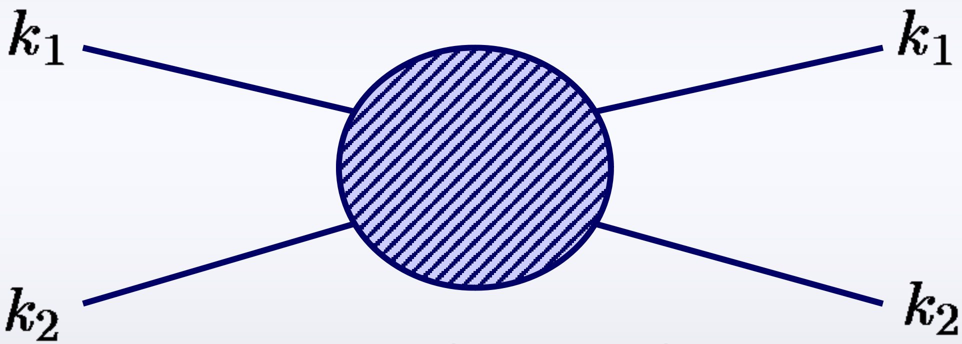
$$\psi_{GS} = \prod_{j<l}\left|\sin\frac{\pi}{L}(x_j - x_l)\right|^{\lambda}$$

$$\prod_{j<l}\left[\sin\frac{\pi}{L}(x_j - x_l)\right]^{\delta(\sigma_j,\sigma_l)} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathrm{sgn}(\sigma_j-\sigma_l)}$$

"anyonic" (Laughlin-type) wave-function

Scattering Phase

- For integrable systems, no true 3-body processes
- 2-body scattering characterized just by a phase



$$-e^{i\theta(k_1-k_2)}$$

- Calogero-Sutherland model: $\boxed{\theta(k) = \pi\lambda\,\mathrm{sgn}(k)}$

$\rightarrow$ dynamical phase like a statistical phase

(Asymptotic) Bethe Ansatz Solution

- States defined by set of integer numbers $\kappa_{\uparrow,\downarrow}$ for spin up/down particles
- Hydrodynamic distribution $\nu(\kappa)$: $\begin{cases} \kappa_\uparrow = \kappa_{L\uparrow}, \ldots, \kappa_{R\uparrow} \\ \kappa_\downarrow = \kappa_{L\downarrow}, \ldots, \kappa_{R\downarrow} \end{cases}$
- Distribution $\tau(k)$ of quasi-momenta piece-wise constant (peculiar to Calogero interaction)

Sutherland & Shastry (1993)
Kato & Kuramoto (1995)

- Proceeding as for FF, we make the identification:

$$v_\alpha \pm \pi \rho_\alpha \equiv \frac{2\pi}{L} \kappa_{(R,L);\alpha}$$

The 3 regimes

$$\epsilon = \sum_{\kappa=-\infty}^{+\infty} \kappa^2 \nu(\kappa) + \frac{\lambda}{2} \sum_{\kappa,\kappa'} |\kappa - \kappa'| \nu(\kappa) \nu(\kappa')$$

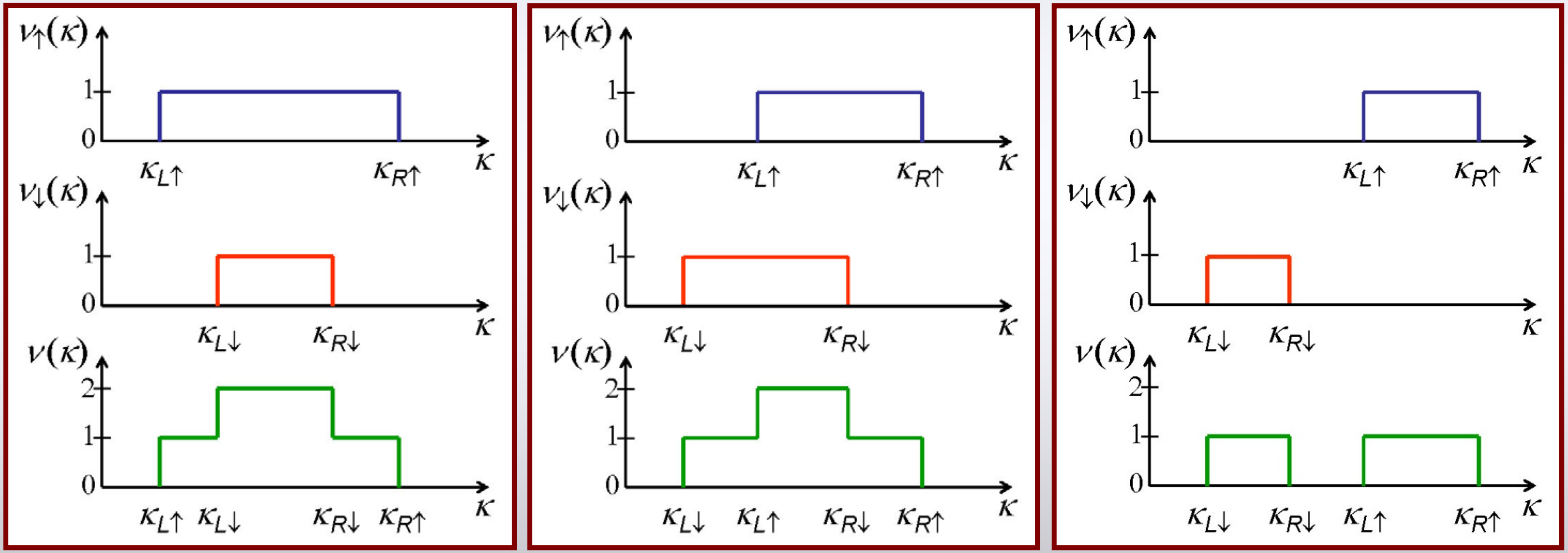
Sutherland & Shastry (1993)

Kato & Kuramoto (1995)

The 3 regimes

$$\epsilon = \sum_{\kappa=-\infty}^{+\infty} \kappa^2 \nu(\kappa) + \frac{\lambda}{2} \sum_{\kappa,\kappa'} |\kappa - \kappa'| \nu(\kappa)\nu(\kappa')$$

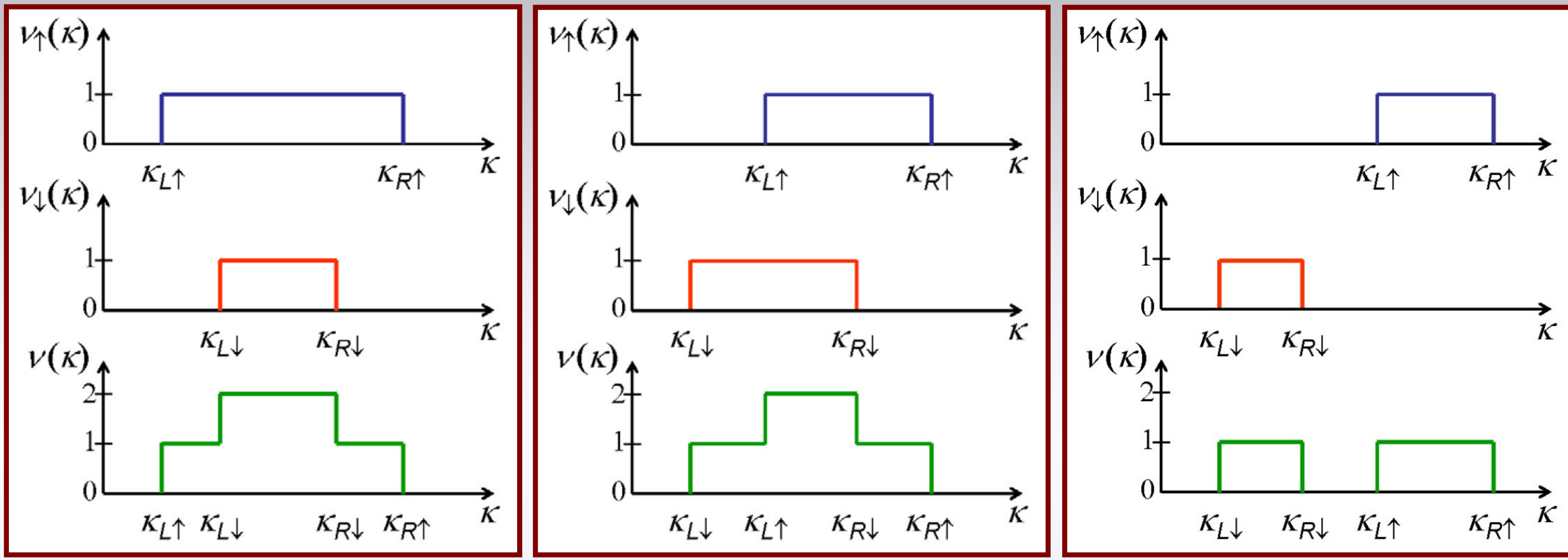
- Because of the absolute value, we identify 3 regimes:



Complete Overlap (CO) Partial Overlap (PO) No Overlap (NO)

x2: Exchanging $\uparrow \leftrightarrow \downarrow$

The 3 regimes



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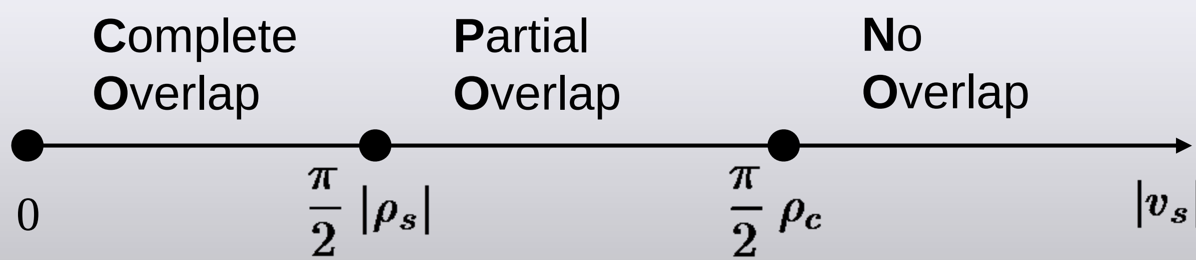
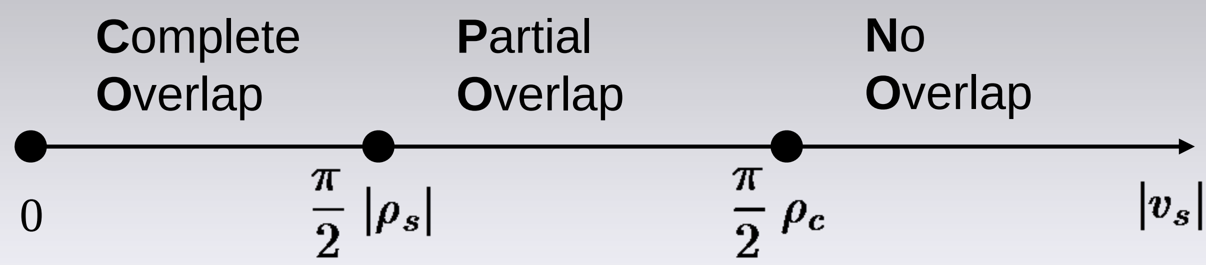


$$v_\alpha \pm \pi\rho_\alpha \equiv \frac{2\pi}{L}\kappa_{(R,L);\alpha}$$

x2: Exchanging $\uparrow \leftrightarrow \downarrow$

The 3 regimes



$$\text{Complete Overlap} \rightarrow |v_s| < \frac{\pi}{2}|\rho_s|$$

$$\text{Partial Overlap} \rightarrow \frac{\pi}{2}|\rho_s| < |v_s| < \frac{\pi}{2}\rho_c$$

$$\text{No Overlap} \rightarrow \frac{\pi}{2}\rho_c < |v_s|$$

- CO & PO: small deviation from AFM Ground state
- Here, I'll concentrate only on the **CO regime**

The CO regime

$$H_{\mathrm{CO}} = \int \mathrm{d}x \left\{ \frac{1}{2}\rho_c v_c^2 + \frac{\pi^2}{6}\left(\lambda + \frac{1}{2}\right)^2 \rho_c^3 + \rho_s v_c v_s + \left[\left(\lambda + \frac{1}{2}\right)\rho_c - \lambda \rho_s\right] v_s^2 + \frac{\pi^2}{4}\left(\lambda + \frac{1}{2}\right)\rho_c \rho_s^2 - \frac{\pi^2}{12}\lambda \rho_s^3 \right\}$$

$$\{\rho_\alpha(x), v_\beta(y)\} = \delta_{\alpha\beta}\delta'(x-y)$$

- Non-linear dynamics **couples** spin & charge!

The CO regime

- Introduce the following linear combination of fields:

$$k_{R\uparrow,L\uparrow} = (v_\uparrow \pm \pi\rho_\uparrow) \pm \pi\lambda\rho_c,$$
$$k_{R\downarrow,L\downarrow} = (v_\downarrow \pm \pi\rho_\downarrow) \pm \pi\lambda\rho_c - \lambda(2v_s \pm \pi\rho_s)$$

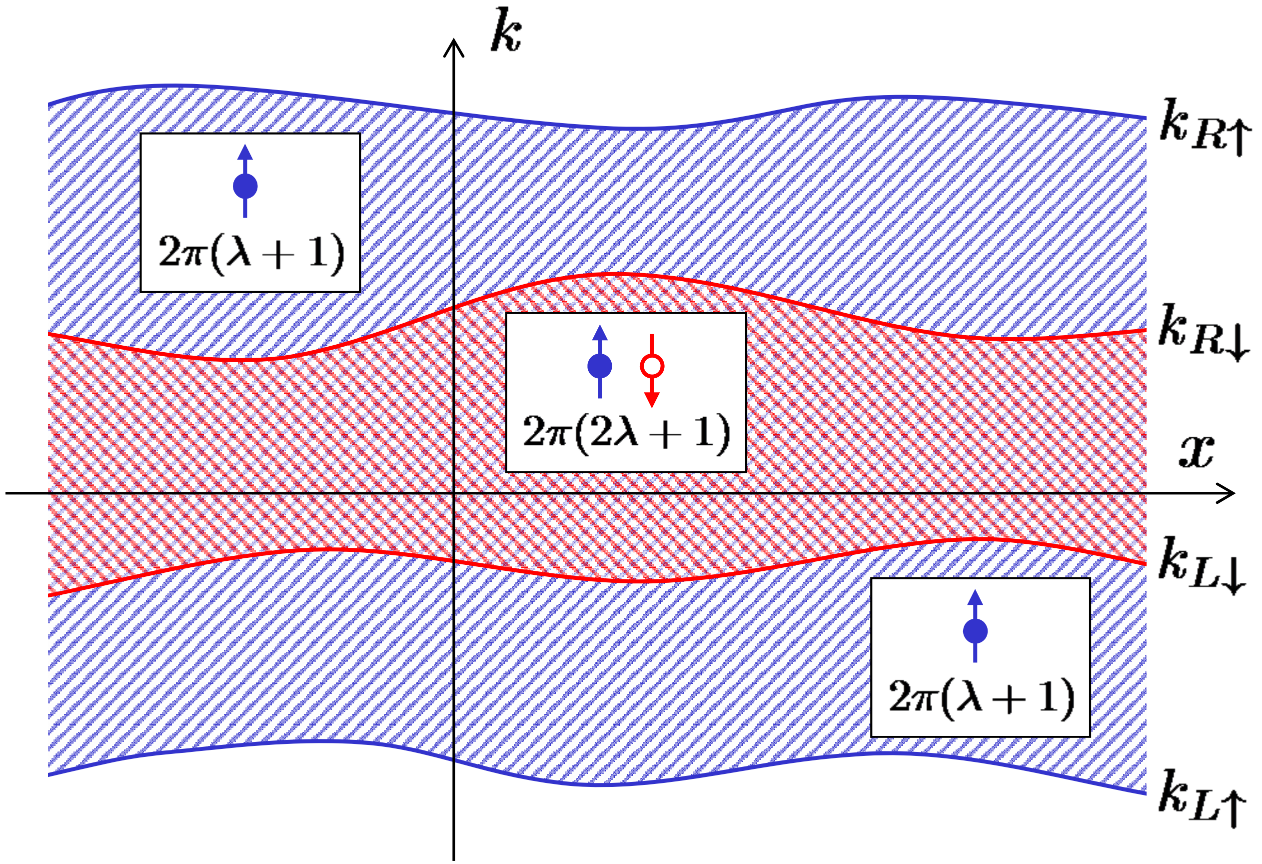
- This transformation decouples the dynamics into 4 Riemann-Hopf equations:

$$H = \sum_{\alpha=\uparrow,\downarrow} \sum_{\chi=L,R} s_{\chi;\alpha} \int \mathrm{d}x \, k^3_{\chi;\alpha}$$

$$u_t + u u_x = 0, \; u = k_{R,L;\uparrow,\downarrow}$$

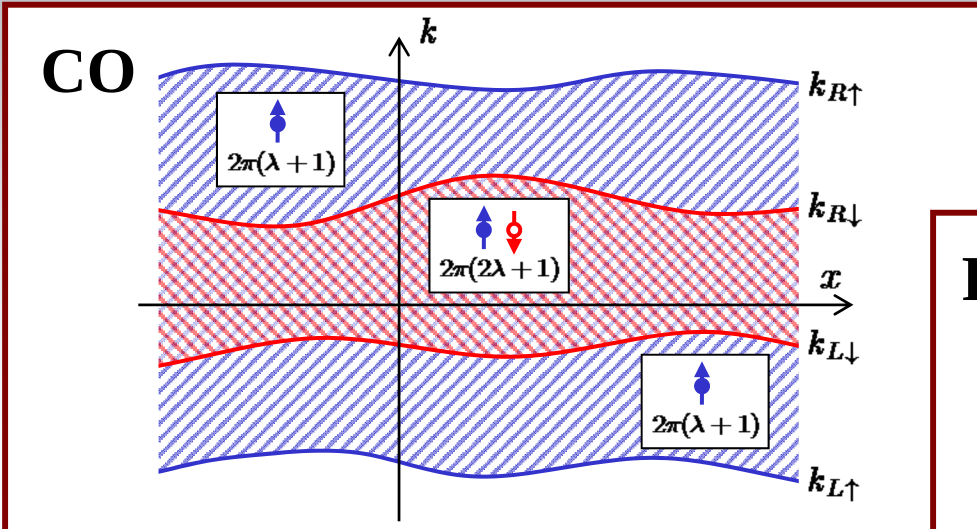
- These k's are the BA dressed "Fermi" momenta

The CO regime

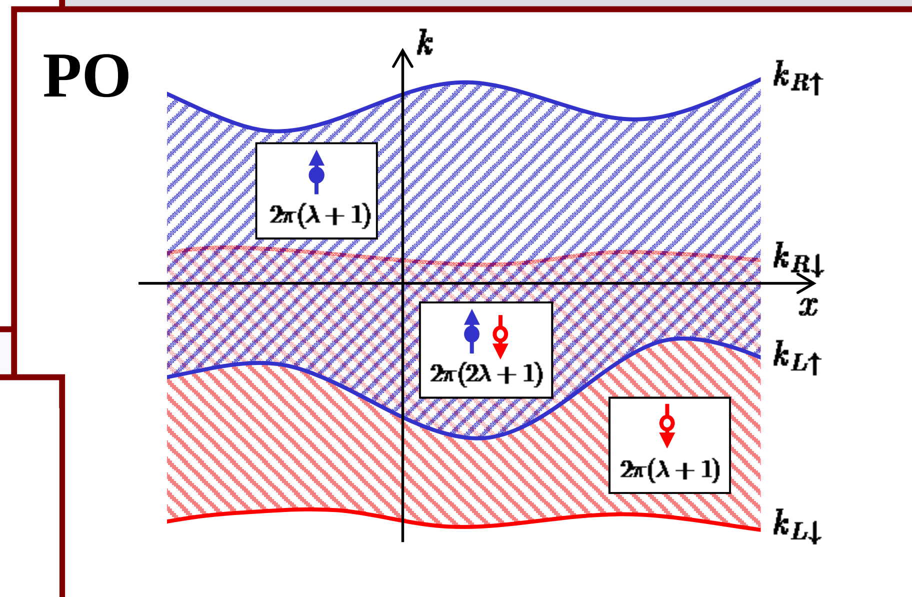


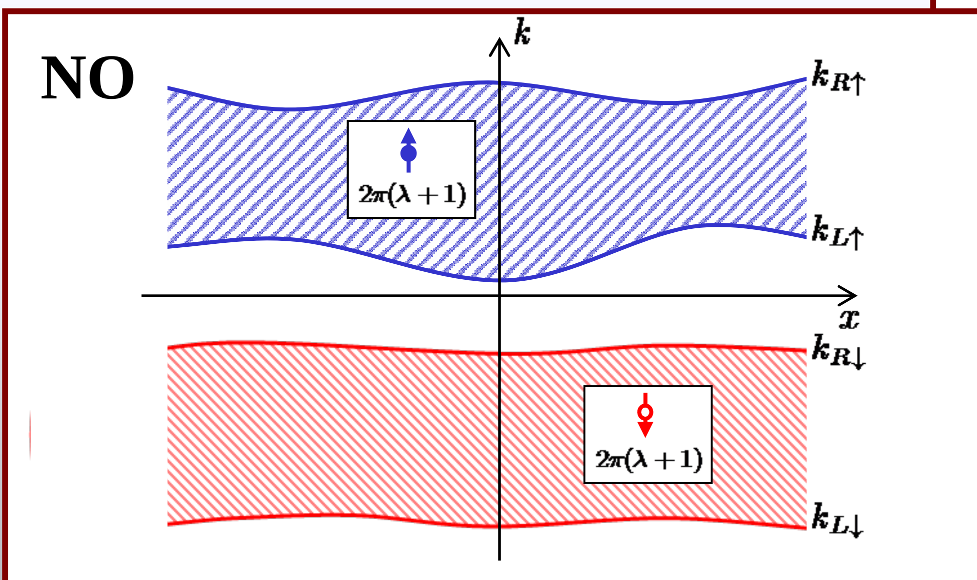
$$u_t + u u_x = 0, \;\; u = k_{R,L;\uparrow,\downarrow}$$

Phase-Space picture for the 3 regimes



$$u_t + uu_x = 0, \;\; u = k_{R,L;\uparrow,\downarrow}$$



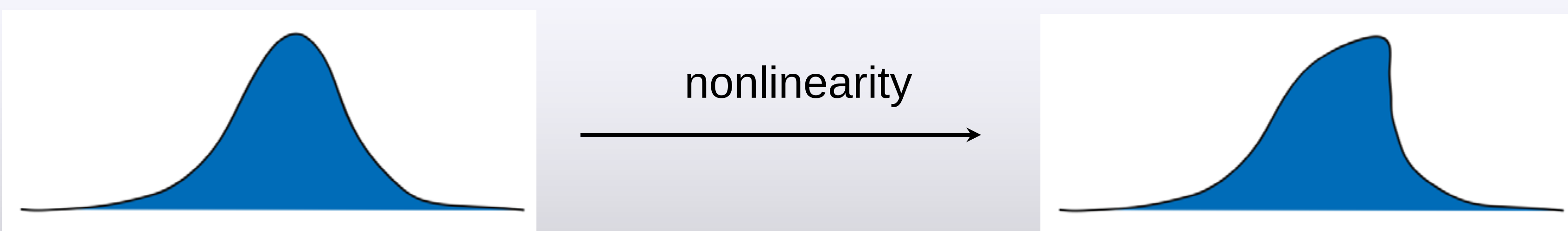


Dressed "Fermi" momenta picture works for every regime

A few words on the Riemann-Hopf Eq.

$$u_t + u u_x = 0$$

- Simplest non-linear equation
- Given an initial condition $u(\mathrm{x},0) = u_0(\mathrm{x})$, its solution is implicitly given by $u = u_0(\mathrm{x} - u\mathrm{t})$ (easy to handle numerically)
- Ill defined for long times (gradient catastrophe):



- Should be corrected by gradient terms like:

$$\begin{aligned} u_t + u u_x + u_{xxx} &= 0, \\ u_t + u u_x + u^H_{xx} &= 0 \ldots \end{aligned}$$

And a few words on our solutions

$$u_t + u u_x = 0, \; u = k_{R,L;\uparrow,\downarrow}$$

- We neglected gradient corrections from the start
- Decoupling of Fermi momenta into RH equations probably broken by gradient correction
- Our hydrodynamics valid for "small" times (« than gradient catastrophe)
- We require all gradients to be small compared to interparticle distance

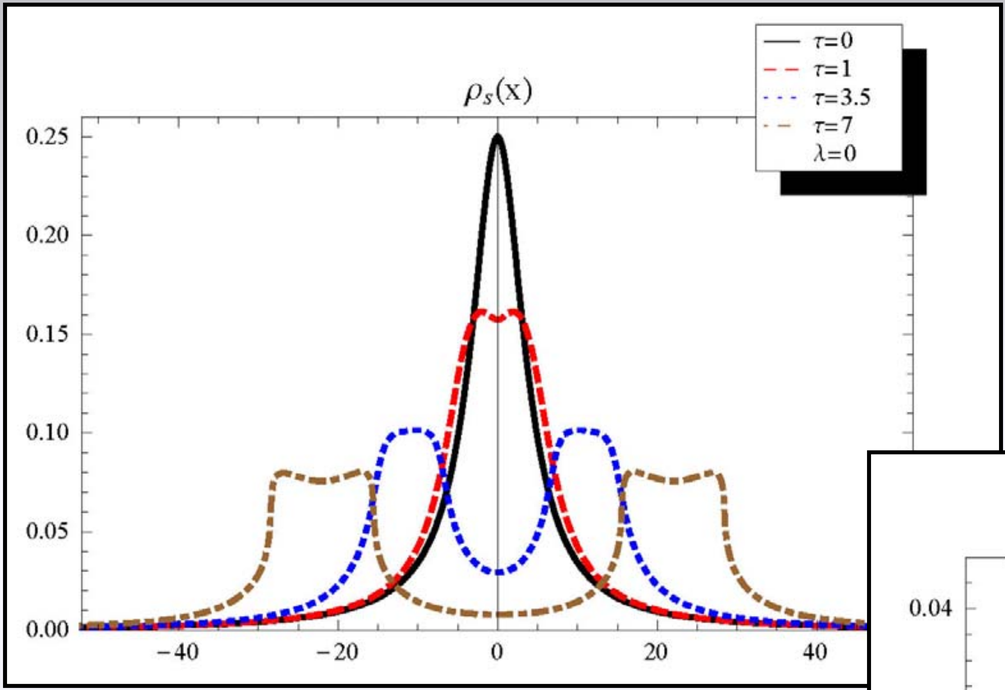
Spin singlet dynamics

- Consider initial condition: $\rho_s, v_s = 0$
- Any configuration of charge sector will not perturb this spin singlet state:

$$\begin{aligned}
\dot{\rho}_c &= -\partial_x(\rho_c v_c) \\
\dot{\rho}_s &= 0 \\
\dot{v}_c &= -\partial_x \left\{ \frac{v_c^2}{2} + \frac{\pi^2 \left(\lambda + \frac{1}{2}\right)^2 \rho_c^2}{2} \right\} \\
\dot{v}_s &= 0
\end{aligned}$$

(Spinless Calogero-Sutherland with $\lambda+1 \to \lambda+1/2$)

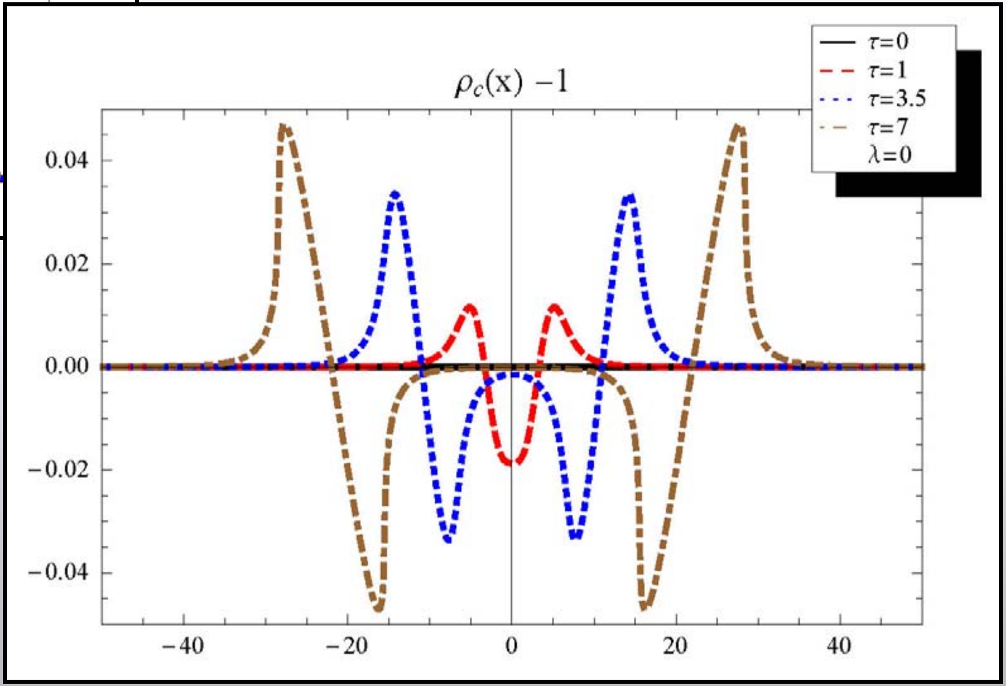
Dynamics of a polarized center: FF



Initial condition at $t = 0$:

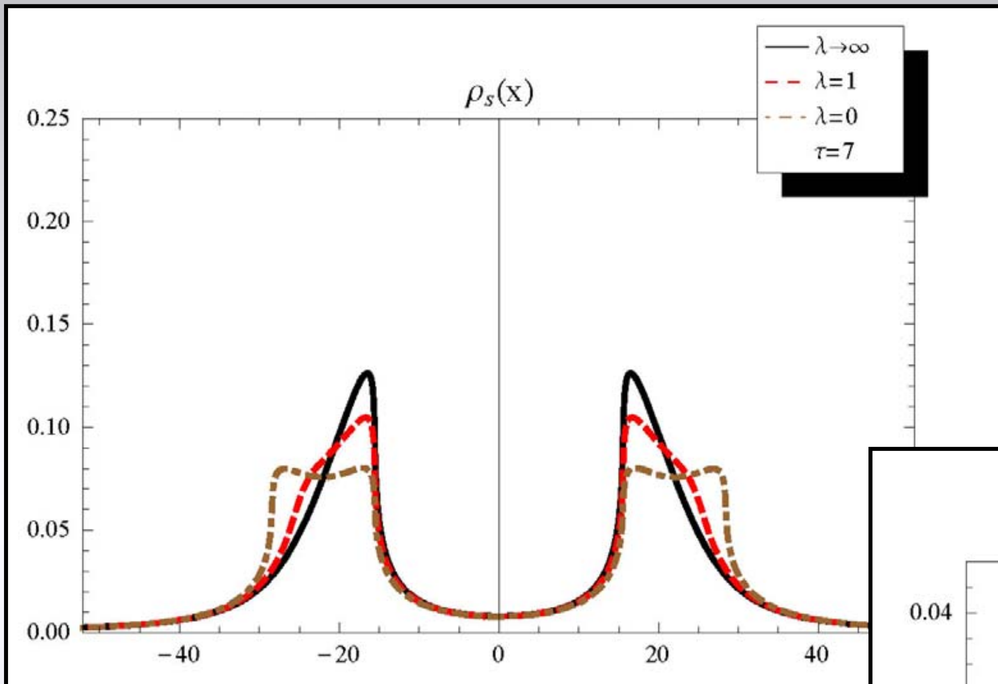
$$\rho_c = 1, v_c = 0,$$

$$v_s = 0, \rho_s = \frac{h}{1 + (x/a)^2}$$



- N.B. λ=0: Free Fermions!
- Essential non-linear dynamics

Dynamics of a polarized center: sCM



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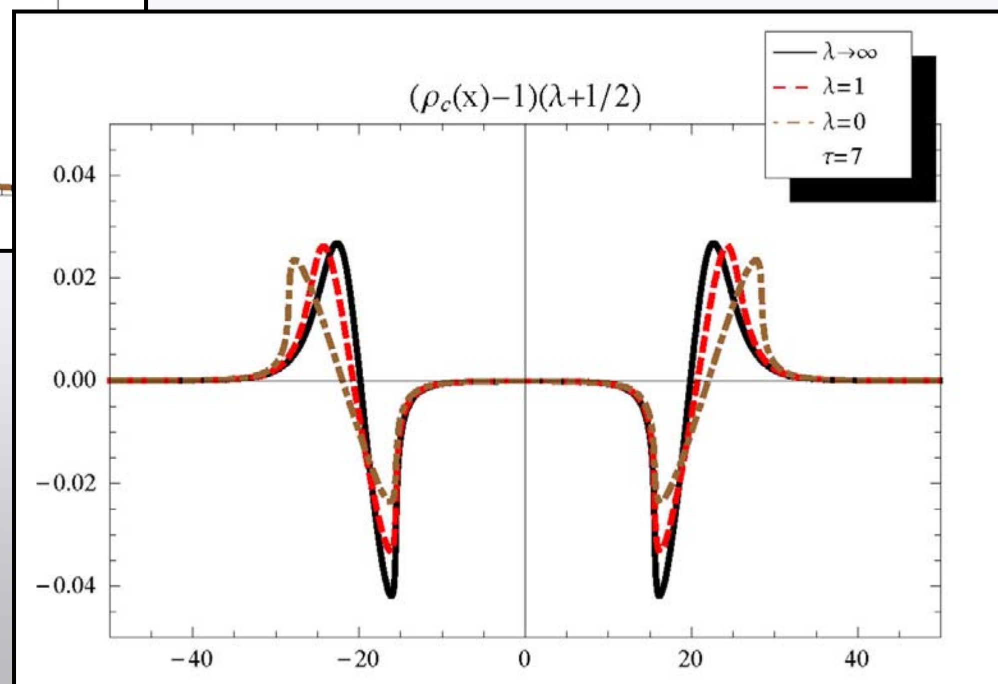

- Qualitatively similar behaviors for rescaled quantities ($\tau = (\lambda+1/2)t$)
- Charge freezing

The Haldane-Shastry model

$$H = -\frac{\hbar^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{2}\left(\frac{\pi}{L}\right)^2 \sum_{j\neq l}\frac{\lambda(\lambda - \mathbf{P}_{jl})}{\sin^2\frac{\pi}{L}(x_j - x_l)}$$

$\lambda \to \infty$ ↓ "freezing trick"

(Polychronakos, 1993)

$$H_{\mathrm{HSM}} = \frac{1}{2}\left(\frac{\pi}{N}\right)^2 \sum_{j<l}\frac{\mathbf{K}_{jl}}{\sin^2\frac{\pi}{N}(j-l)}$$

(Haldane, 1988; Shastry, 1988)

- Spectrum of HS equal to spinless CS with λ=2, but with high degeneracy (Yangian symmetry) (Haldane & Ha, 1992; Ha & Haldane, 1993)

Connection to Haldane-Shastry model

Hydrodynamics of sCM from its Bethe Ansatz solution

$$H \simeq \int \mathrm{d}x \left\{ \frac{\pi^2}{6}\mu^2\rho_c^3 + \mu \left[\rho_c v_s^2 - \rho_s v_s^2 + \frac{\pi^2 \rho_c \rho_s^2}{4} - \frac{\pi^2 \rho_s^3}{12} \right] + O(\mu^0) \right\}$$

$$\mu = \lambda + 1/2$$

$$H_{\mathrm{HSM}} = \int \mathrm{d}x \left\{ \rho_0 v_s^2 - \rho_s v_s^2 + \frac{\pi^2 \rho_0 \rho_s^2}{4} - \frac{\pi^2 \rho_s^3}{12} \right\}$$

Hydrodynamics of HSM from its Bethe Ansatz solution

Connection to Haldane-Shastry model

Hydrodynamics of sCM from its Bethe Ansatz solution

$$H \simeq \int \mathrm{d}x \left\{ \frac{\pi^2}{6}\mu^2\rho_c^3 + \mu\left[\rho_c v_s^2 - \rho_s v_s^2 + \frac{\pi^2\rho_c\rho_s^2}{4} - \frac{\pi^2\rho_s^3}{12}\right] + O(\mu^0) \right\}$$

$$\mu = \lambda + 1/2$$

$$H_{\mathrm{HSM}} = \int \mathrm{d}x \left\{ \rho_0 v_s^2 - \rho_s v_s^2 + \frac{\pi^2\rho_0\rho_s^2}{4} - \frac{\pi^2\rho_s^3}{12} \right\}$$

$$= \int \mathrm{d}x \left[\frac{1}{2}\,\rho\, v^2 + \frac{2}{3}\,\pi^2\,\rho^3 \right]$$

$$\lambda = 2$$

$$\boxed{\rho = \rho_\downarrow = \frac{\rho_0 - \rho_s}{2},\ v = -2v_s,\ \rho_0 = 1}$$

- Higher orders give corrections to freezing…

Correlation functions

- So far: non-linear dynamics couples spin & charge
- Asymptotics of correlation functions easy from field theory
- 2-point correlation functions: Luttinger Liquid is sufficient
- For extended objects non-linear theory is needed
- To leading order, gradient-less theory is enough

Emptiness Formation Probability

- It measures the probability $P(R)$ that there are no particles for $-R < x < R$
- Simplest correlator in integrable models
- For sCM different EFPs: $P_\alpha(R)$, $\alpha = \uparrow, \downarrow, c, s \ldots$
- Easy to calculate in instanton formalism
- Non-local correlation function

$\Rightarrow$ linear bosonization not sufficient: full hydrodynamics

Instanton Approach to EFP

- EFP as probability of rare fluctuation in imaginary time

$$P(R) \simeq \mathrm{e}^{-S[\phi_{\mathrm{EFP}}]}$$

- Instanton: solution of equation of motion with b.c.'s

$$\rho_\alpha(\tau = 0, -R < x < R) = \bar{\rho}_\alpha, \quad \alpha = \uparrow, \downarrow$$

$$\rho_\alpha(x, \tau \to \infty) \to \rho_{0\alpha}, \; v_\alpha(x, \tau \to \infty) \to 0$$

- $\bar{\rho}_\alpha = 0$: Emptiness, otherwise Depletion Formation Probability (DFP)

- Gradient-less theory sufficient for leading order

EFP/DFP for sCM

- Using our hydrodynamic description for sCM:

$$P(R) \simeq \exp\left\{ -\frac{\pi^2}{2} \left[(\lambda + \frac{1}{2}) \left(\rho_{0c} - \bar{\rho}_c\right)^2 + \frac{1}{2} \left(\rho_{0s} - \bar{\rho}_s\right)^2 \right] R^2 \right\}$$

- Result factorizes: **effective spin-charge separation** in non-linear dynamics!
- Spin and charge sectors as independent spin-less Calogero fluids with couplings $\lambda' = \lambda + 1/2$ and $\lambda' = 2$
- Spin-charge separation true (at Gaussian Order) for other extended correlators: why?

Conclusions

- Derived gradient-less hydrodynamics for **spin Calogero-Sutherland** & **Haldane Shastry** model
- Showed freezing limit and corrections
- Captured essentially non-linear spin-charge coupling
- EFP restores effective spin-charge separation

Outlook

- Ferromagnetic Fermions, Bosons
- Gradient terms

Thank you!