Universal Quantum Simulator, Local Convertibility and Edge States in Many-Body Systems

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Entanglement

- **Entanglement**: fundamental quantum property

- **Different reasons for interest**:
  1. Quantum information $\rightarrow$ quantum computers
  2. Quantum Phase Transitions $\rightarrow$ universality
  3. Condensed matter $\rightarrow$ non-local correlator
  4. Integrable Models $\rightarrow$ new playground
  5. Cosmology $\rightarrow$ Black Holes
  6. ...
Entanglement: what is it good for?

- Characterization of quantum states and how to simulate them (DMRG, MPS.....)
- Detection of novel quantum phases (topological phases)
- Can determine computational power of a quantum phase?
- Does a quantum phase transition change such comp. power?

→ Our answer: if QPT yields degeneracy from edge states
⇒ the long-range order of these boundary states gives phase a greater quantum computational power
Computational power

- Divide system into two subsystems A & B

- Consider only Local Operations & Classical Communications (LOCC) with respect to (A|B)

**Question:**

Can an adiabatic evolution be rendered through LOCC?

- If Yes, no advantage from quantum manipulation

- If Not, more powerful quantum phase!
Consider a pure ground state and bi-partition \((A|B)\).

If system wave-function:

\[
|\Psi^{A,B}\rangle = |\Psi^A\rangle \otimes |\Psi^B\rangle \quad \rightarrow \text{No Entanglement}
\]

\[
|\Psi^{A,B}\rangle = \sum_{j=1}^{D} \sqrt{\lambda_j} |\Psi^A_j\rangle \otimes |\Psi^B_j\rangle \quad \rightarrow \text{Entangled}
\]

(with \(D > 1, |\Psi^A_j\rangle \& |\Psi^B_j\rangle \text{ linearly independent}): 

Entangled: Measurements on \(B\) affect \(A\)
Von Neumann & Renyi Entropies

\[ |\Psi^{A,B}\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j} |\Psi^A_J \rangle \otimes |\Psi^B_J \rangle \]

\[ \rho_A = \text{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}| = \sum \lambda_j |\Psi^A_J \rangle \langle \Psi^A_J| \]

- **Von Neumann** (Quantum analog of Shannon Entropy):

\[ S = -\text{tr}_A (\rho_A \log \rho_A) = - \sum \lambda_j \log \lambda_j \]

- **Renyi Entropy \rightarrow Entanglement spectrum**

\[ S_\alpha = \frac{1}{1-\alpha} \log \text{tr} (\rho_A^\alpha) = \frac{1}{1-\alpha} \log \sum_j \lambda_j^\alpha \]

(equal to Von Neumann for \( \alpha \rightarrow 1 \))
• Consider bi-partite states \((A \mid B)\): \(|\Psi_{A,B}\rangle \& |\Phi_{A,B}\rangle\)

• **Entanglement cannot increase** under Local Operations &

  **Classical Communications (LOCC)**

\[ \Rightarrow \text{ if } S_\alpha ([\Phi]) < S_\alpha ([\Psi]) \quad \forall \alpha \]

\(|\Psi_{A,B}\rangle \text{ can be converted to } |\Phi_{A,B}\rangle \text{ but not vice-versa!} \]

( Depends upon partition choice! )

• **A state can only be converted to one of lower entanglement**

  S. Turgut JPA (2007)
Local Convertibility

- Take two bipartite states: $|\Psi_{A,B}\rangle \& |\Phi_{A,B}\rangle$

- If $\exists \alpha_1$ such that $S_{\alpha_1} ([\Phi]) < S_{\alpha_1} ([\Psi]) \&$
  $\exists \alpha_2$ such that $S_{\alpha_2} ([\Phi]) > S_{\alpha_2} ([\Psi])$

$\Rightarrow$ the two states cannot be transferred locally (by LOCC) one into the other
Local Convertibility & Adiabatic Evolution

- Adiabatic evolution: $|\Psi_{A,B}\rangle$ ground state of $H(g)$ and $|\Phi_{A,B}\rangle$ ground state of $H(g + \Delta g)$

• Study Renyi entropy derivative w.r.t $g$ as function of $\alpha$

$S_\alpha(g)$ monotonous

$S_\alpha(g)$ non-monotonous

• Study Renyi entropy derivative w.r.t $g$ as function of $\alpha$

→ Differential Local Convertibility
Adiabatic evolution: Renyi entropy of instantaneous ground state of Hamiltonian $H(g)$ as function of $g$ and $\alpha$

If $\frac{dS_\alpha}{dg}$ changes sign as $\alpha$ varies

$\Rightarrow$ LOCC cannot simulate evolution

**Sign of entropy derivative**

distinguishes computational power of different phases
• Naively, we expect all entanglement entropies to increase with the correlation length

\[ |\Psi^{A,B}\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j} |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle \quad \Rightarrow \quad \rho_A = \sum \lambda_j |\Psi_j^A\rangle \langle \Psi_j^A| \]

\[ S_{\alpha} = \frac{1}{1-\alpha} \log \text{tr} (\rho_A^{\alpha}) = \frac{1}{1-\alpha} \log \sum \lambda_j^{\alpha} \]

• Approaching a QPT, scale invariance require more eigenvalues to contribute equally:

\[ \text{Tr} \rho_A = \sum_{j=1}^{D} \lambda_j = 1 , \quad \Rightarrow \quad \lambda_j \approx \frac{1}{D} \]
Quantum phases with differing computational power

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Numerical Results

\[ H_I = - \sum_{j=1}^{N} \left( \sigma_j^x \sigma_{j+1}^x + g \sigma_j^z \right) \]

- Ising model for \( N=12 \) and bipartitions \((6|6), (7|5), (8|4)\)
- Sign of entropy derivative:  
  Blue = Negative; Red = Positive
- Ferromagnetic phase more powerful for adiabatic quantum computation!
- Not true for large subsystems!
Local Convertibility & Topological Order

Entanglement and Quantum Computation in Ising Chain

Local characterization of one-dimensional topologically ordered states

Jian Cui,1,2 Luigi Amico,3,4 Heng Fan,1 Mile Gu,4,5 Alioscia Hamma,5,6 and Vlatko Vedral4,7,8

• Cluster Ising Model:

\[ H(g) = -\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + g \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y \]

(50|50)

(48|3|49)

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Local Convertibility & Topological Order

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Local characterization of one-dimensional topologically ordered states

Jian Cui,1,2 Luigi Amico,3,4 Heng Fan,1 Mile Gu,4,5 Alioscia Hamma,5,6 and Vlatko Vedral4,7,8

• The $\lambda$-D Model:

$$H = \sum_i \left[ (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \lambda S_i^z S_{i+1}^z + D(S_i^z)^2 \right]$$

(a1) (a2) (a3) (a4) (b1) (b2) (b3) (b4)

(50|50) (96|4)
Local Convertibility & Topological Order

Local Response of Topological Order to an External Perturbation

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• Perturbed 2-D Toric Code:

$$\mathcal{H} = -\sum_s \prod_i \sigma_i^x - \sum_p \prod_i \sigma_i^z + V(\lambda)$$

<table>
<thead>
<tr>
<th>Perturbation $V(\lambda)$</th>
<th>G.I.</th>
<th>DLC</th>
<th>Exact</th>
<th>$\xi$</th>
</tr>
</thead>
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<td>$\sum_s e^{-\lambda_s \sum_i \sigma_i^z}$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>0</td>
</tr>
<tr>
<td>$\lambda_h \sum_{i \in H} \sigma_i^z$</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>$\lambda_z \sum_i \sigma_i^z$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>$\lambda_z \sum_i \sigma_i^z + \lambda_x \sum_j \sigma_j^z$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>$\neq 0$</td>
</tr>
</tbody>
</table>
The Quantum Ising Chain

\[ H_I = - \sum_{j=1}^{N} \left( t \sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right) \]

\[ H_I = \sum_q \varepsilon_q \left( \chi_q^\dagger \chi_q - \frac{1}{2} \right), \quad \varepsilon_q = \sqrt{t^2 + h^2 - 2ht \cos q} \]

\[
\begin{cases} 
  h/t > 1 \rightarrow \langle \sigma^x \rangle = 0 & \text{Paramagnetic phase} \\
  h/t < 1 \rightarrow \langle \sigma^x \rangle \neq 0 & \text{Ferromagnetic phase} \\
  h/t = 1 & \text{Ising QPT: } c=1/2 
\end{cases}
\]
Kitaev Chain

\[ H_I = -\sum_{j=1}^{N} \left( t \sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right) = -\sum_{j=1}^{N} \left( t f_j^{(2)} f_{j+1}^{(1)} + h f_j^{(1)} f_j^{(2)} \right) \]

Majorana Fermion \( f_j^{(1)} = \sigma_j^x \prod_{l<j} \sigma_l^z \)

Majorana Fermion \( f_j^{(2)} = \sigma_j^y \prod_{l<j} \sigma_l^z \)

For large \( h/t \):

\[ c_j = f_j^{(1)} + i f_j^{(2)} \]

For small \( h/t \):

\[ c_j = f_{j+1}^{(1)} + i f_j^{(2)} \]

Entanglement and Quantum Computation in Ising Chain

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Edge States

\[ H_I = - \sum_{j=1}^{N} \left( t \sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right) \]

\[ = - \sum_{j=1}^{N} \left( t f_j^{(2)} f_{j+1}^{(1)} + h f_j^{(1)} f_{j+1}^{(2)} \right) \]

\[
\frac{h}{t} > 1 \quad \text{subsystem B} \quad \boxed{\text{subsystem A}} \quad \text{subsystem B}
\]

\[
\frac{h}{t} < 1 \quad \text{subsystem B} \quad \boxed{\text{subsystem A}} \quad \text{subsystem B}
\]
• Edge states combined into a complex fermion:
  occupied/empty ⇒ two-fold degeneracy

→ Long-range entanglement among edge states

• Edge states also generated by partitioning

• Grow closer as correlation length increases
EPR Analogy

\[ S = 0 \]

\[ S = \ln 2 \]

\[ 0 < S < \ln 2 \]

- Approaching the QPT, edge states effectively grow closer

\[ |0\rangle \]

\[ |\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle \]

\[ j \# \rangle \]

\[ \text{their entanglement can decrease} \]

(while bulk states entanglement increases)
Conclusions

• Entanglement derivative to study non-local convertibility

• Local way of detecting long-range entanglement!

• Edge state recombination explains it

• Approaching a QPT:
  1. Correlation length increases
  2. Bulk states entanglement increases
  3. Edge states entanglement decreases

• Universal quantum simulator cannot be locally convertible

Thank you!
• Certain problems too complex for classical computers: factorization, searches, simulation of quantum systems...

• Quantum algorithms give exponential speed-up, but implementation of quantum computers is hard

• Quantum systems as computers

→ Universal quantum simulator
Quantum Adiabatic Algorithm

- Ground state of $H_I$ is the output of given problem
- Start from ground state of easy Hamiltonian $H_0$
- Adiabatically evolved it to desire state

$$H(t) = \left( 1 - \frac{t}{T} \right) H_0 + \frac{t}{T} H_I$$

- If velocity sufficiently small ($T \ll \Delta_{\text{min}}^{-2}$), system stays in instantaneous ground state
Computational power

• Any efficient quantum algorithm can be casted as a Quantum Adiabatic Algorithm

• Adiabatic evolution performs quantum computation

  → computational power of a quantum phase

• How to extract this computational power

  → Entanglement!
Entropy as a measure of entanglement

- Assume Bell State as unity of Entanglement:

\[ |\text{Bell}\rangle = \frac{|\downarrow \downarrow\rangle \pm |\uparrow \uparrow\rangle}{\sqrt{2}} , \frac{|\downarrow \uparrow\rangle \pm |\downarrow \uparrow\rangle}{\sqrt{2}} \]

- Von Neumann Entropy measures how many Bell-Pairs can be distilled using LOCC from a given state \(|\Psi^{A,B}\rangle\) (i.e. closeness of state to maximally entangled one)

What can entanglement entropy teach us about a system?
\[ H_I = -\sum_{j=1}^{N} \left( t \sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right) \]

- Ising model: prototype of $\mathbb{Z}_2$ symmetry
- Realized non-locally: string order parameter: 
  \[ \mu_N^x = \prod_{j=1}^{N} \sigma_j^z \]
- Eigenstates with $\mathbb{Z}_2$ symmetry: \( \langle \sigma^x \rangle = 0 \) \text{ → thermal ground state}
- Symmetry broken states: \( \langle \sigma^x \rangle \neq 0 \)
Entanglement

\[ |0\rangle = \sum_{\kappa=1}^{2^L} \sqrt{\lambda_\kappa} |\Psi^A_\kappa\rangle \otimes |\Psi^B_\kappa\rangle \]

\[ S_\alpha = \frac{1}{1 - \alpha} \log \sum_\kappa \lambda_\kappa^\alpha \]

- **Quadratic Theory:** Block eigenstates from block excitations

\[ |\Psi^A_\kappa\rangle = |n_1, n_2, \ldots, n_L\rangle, \quad n_l = 0, 1 \]

\[ \lambda_\kappa = |\langle \Psi^A_\kappa |0\rangle|^2 = \prod_{j=1}^{L} \langle 0|n_j\rangle \langle n_j|0\rangle \]

- Measure overlap of block excitations with G.S.:

**Whole system excitations:**

\[ c_j, c_j^\dagger \rightarrow c_j |0\rangle = 0 \]

**Block excitation:**

\[ \tilde{c}_l, \tilde{c}_l^\dagger \rightarrow \tilde{c}_j |0\rangle \neq 0 \]
Entanglement

\[ |0\rangle = \sum_{\kappa=1}^{2^L} \sqrt{\lambda_\kappa} |\Psi_\kappa^A\rangle \otimes |\Psi_\kappa^B\rangle \]

\[ S_\alpha = \frac{1}{1 - \alpha} \log \sum_{\kappa} \lambda_\kappa^\alpha \]

- **Block excitations from correlation matrix:**

  \[ \langle f_k^{(a)} f_j^{(b)} \rangle = \delta_{j,k} \delta_{a,b} + i (B_L)^{(a,b)}_{(j,k)} \]

  \[ \langle 0|0_j\rangle \langle 0_j|0 \rangle = \langle 0|\tilde{c}_j \tilde{c}_j^\dagger |0 \rangle = \frac{1 + \nu_j}{2} \]

  \[ \langle 0|1_j\rangle \langle 1_j|0 \rangle = \langle 0|\tilde{c}_j^\dagger \tilde{c}_j |0 \rangle = \frac{1 - \nu_j}{2} \]

\[ \lambda_\kappa = \prod_{j=1}^{L} \langle 0|n_j\rangle \langle n_j|0 \rangle = \prod_{j=1}^{L} \left( \frac{1 \pm \nu_j}{2} \right) \]


**Overlap between block excitations and ground state**
Correlation Matrix Eigenvalues

\[ \langle f_k^{(a)} f_j^{(b)} \rangle = \delta_{j,k} \delta_{a,b} + i (B_L)^{(a,b)}_{(j,k)} \]

- One edge state for \( h < 1 \): partial overlap
- Approaching QPT: bulk states overlap decreases, edge states overlap increases (edge state recombination)

\[ \langle 0 | d_j d_j^\dagger | 0 \rangle = \frac{1 + \nu_j}{2} \]
\[ \langle 0 | d_j^\dagger d_j | 0 \rangle = \frac{1 - \nu_j}{2} \]
2-Sites Block Entanglement

- Lack of **local convertibility** due to edge state recombination
- 2-sites classical gates destroy long-range correlations!
Large Block Entanglement

- For $L \to \infty$ we have **full analytical knowledge of entanglement (spectrum)**: Its & al. (2005); F.F. & al. (2008); F.F & al. (2011)
- For $h/t < 1$ edge states give **double degeneracy**
- **Local convertibility restored!**
- **Numerics confirm**
Symmetry broken Ground State

- So far, ground state as eigenstate of $\mathbb{Z}_2 : \mu_N^x = \prod_{j=1}^{N} \sigma_j^z$

- For $h < 1, \langle \sigma^x \rangle \neq 0$ : symmetry broken state
  → no edge states → locally convertible!

- No analytical approaches, just numerics

$Z_2 : \mu_N^x = \prod_{j=1}^{N} \sigma_j^z$

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$Z_2 : \mu_N^x = \prod_{j=1}^{N} \sigma_j^z$
Conclusions

• Non-local convertibility from entanglement derivative

• Local way of detecting long-range entanglement!

• Edge state recombination explains it

• Approaching a QPT:
  1. Correlation length increases
  2. Bulk states entanglement increases
  3. Edge states entanglement decreases

• Universal quantum simulator cannot be locally convertible

Thank you!
Entanglement Spectrum

First few eigenvalues of the reduced density matrix (multiplicities not shown)

Finite Size Numerical results

Theromodynamic Limit Analytical results
• Entropy depends on single parameter $\varepsilon$

• $\varepsilon$ vanishes at phase transitions, large in gapped phase

• Microscopics of the model through $\varepsilon(k)$

\[
\langle \sigma^x \rangle = 0 \quad \text{(2)} \quad \Gamma_i
\]

\[
\langle \sigma^x \rangle \neq 0 \quad \text{(1a)} \quad \Omega_o
\]

\[
\text{h} > 1 : \quad k = \frac{\gamma}{\sqrt{h^2 + \gamma^2 + 1}} \rightarrow \frac{dk}{dh} < 0
\]

\[
\text{h} < 1 : \quad k = \frac{\sqrt{h^2 + \gamma^2 + 1}}{\gamma} \rightarrow \frac{dk}{dh} > 0
\]
Entanglement Derivative

$\frac{dS_R}{dk}$

**Paramagnetic Phase**

$\times \frac{dk}{dh} < 0$

**Ferromagnetic Phase**

$\times \frac{dk}{dh} > 0$

Entanglement and Quantum Computation in Ising Chain

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L-spins subsystem

- Diagonalize $L \times L$ Hankel matrix:

$$\tilde{B} = \begin{pmatrix} g_{L-1} & g_{L-2} & \cdots & g_0 \\ g_{L-2} & g_{L-3} & \cdots & g_{-1} \\ \vdots & \ddots & \vdots & \vdots \\ g_0 & g_{-1} & \cdots & g_{1-L} \end{pmatrix}, \quad g_j \equiv \frac{1}{2\pi} \int_{0}^{2\pi} e^{ij\theta} \frac{\cos \theta - h + i\gamma \sin \theta}{\sqrt{(\cos \theta - h)^2 + \gamma^2 \sin^2 \theta}} \, d\theta$$

- Use $L$ eigenvalues $\lambda_j$ to compute Renyi entropy as sum of entropies of 2-levels systems:

$$S(\alpha) = \frac{1}{1-\alpha} \sum_{l=1}^{L} \ln \left[ \left( \frac{1 + \lambda_l}{2} \right)^\alpha + \left( \frac{1 - \lambda_l}{2} \right)^\alpha \right]$$

$$dS(\alpha) = \frac{\alpha}{1-\alpha} \sum_{l=1}^{L} \frac{(1 + \lambda_l)^{\alpha-1} - (1 - \lambda_l)^{\alpha-1}}{(1 + \lambda_l)^\alpha + (1 - \lambda_l)^\alpha} \, d\lambda_l$$
2-spins subsystem: Ising line

- Entropy derivative vanishes in ferromagnetic phase!
  \[ \Rightarrow \text{ the two phases have different computation power!} \]
- Role of Majorana edge states?