Approaching critical points through entanglement: why take one, when you can take them all?

Fabio Franchini (M.I.T./SISSA)

Collaborators:

A. De Luca;

E. Ercolessi, S. Evangelisti, F. Ravanini;

V. E. Korepin, A. R. Its, L. A. Takhtajan ...

- arXiv:1205:6426
- PRB 85: 115428 (2012)
- PRB 83: 12402 (2011)
- Quant. Inf. Proc. 10: 325 (2011)
- JPA 41: 2530 (2008)
- JPA 40: 8467 (2007)

Entanglement Entropy in 1-D exactly solvable models

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Motivation

- Entanglement Entropy: non-local correlator → area law
- 1+1-D CFT prediction (universal behavior):

$$S_\alpha = \frac{c+\bar{c}}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln \ell + c'_\alpha + b_\alpha \ell^{-2h/\alpha} + \ldots$$

where c central charge,

h dimension of (relevant) operator

- Exactly solvable, lattice models efficient testing tools

Aims

- Gapped systems: entropy saturates
- We'll test:

1. Expected simple scaling law: $\ell \leftrightarrow \xi$

$$S_\alpha = \frac{c}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \ldots$$

   with the same dimension h ?

2. Close to non-conformal points: competition between different length scales → essential singularity

Outline

- Introduction: Von Neumann and Renyi Entropy as a measure of Entanglement
- Entanglement Entropy in 1-D systems
- Integrability & Corner Transfer Matrices
- Restriced Solid-On-Solid Models: integrable deformation of minimal & parafermionic CFT
- Essential Critical Point for the entropy: XYZ chain
- Conclusions

Introduction

- Entanglement: fundamental quantum property
- Different reasons for interest:

1. Quantum information → quantum computers
2. Quantum Phase Transitions → universality
3. Condensed matter → non-local correlator
4. Integrable Models → new playground
5. Cosmology → Black Holes
6. …

Understanding Entanglement: A simple Example

- Two spins 1/2 in triplet state $\rightarrow S_z = 1$:

$$| \uparrow \rangle \otimes | \uparrow \rangle$$

No entanglement

- Middle component with $S_z = 0$:

$$| \uparrow \rangle \otimes | \downarrow \rangle + | \uparrow \rangle \otimes | \downarrow \rangle$$

Maximally entangled

Entanglement Entropy

- Whole system in a pure quantum state
- Compute Density Matrix of subsystem:

$$\rho_A = tr_B \left(|\Psi^{A,B}\rangle \langle \Psi^{A,B}| \right)$$

- Entanglement for pure state as Quantum Entropy (Bennett, Bernstein, Popescu, Schumacher 1996):

$$S = -tr_A \left(\rho_A \ln \rho_A \right)$$

Von Neumann Entropy

Entropy as a measure of entanglement

- Quantum analog of Shannon Entropy: Measures the amount of "quantum information" in the given state

- Assume Bell State as unity of Entanglement:

$$|\mathrm{Bell}\rangle = \frac{|\downarrow\downarrow\rangle \pm |\uparrow\uparrow\rangle}{\sqrt{2}} \, , \, \frac{|\downarrow\uparrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}$$

- Von Neumann Entropy measures how many Bell-Pairs are contained in a given state $|\Psi^A\rangle$ (i.e. closeness of state to maximally entangled one)

More Entanglement Estimators

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}|$$

- Von Neumann Entropy: $S_A = -\mathrm{tr}\,(\rho_A \log \rho_A)$
- Renyi Entropy: $S_\alpha = \dfrac{1}{1-\alpha} \ln \mathrm{tr}\,(\rho_A^\alpha)$
  (equal to Von Neumann for $\alpha \to 1$)
- Tsallis Entropy
- Concurrence (Two-Tangle)
- ...

Bi-Partite Entanglement

- Consider the Ground state of a Hamiltonian H
- Space interval $[1, \ell\,]$ is subsystem A
- The rest of the ground state is subsystem B.

$\rightarrow$ Entanglement of a block of spins in the space interval $[1, \ell\,]$ with the rest of the ground state <u>as a function of ℓ</u>

General Behavior (Area Law)

- Asymptotic behavior (block size $\ell \to \infty$)

(Double scaling limit: $0 << \ell << N$)

$$S(\ell) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

- For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)

$$S(\ell) \simeq \text{Constant} + \ldots$$

- For critical conformal phases: (Calabrese, Cardy 2004)

$$S(\ell) \simeq \frac{c+\bar{c}}{6} \ln \ell + \ldots$$

Subleading corrections

$$S_\alpha(\ell) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$

- Integers Powers of ρ accessible in CFT (replica)

(Cardy, Calabrese 2010)

$$S_\alpha(\ell) = \frac{c+\bar{c}}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln \ell + c'_\alpha + b_\alpha \ell^{-2h/\alpha} + \ldots$$

From cut-off regularization

- Close to criticality: $\xi \sim \Delta^{-1}, n \to \infty$

(Calabrese, Cardy, Peschel 2010)

Conjecture

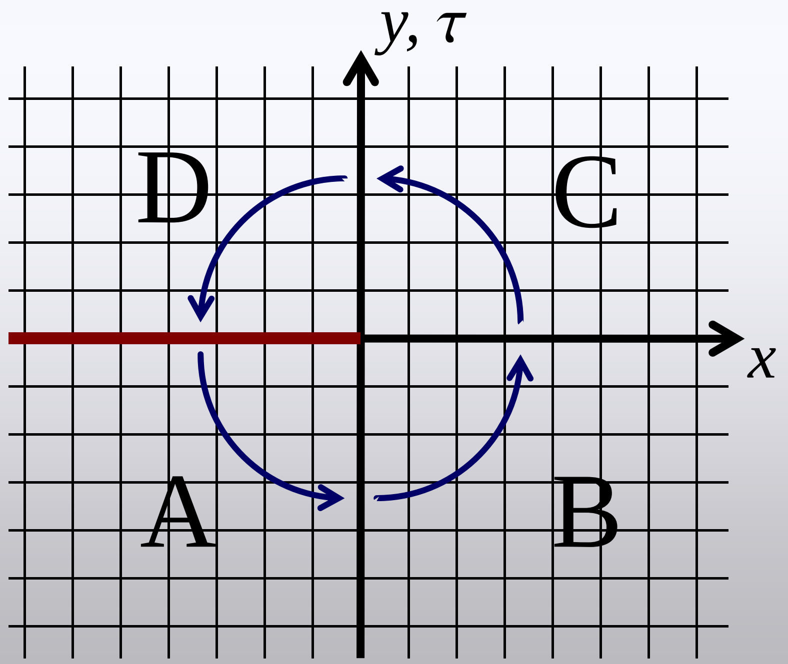
$$S_\alpha = \frac{c}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln \xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \ldots$$

Corner Transfer Matrices

- Consider 2-D classical system whose transfer matrices commutes with Hamiltonian of 1-D quantum model
- Use of Corner Transfer Matrices (CTM) to compute reduced density matrix

$$\mathcal{Z} = \mathrm{tr}\,(ABCD)$$

$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$



Entanglement of one half-line with the other

Entanglement & Integrability

- Baxter diagonalized CTM's of integrable models

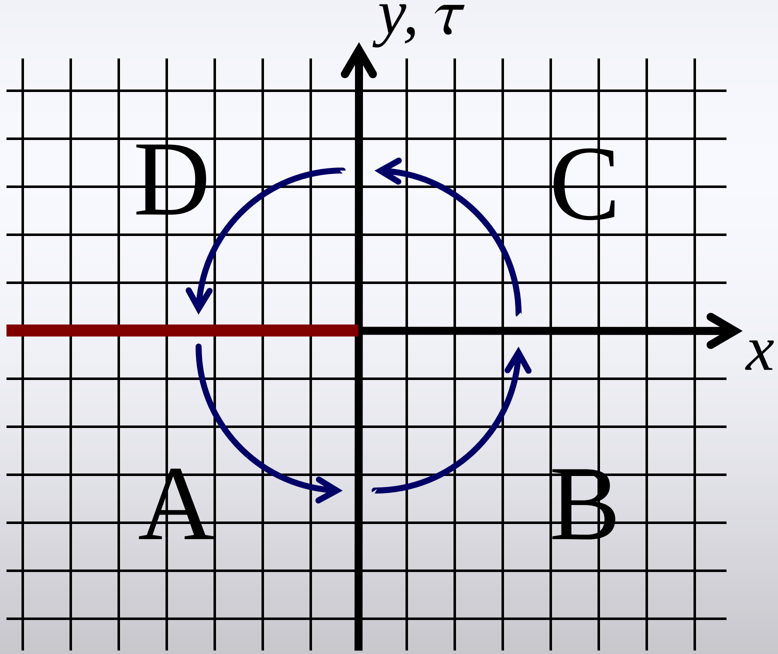
  $\Rightarrow$ regular structure of the entanglment spectrum

$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$

$$\mathcal{Z}_\alpha \equiv \mathrm{Tr}\rho^\alpha$$

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha-1}\ln\mathcal{Z}_1 + \frac{1}{1-\alpha}\ln\mathcal{Z}_\alpha$$

α real (or even complex)!



CTM & Integrability

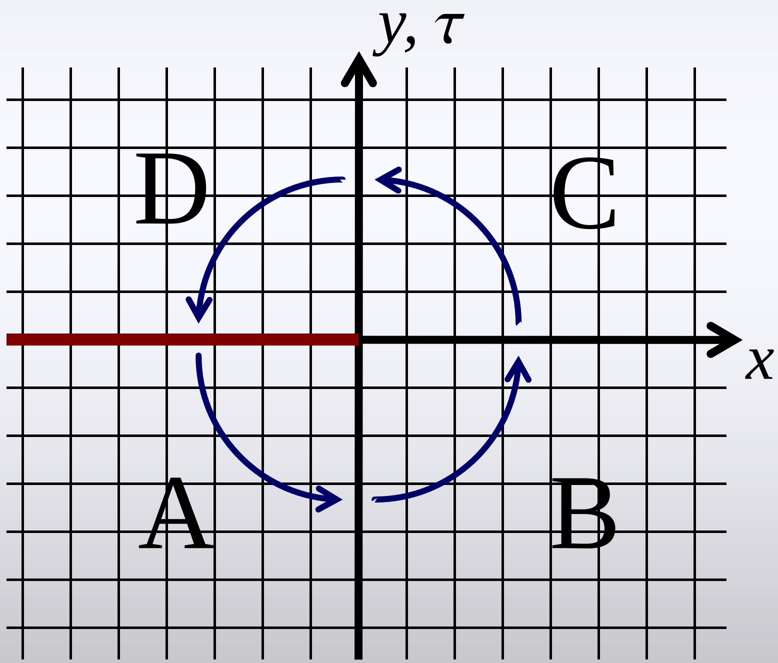
- CTM spectrum in integrable models <u>same as</u> certain Virasoro representations (unknown reason!)

$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$

$$\boxed{\mathcal{Z}_\alpha = \mathrm{Tr}\rho^\alpha = \sum_x N_x^\alpha \, \chi_x(q^\alpha)}$$

Only formal: q measures "mass gap", not same as CFT!

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha - 1} \ln \mathcal{Z}_1 + \frac{1}{1-\alpha} \ln \mathcal{Z}_\alpha$$



Integrable Models

- Restricted Solid-On-Solid (RSOS) Models

  $\rightarrow$ Minimal & Parafermionic CFTs

  with Andrea De Luca

- Two integrable chains (8-vertex model)

  1) XY in transverse field ($J_z = 0$)

  with Korepin, Its, Takhtajan

  2) XYZ in zero field ($h = 0$)

  with Stefano Evangelisti, Ercolessi, Ravanini

$$H = \sum_{i=1}^{N} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - h \sum_i \sigma_i^z$$

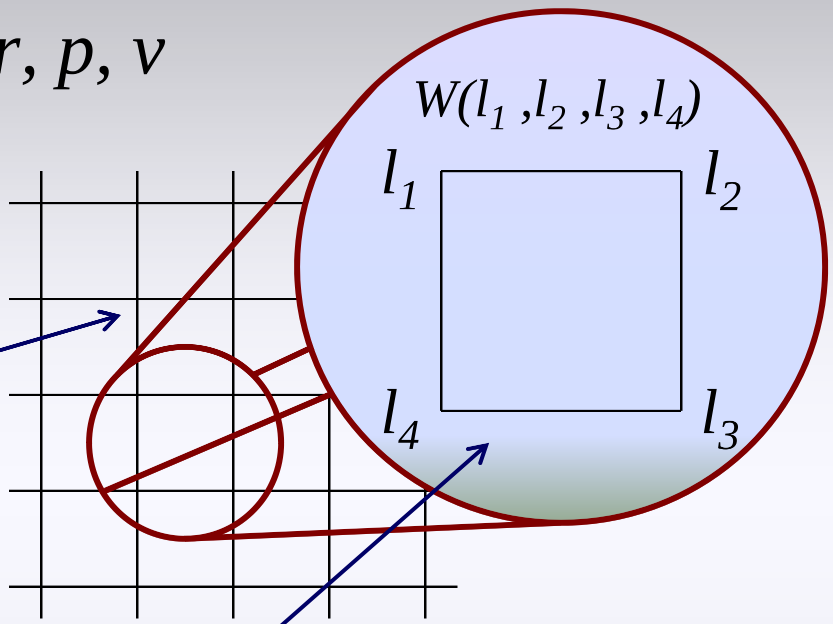
Restricted Solid-On-Solid Models

- Specified by 3 parameters: r, p, v
- 2-D square lattice
- Heights at vertices:

  $$1 \le l_i \le (r-1)$$

  with local constraint

  $$|l_i - l_j| = 1, \ for\ n.n.$$



- Interaction Round-a-Face: weight for each plaquette
- Choice of weights makes model integrable

  (satisfy Yang-Baxter of 8-vertex model: p, v parametrize weights)

RSOS Phase Diagram

- At fixed *r*

$$1 \le l_i \le (r-1)$$

l_1 l_2

l_4 l_3

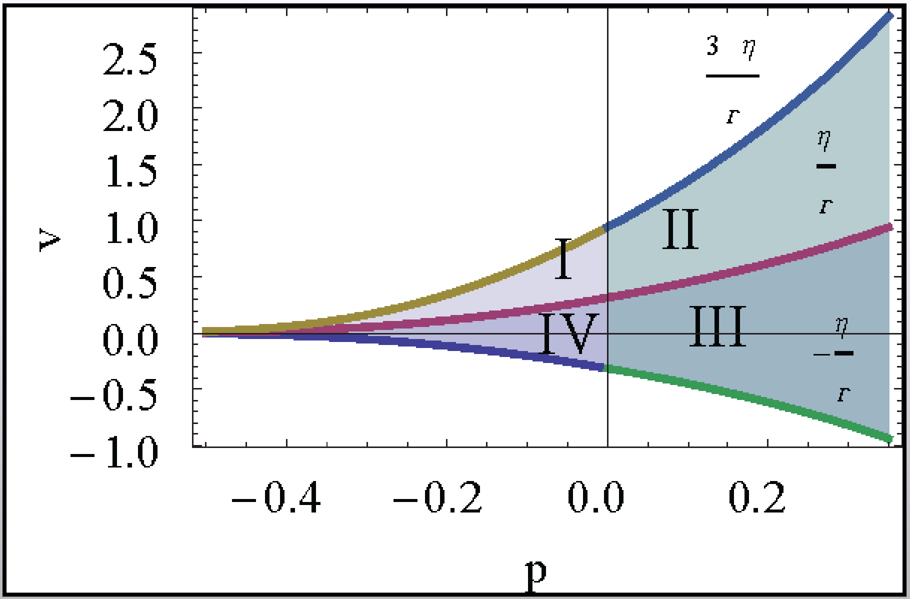
$$= W^{l_1, l_2}_{l_3, l_4}(p, v)$$

- 4 Phases:

$$-1 < p < 1 \, , -\eta < v < 3\eta$$

I	$-1 < p < 0$	$\eta < v < 3\eta$
II	$0 < p < 1$	$\eta < v < 3\eta$
III	$0 < p < 1$	$-\eta < v < \eta$
IV	$-1 < p < 0$	$-\eta < v < \eta$

$$\eta \equiv \frac{K(p)}{r}$$

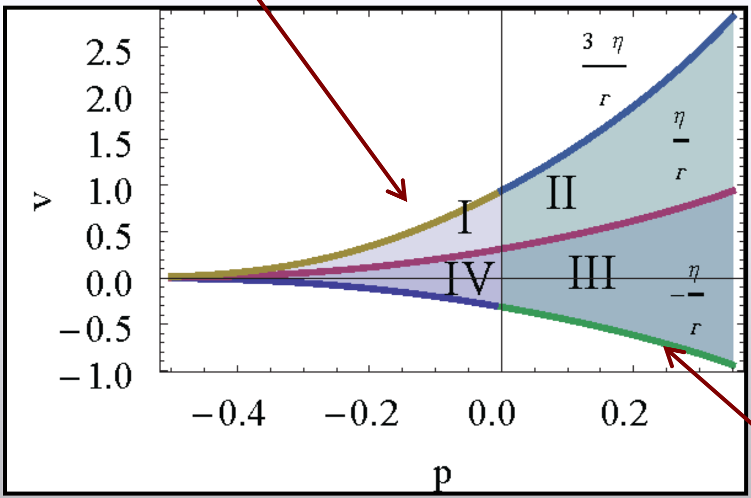


RSOS: Phases I & III

Phase I $-1 < p < 0 \quad \eta < v < 3\eta$

- 1 ground state → Disordered
- For $p \to 0$: parafermion CFT (Virasoro + $\mathbb{Z}_{r-2}$)

$$c_r = \frac{2(r-3)}{r}$$



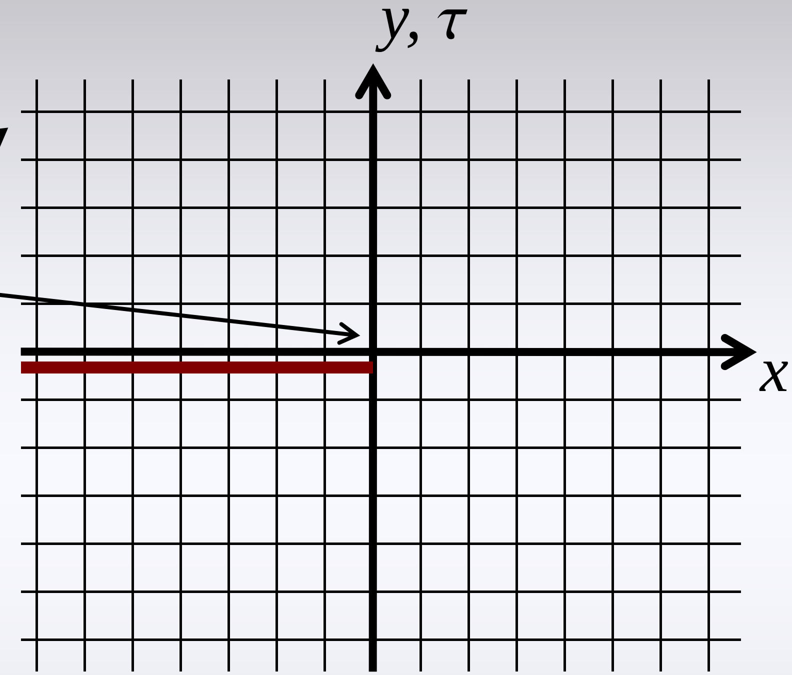
Phase III: $0 < p < 1 \quad -\eta < v < \eta$

- r - 2 ground states → Ordered
- For $p \to 0$: minimal CFT

$$c_r = 1 - \frac{6}{r(r-1)}$$

Sketch of the calculation

- Diagonal reduced ρ depends on b.c. at origin (a) & infinity (b)



$$\rho_{\mathrm{diag}'} = \mathbf{R}(a)\mathbf{T}(a,b)$$

$$\begin{aligned} \mathcal{Z}_\alpha &= \mathrm{Tr}\rho^\alpha \\ &= \sum_a E^\alpha(a;\tilde{p})\, X(a,b;\tilde{p}^\alpha) \end{aligned}$$

$\tilde{p} \to 1$: at criticality: Poisson Summation formula (S-Duality)

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha-1}\ln \mathcal{Z}_1 + \frac{1}{1-\alpha}\ln \mathcal{Z}_\alpha$$

Regime III: Minimal models

$$\mathcal{Z}_\alpha = \sum_a E^\alpha(a;\tilde{p})\, X(a,b;\tilde{p}^\alpha)$$

- Fixing a & b: single minimal model character:

$$X(a,b,\tilde{p}^\alpha) \propto \chi_{b,a}^{(r-1)}(\tilde{p}^\alpha) = \sum_{t,s} S_{b,a}^{t,s}\, \chi_{t,s}^{(r-1)}(p^{1/\alpha})$$

- After S-Duality (Poisson) duality and logarithm

$$S_\alpha = \frac{c_r}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \dots$$

where h dimension of most relevant operator here

(generally $h = 2\,\Delta_{2,2} = \dfrac{3}{2r(r-1)}$)

Regime III: Minimal models

$$\mathcal{Z}_\alpha = \sum_a E^\alpha(a;\tilde{p})\, X(a,b;\tilde{p}^\alpha)$$

- Fixing a equivalent to projecting Hilbert space

- True ground state by summing over a:

$$S_\alpha = \frac{c_r}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \ldots$$

- $\mathbb{Z}_2$ dicates most relevant operator vanishes (odd):

$$\boxed{h = 2\Delta_{3,3} = \frac{4}{r(r-1)}}$$

Regime III: conclusion

- RSOS as integrable deformation of minimal models
- Integrability fixes coefficients: $\mathcal{Z}_\alpha = \sum_x N_x^\alpha \, \chi_x(q^\alpha)$
- Corrections from relevant operators

$$S_\alpha = \frac{c_r}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \ldots$$

$$\boxed{h = 2\Delta_{2,2}, 2\Delta_{3,3}}$$

- Same scaling function in ξ & ℓ ?
- $\mathbb{Z}_2$ role at criticality?

Regime I: Parafermions

$$\mathcal{Z}_\alpha = y(b;\tilde{p}^\alpha) \sum_a E^\alpha(a;\tilde{p})\, Y(a;\tilde{p}^\alpha)$$

- b.c. at infinity factorize out
- a selects a combination of operators neutral for $\mathbb{Z}_{r-2}$

$$S_\alpha^{\mathrm{bulk}} = \frac{c_r^{\mathrm{Pf}}}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \ldots$$

- In general: $h = 4/r$ (most relevant neutral op)
- b can give logarithmic corrections (marginal fields?)

$$S_\alpha^{(b)} = \ln b + \frac{(b^2-1)\pi^4\alpha}{24(\ln\xi)^2} + O\left(\frac{1}{\ln\xi}\right)^4$$

RSOS Round-up

- RSOS as integrable deformations of CFT
- CTM spectrum mimics critical theory (accident?)

$\Rightarrow$ same scaling function for entanglement in ξ & ℓ ?

$$S_\alpha = \frac{c}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln\xi + A_\alpha + B_\alpha \xi^{-h/\alpha} + \dots$$

- Logarithmic corrections for parafermions?

Let's look directly at some

1-D quantum models

Subtle Puzzle

- For $c=1$ CFT, it is by now established: $h = K$

$$S_\alpha(n) = \frac{1}{6}\left(\frac{1+\alpha}{\alpha}\right)\ln n + c'_\alpha + b_\alpha n^{-2K/\alpha} + \ldots$$

- Off criticality, expected?

$$S_\alpha = \frac{1}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln \xi + A_\alpha + B_\alpha \xi^{-K/\alpha} + \ldots$$

- Close to Heisenberg AFM point, observed

(Calabrese, Cardy, Peschel 2010)

$$S_\alpha = \frac{1}{12}\left(\frac{1+\alpha}{\alpha}\right)\ln \xi + A_\alpha + B_\alpha \xi^{-2/\alpha} + \ldots$$

$\rightarrow$ $h=2$? ($K=1/2$)

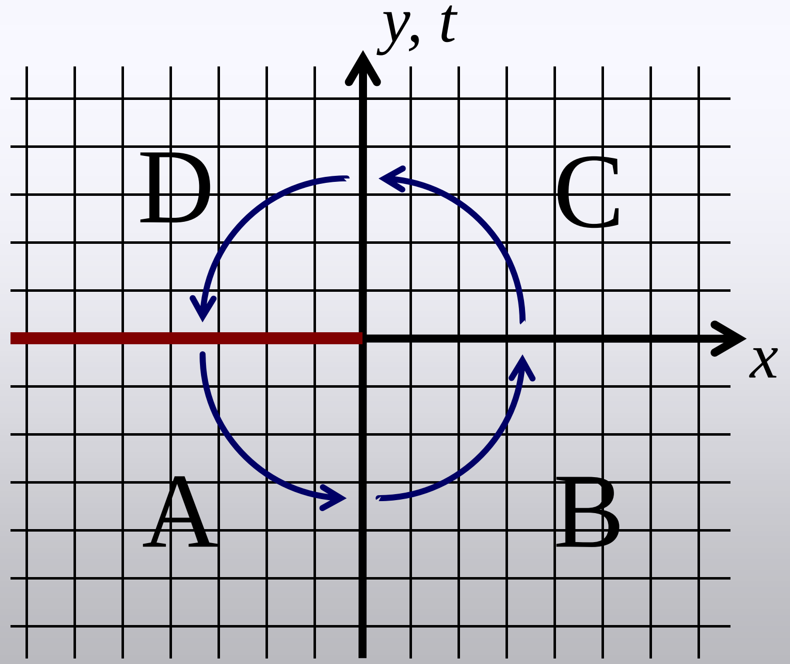
XYZ Spin Chain

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

- Commutes with transfer matrices of 8-vertex model
- Use of Baxter's Corner Transfer Matrices (CTM)

$$\mathcal{Z} = \mathrm{tr}\,(ABCD)$$

$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$



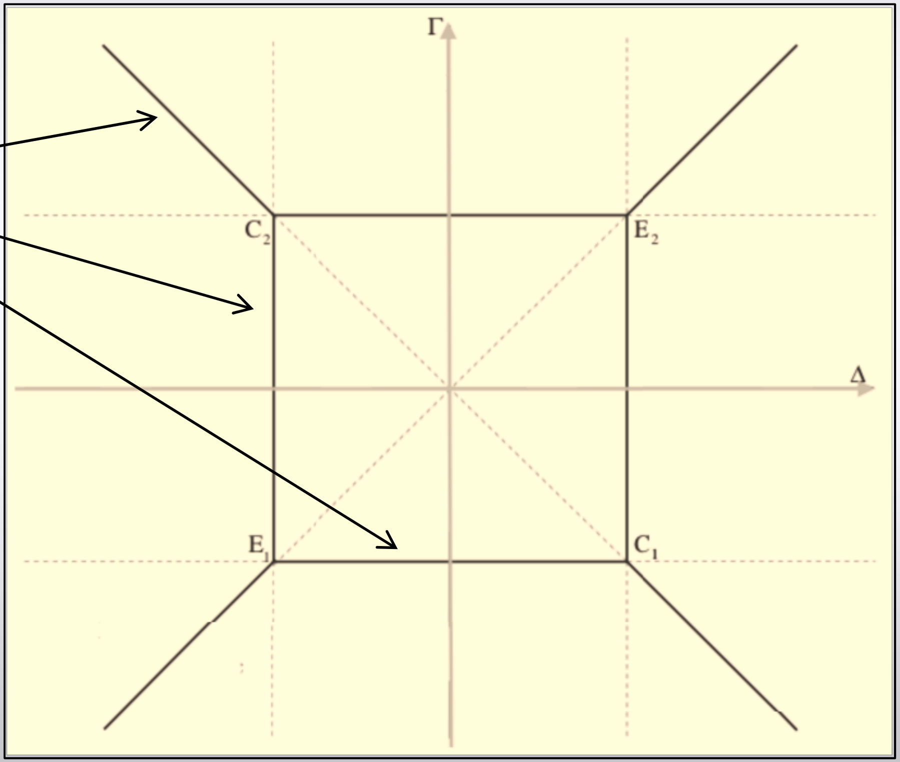
Phase Diagram of XYZ model

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

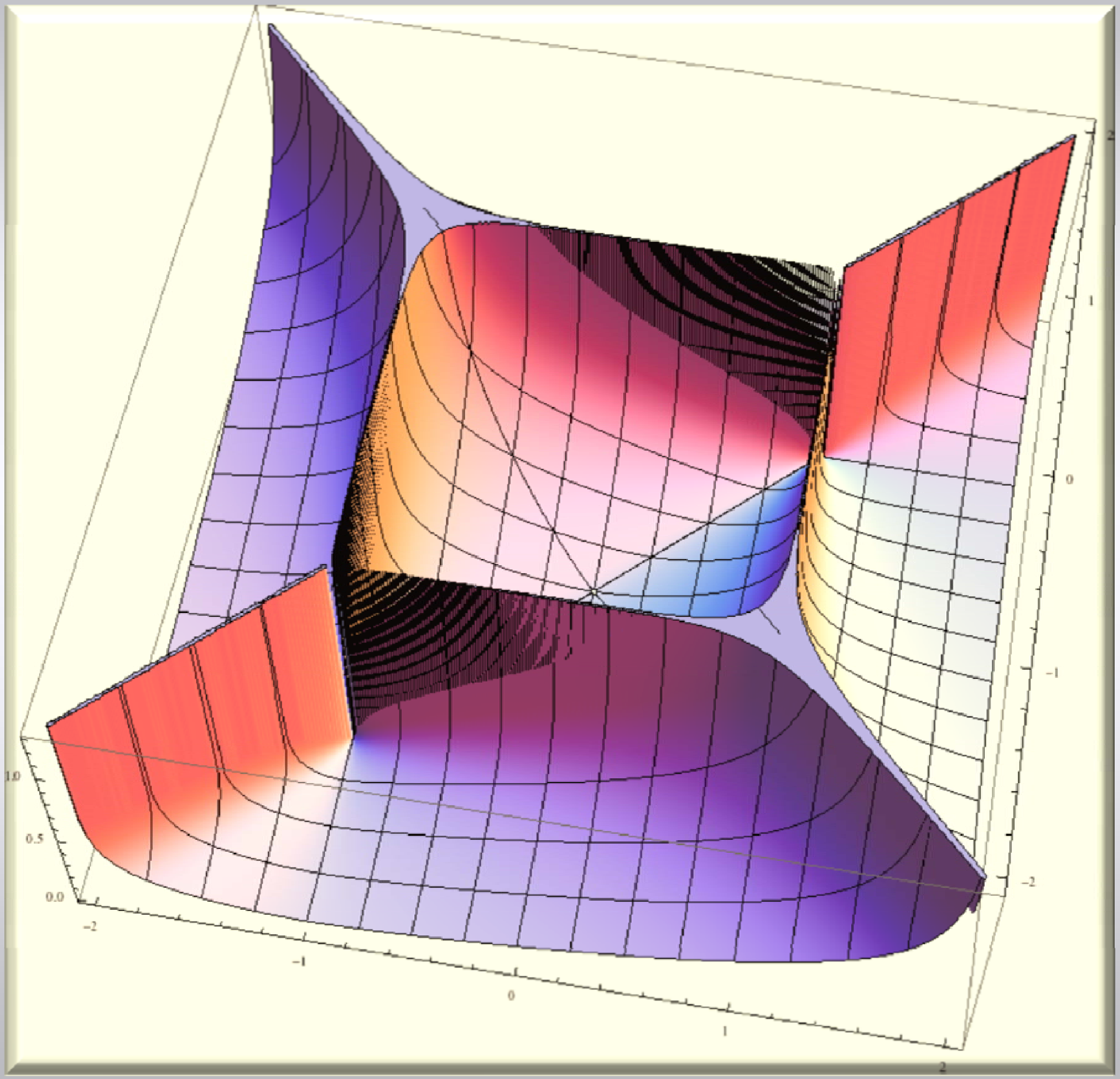
- Gapped in bulk of plane
- Critical on dark lines (rotated XXZ paramagnetic phases)
- 4 "tri-critical" points:

  $C_{1,2}$ conformal

  $E_{1,2}$ quadratic spectrum



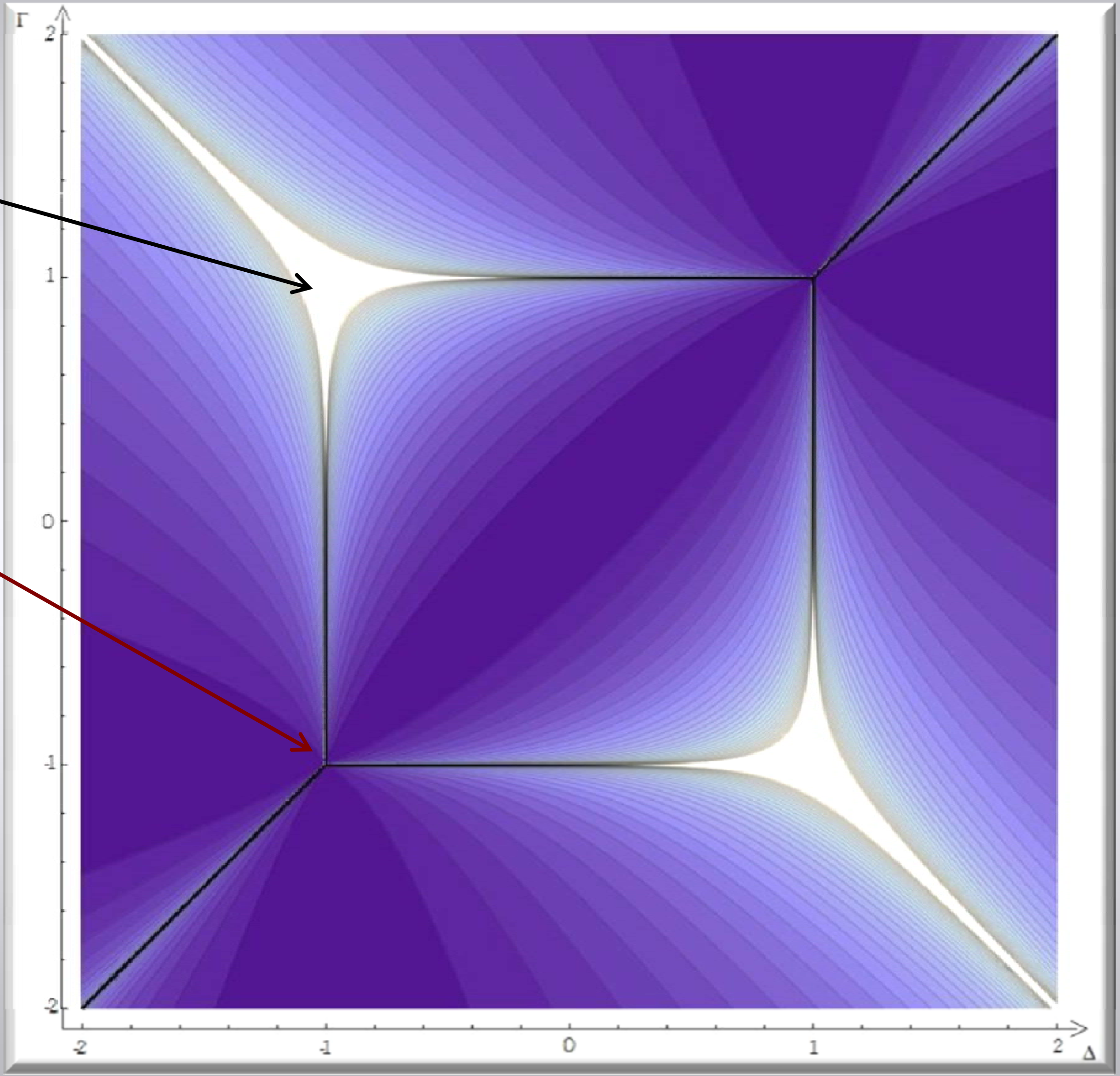
3-D plot of entropy



Iso-Entropy lines

- Conformal point: entropy diverges close to it

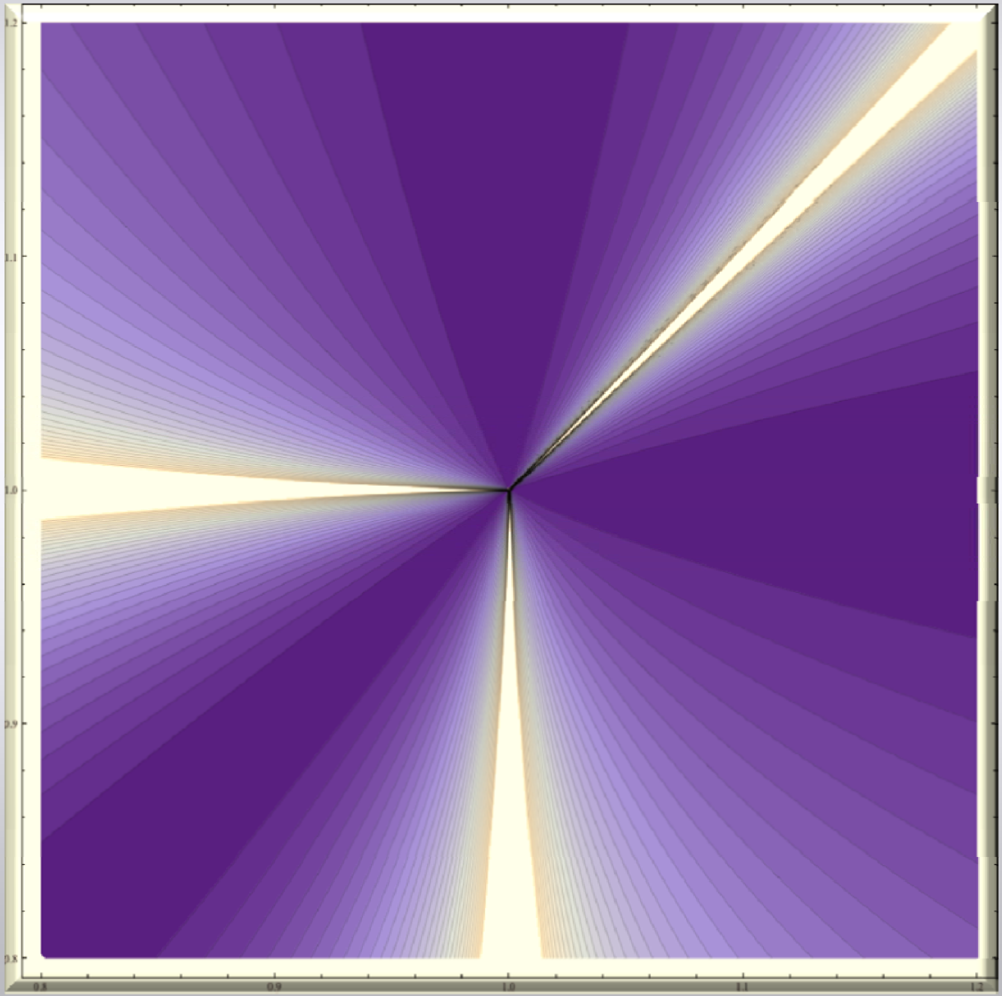
- Non-conformal point (ECP): entropy goes from 0 to ∞ arbitrarily close to it (depending on direction)



Close-up to non-conformal point

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

- Isotropic Ferromagnetic Heisenberg: quadratic spectrum
- Curves of constant entropy pass through it
- Similar physics as XY model

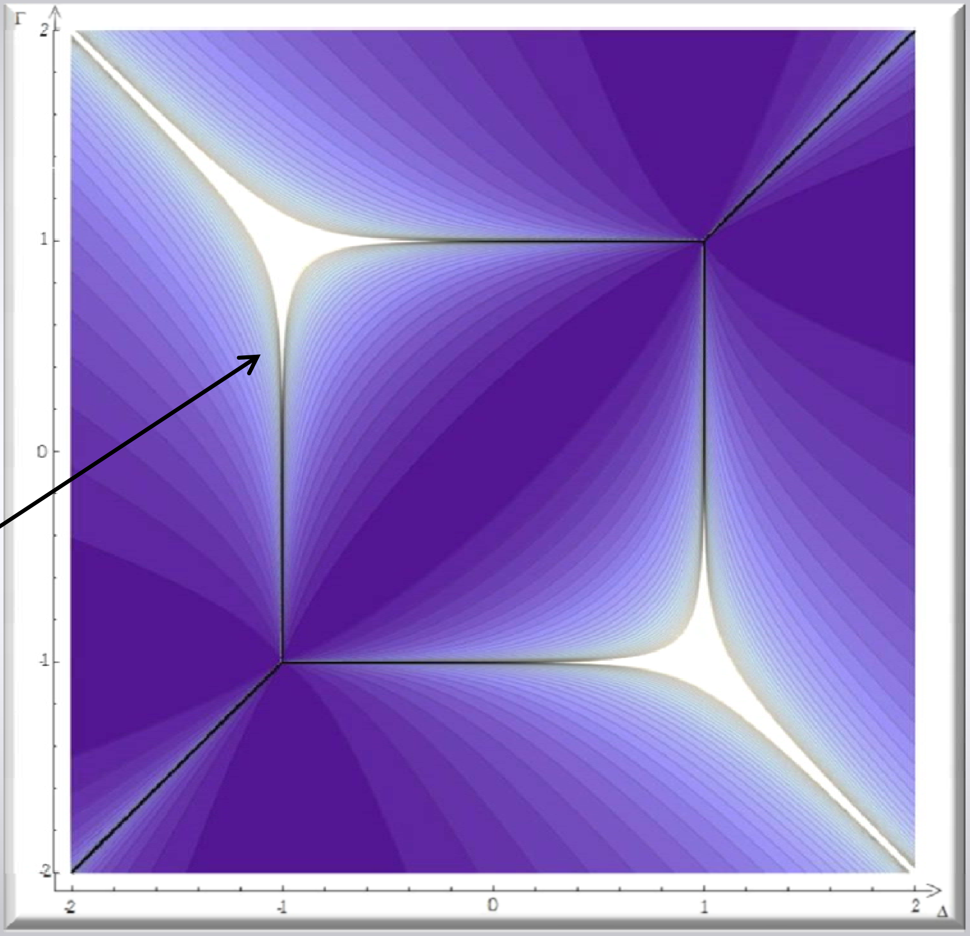


Conformal check

- Expansion close to conformal points agree with expectations:

$$S_\alpha = \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\ln(\xi) + \ldots$$

- Plus the corrections…

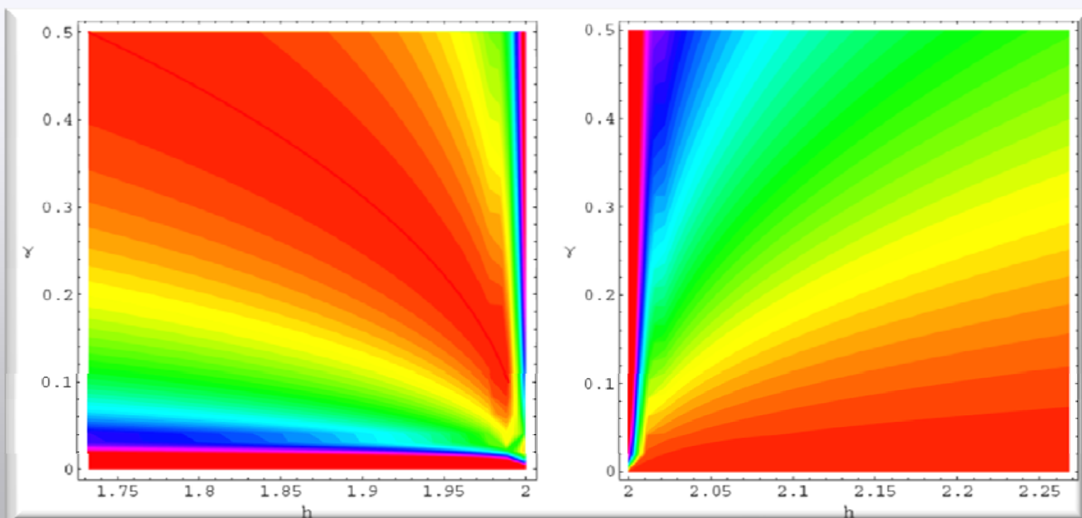


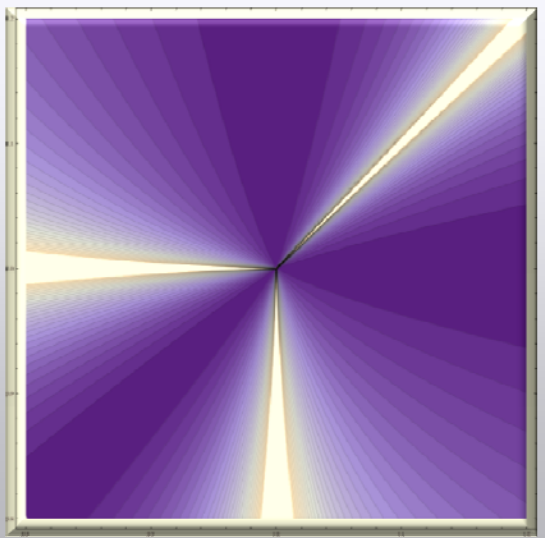
1^{st} round-up

- Gapped phases saturate
- Close to conformal points: logarithmic divergence
- Close to non-conformal points: <u>essential singularity</u> (entropy depends on direction of approach)

  → **Entanglement to discriminate non-conformal QPTs**

  (finite size scaling?)





The Reduced Density Matrix

- All spin 1/2 integrable chain systems have the same diagonal structure for ρ (Baxter's Book, Peschel et al 2009, …):

$$\rho_d = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_3 \end{pmatrix} \otimes \dots$$

$$x_j = \begin{cases} \mathrm{e}^{-2j\epsilon}, & \text{ordered} \\ \mathrm{e}^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

where ϵ is characteristic of the model

- Origin in CTM of 8-vertex model

Entanglement Spectrum

$$\rho_d = \bigotimes_{j=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & x_j \end{pmatrix} \qquad x_j = \begin{cases} e^{-2j\epsilon}, & \text{ordered} \\ e^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

- Eigenvalues form a geometric series
- Degeneracies from partitions of integers

  (Okunishi et al. 1999; Franchini et al. 2010; ...)
- All these models have the same entanglement spectrum

$$\rho = e^{-\mathcal{H}_{\text{Entanglement}}}$$

$\rightarrow \mathcal{H}_{\text{Entanglement}}$: free fermions with spectrum ϵ

- Microscopic of the model only in ϵ

Characters

- For integrable models: entropy reads characters
- CTM spectrum = Virasoro representation (Tokyo Group, Cardy...)

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha-1} \ln \prod_{j=1}^{\infty} \left(1 + x^{2j}\right) + \frac{1}{1-\alpha} \ln \prod_{j=1}^{\infty} \left(1 + x^{2j\alpha}\right)$$

$$x \equiv \mathrm{e}^{-\epsilon}$$

- For XYZ: $\displaystyle\prod_{j=1}^{\infty} \left(1 + x^{2j}\right) = x^{-\frac{1}{12}} \chi_{1/16}^{\mathrm{Ising}} \left(\mathrm{i}\epsilon/\pi\right)$
- Close to QPT: expansion in the S-dual variable: $\tilde{x} = \mathrm{e}^{-\pi^2/\epsilon}$

$$\prod_{j=1}^{\infty} \left(1 + x^{2j}\right) \propto \chi_0^{\mathrm{Ising}}(\mathrm{i}\pi/\epsilon) - \chi_{1/2}^{\mathrm{Ising}}(\mathrm{i}\pi/\epsilon)$$

Entropy & Characters

$$\boxed{\tilde{x} = \mathrm{e}^{-\pi^2/\epsilon}}$$

$$\begin{aligned} S_\alpha &= -\frac{1+\alpha}{24\alpha}\ln\tilde{x} - \frac{1}{2}\ln 2 \\ &\quad -\frac{1}{1-\alpha}\sum_{n=1}^{\infty}\sigma_{-1}(n)\left[\tilde{x}^{\frac{n}{\alpha}} - \alpha\tilde{x}^{n} - \tilde{x}^{\frac{2n}{\alpha}} + \alpha\tilde{x}^{2n}\right] \end{aligned}$$

Divisor function

$$\sigma_{-1}(n) \equiv \frac{1}{n}\sum_{\substack{j<k=1 \\ j\cdot k=n}}^{\infty}(j+k) + \sum_{\substack{j=1 \\ j^2=n}}^{\infty}\frac{1}{j}$$

- Need to express as universal paramter
- In scaling limit: $\tilde{x} \approx \xi^{-2}$

Entropy & Characters

$$S_\alpha = \frac{1+\alpha}{12\alpha}\ln\xi - \frac{1}{2}\ln 2 - \frac{1}{1-\alpha}\sum_{n=1}^{\infty}\sigma_{-1}(n)\left[\xi^{-\frac{2n}{\alpha}} - \xi^{-\frac{4n}{\alpha}} - \alpha\xi^{-2n} + \alpha\xi^{-4n}\right]$$

$$S_\alpha = \frac{1}{1-\alpha}\ln\frac{\mathcal{Z}(\xi^\alpha)}{\mathcal{Z}^\alpha(\xi)}$$

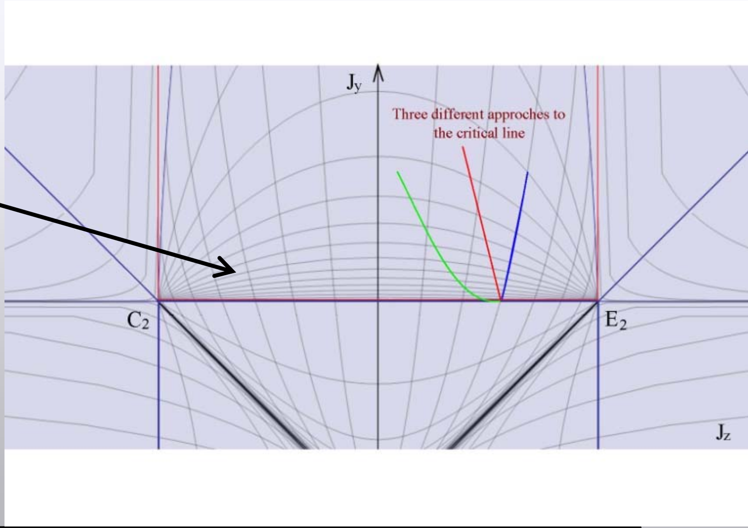
$$\mathcal{Z}(\xi) = \frac{1}{\sqrt{2}}x^{-\frac{1}{12}}\xi^{\frac{1}{12}}\prod_{k=1}^{\infty}\left(1-\xi^{1-2k}\right)\left(1+\xi^{1-2k}\right) = \frac{x^{-\frac{1}{12}}}{\sqrt{2}}\left[\left|\chi_0\right|^2 - \left|\chi_{1/2}\right|^2 - \chi_0\bar{\chi}_{1/2} + \chi_{1/2}\bar{\chi}_0\right] .$$

- Partition function of a Bulk Ising Model !

Lattice effects

- For a finite ultra-violet cut-off a_0: $\frac{a_0}{\xi} = f(\tilde{x}) \propto \tilde{x}^{1/2} + \dots$
- New subleading corrections to entanglement entropy
- Low energy states: free excitations

$$S_\alpha = \frac{1+\alpha}{12\alpha} \ln \frac{\xi}{a_0} + \frac{1-2\alpha}{6\alpha} \ln 2 + B_\alpha \xi^{-\frac{2}{\alpha}} + C_\alpha \xi^{-2\frac{1+\alpha}{\alpha}} + B'_\alpha \xi^{-\frac{4}{\alpha}} - \alpha B_\alpha \xi^{-2} - \alpha B'_\alpha \xi^{-4} + \dots$$

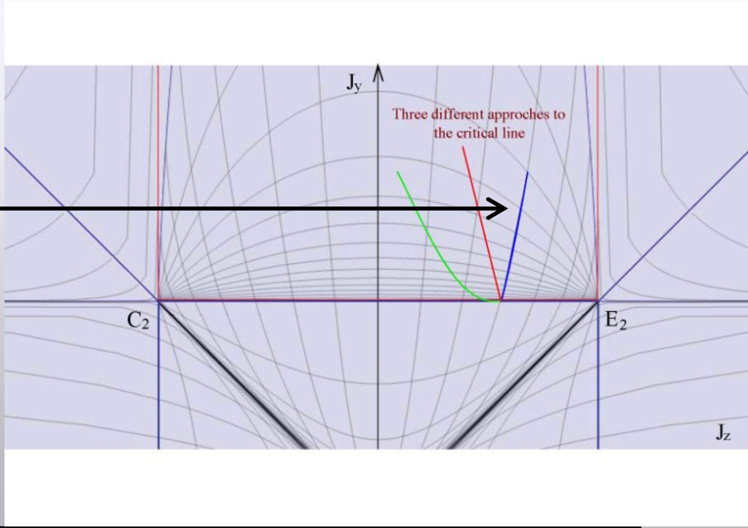


Lattice effects

- For a finite ultra-violet cut-off a_0: $\dfrac{a_0}{\xi} = f(\tilde{x}) \propto \tilde{x}^{1/2} + \dots$
- New subleading corrections to entanglement entropy
- Low energy states: bound states

  $\rightarrow$ direction dependent

$$
\begin{aligned}
S_\alpha \quad &= \quad \frac{1+\alpha}{12\alpha} \ln \xi + A_\alpha(\mu_0) + B_\alpha(\mu_0)\xi^{-\frac{2}{\alpha}} \\
&\quad -\alpha B_\alpha(\mu_0)\xi^{-2} + C_\alpha(\mu_0)\xi^{-2-\frac{2}{\alpha}} + \dots
\end{aligned}
$$



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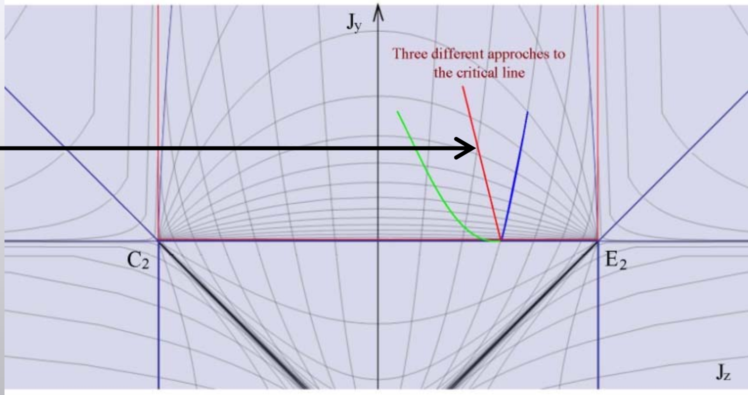
- New subleading corrections to entanglement entropy
- Low energy states: bound states

$\rightarrow$ direction dependent

$$S_\alpha = \frac{1+\alpha}{12\alpha} \ln \xi + A_\alpha(\mu_0) + B_\alpha(\mu_0)\xi^{-2/\alpha} + D_\alpha(m, \mu_0)\xi^{-(2-h)} \dots$$

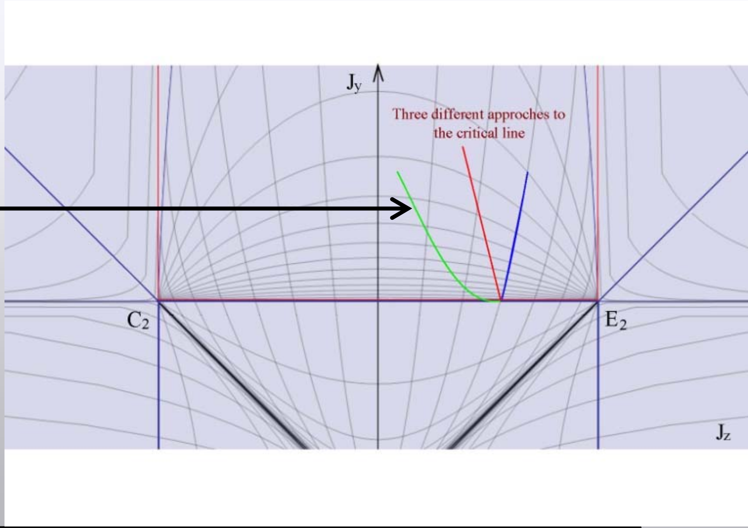


Lattice effects

- For a finite ultra-violet cut-off a_0: $\frac{a_0}{\xi} = f(\tilde{x}) \propto \tilde{x}^{1/2} + \ldots$
- New subleading corrections to entanglement entropy
- Low energy states: bound states

  $\rightarrow$ direction dependent

$$S_\alpha = \frac{1+\alpha}{12\alpha} \ln \xi + A_\alpha(\mu) + \frac{E_\alpha(r,u)}{\ln \xi} + \ldots$$



Conclusions & Outlook

- Analytical study of bipartite entanglement of 1-D integrable models: RSOS, the XY and XYZ models
- CTM/ reduced ρ spectrum & CFT: unusual corrections
- Logarithmic corrections in parafermions?
- Near non-conformal points, entropy has an essential singularity

  $\rightarrow$ Universality close to ECPs? Finite size scaling?
- Lattice corrections (logarithmic)
- Subleading corrections from (bulk) Ising model
- Relation between CTM & critical theory?