Spontaneous Breaking of U(N) Symmetry in Invariant Matrix Models

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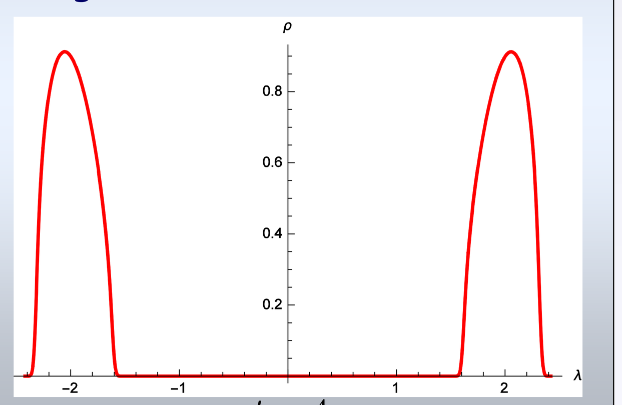
Abstract

Matrix Models have a strong history of success in describing a variety of situations, from nuclei spectra to conduction in mesoscopic systems, from strongly interacting systems to various aspects of mathematical physics. Traditionally, the requirement of base invariance has lead to a factorization of the eigenvalue and eigenvector distribution and, in turn, to the conclusion that invariant models describe extended systems. I show that deviations of the eigenvalue statistics from the Wigner-Dyson universality (in the form of a gap) reflects itself on the eigenvector distribution and that the phase transition observed when the eigenvalue density become disconnected corresponds to a breaking of the U(N) symmetry to a smaller one. This spontaneous symmetry breaking means that the system looses ergodicity, with implications on localization problems, as well as for fundamental theories

Double Well Matrix Models

$$V_{2W}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

- Disjoint (two-cuts) support of eigenvalue distribution for $t > 2$
- Half of eigenvalues around each minima $\pm t$ (assume N even)
- U(N) symmetry broken into U(N/2) x U(N/2)



$t = 4$

Introduction

- Base invariant matrix models:

$$d\mu(\mathbf{M}) = e^{-N \mathrm{Tr} V(\mathbf{M})}\, d\mathbf{M} = e^{-N\sum_j V(\lambda_j)} \prod_j d\lambda_j\, d\mathbf{U}$$

- Eigenvector distribution independent from weight $V(x)$

  $\Rightarrow$ Uniform eigenvector distribution

  $\Rightarrow$ Delocalized phases, Porter-Thomas Distribution

$$\mathcal{P}\left(|U_{ij}|^2\right) = N \exp\left[-N|U_{ij}|^2\right]$$

- Localization by non-invariant ensembles (Banded Matrices)

$$d\mu(\mathbf{M}) \propto e^{-\sum_{j,l} A_{jl}|M_{jl}|^2} \Rightarrow \langle M_{nm}^2 \rangle = A_{nm}^{-1}$$

  - $A_{nm} = e^{|n-m|/B}$ Localized (Fyodorov & Mirlin, PRL '91)
  - $A_{nm} = 1 + \frac{(n-m)^2}{B^2}$ Critical (multi-fractal) (Evers & Mirlin, PRL '00)

- But limited tractability (numerics or perturbative regimes)

Can a non-trivial (non Wigner-Dyson) eigenvalue distribution trigger a spontaneous breaking of U(N) symmetry and lead to (partially) localized eigenstates?

- Like for a ferromagnet, base invariance means that no direction over the N-dimensional unit sphere of the Hilbert space is preferred, but a gap in the eigenvalue distribution freezes the motion of eigenvectors in certain directions

  $\Rightarrow$ U(N) SSB breaks ergodicity

General Considerations

- The de Haar measure not flat in eigenvalue-eigenvector coordinates

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 \left| \left(\mathbf{U}^\dagger d\mathbf{U}\right)_{jl} \right|^2$$

- If two eigenvalues are distant, even a small angular change can produce a large ds
- Dyson Brownian motion representation

Delta-Correlated, independent, stochastic sources

$$d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}dt + \frac{\beta}{2N}\sum_{l\neq j}\frac{dt}{\lambda_j - \lambda_l} + \frac{1}{\sqrt{N}}dB_j$$

$$d\vec{\psi}_j = -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j-\lambda_l)^2}\vec{\psi}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}}{\lambda_j - \lambda_l}\vec{\psi}_l$$

- If two sets of eigenvalues are separated by a gap of the order of unity, the evolution of the eigenvectors toward the subspace spanned by eigenvectors belonging to the distant eigenvalues is suppressed

  $\Rightarrow$ eigenvectors cannot spread ergodically over the whole Hilbert space

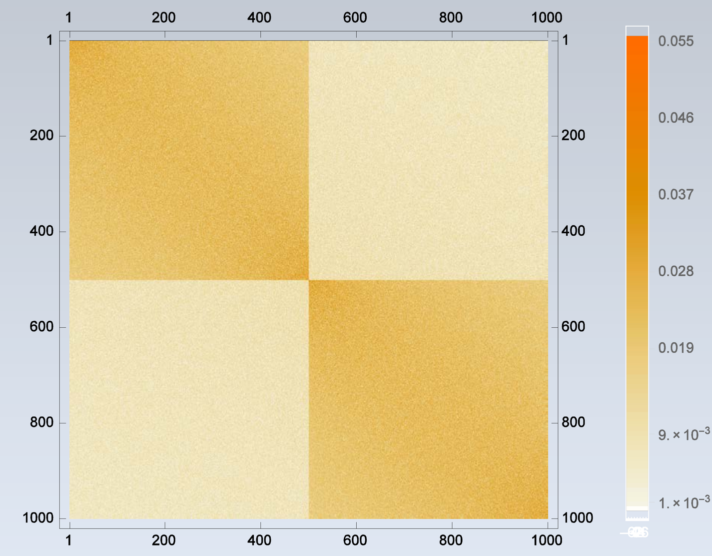
Effect of a Small Perturbation: The Double Well Case

- How do eigenvectors respond to a perturbation?

$$\left.\begin{array}{r}\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U} \\ \mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'\end{array}\right\} \tilde{\mathbf{U}} = \mathbf{U}'\mathbf{U}^\dagger$$

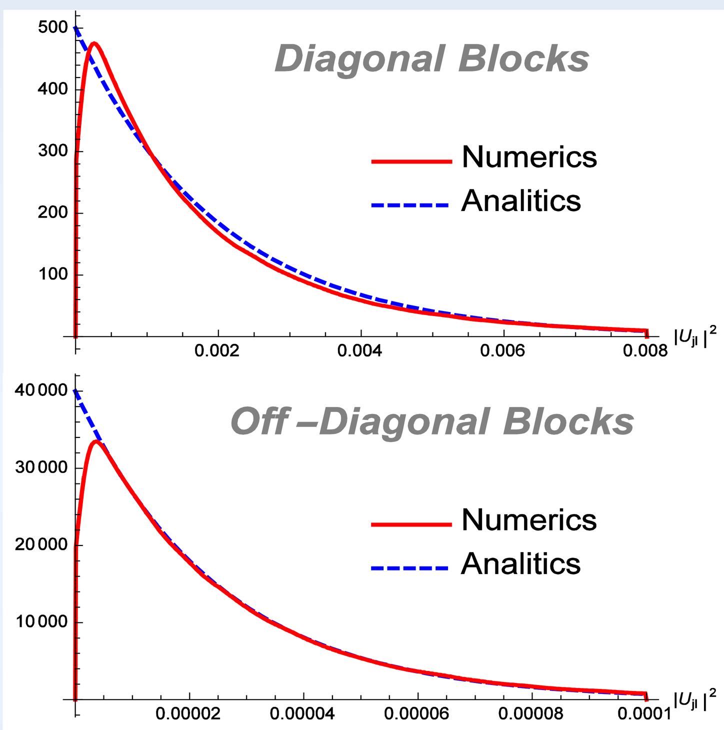
Order N non-zero elements independently sampled from a Gaussian distribution (mean 0 , width $\sqrt{N}$)

- We study the perturbed eigenvectors in the basis where $\mathbf{M}$ is diagonal
- Block structure $\Rightarrow$ SSB



- Diagonal and Off-diagonal elements follow two different distributions

$$\chi_D = \frac{N}{2} \qquad \mathcal{P}\left(|U_{ij}|^2\right) = \chi e^{-\chi |U_{ij}|^2} \qquad \chi_{OD} = \frac{2tN^2}{n}$$

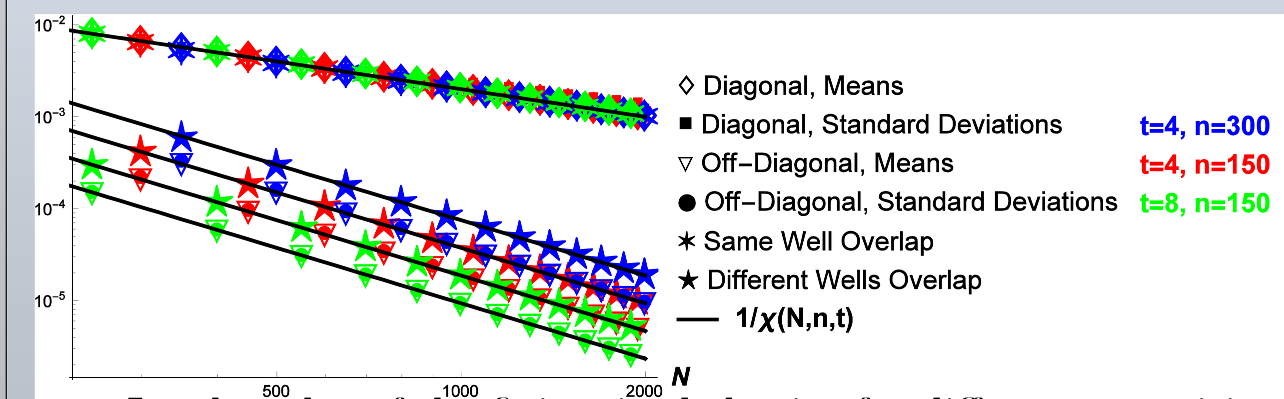


Distribution of diagonal and off-diagonal elements of a typical unitary matrix ($\mathbf{\Delta M}$ has $N \times n$ non-zero elements, with $N = 1000$, $n = 150$; $t = 4$)

- Overlap between eigenstates: $O_{jl} = \sum_m^N \left|\tilde{U}_{mj}\right|^2 \left|\tilde{U}_{ml}\right|^2$

$$\langle |O_{jl}| \rangle_D = \langle |\tilde{U}_{jl}| \rangle_D = \langle |\Delta\tilde{U}_{jl}| \rangle_D = \frac{1}{\chi_D}$$

$$\langle |O_{jl}| \rangle_{OD} = 2\langle |\tilde{U}_{jl}| \rangle_{OD} = 2\langle |\Delta\tilde{U}_{jl}| \rangle_{OD} = \frac{2}{\chi_{OD}}$$



Log-log plot of the finite size behavior for different quantities, averaged over several realizations of the applied perturbation: notice the remarkable aggreement with the analytical expectations

- Off-diagonal blocks suppressed by a power of N: in the thermodynamic limit the eigenvectors are localized over a N/2-dimensional sphere

Symmetry Breaking Term: Double Well Case

- Introduce an explicit symmetry breaking term
- Want to favor alignment of the eigenvectors of $\mathbf{M}$ along those of the given Hermitian matrix $\mathbf{S}$
- Most natural choice: $V_{\mathrm{br}} = \mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$ (but too complicated to handle)
- We use

$$d\mu(\mathbf{M}) \propto e^{-N\mathrm{Tr}\left[V_{2W}(\mathbf{M}) + J|\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}|\right]} \qquad \mathbf{M} = \mathbf{U}^\dagger\mathbf{\Lambda}\mathbf{U}, \quad \mathbf{S} = \mathbf{V}^\dagger\mathbf{T}\mathbf{V}$$

- Double well case: $\mathbf{S}$ with two sets of N/2 degenerate eigenvalues $\pm\tau$
- Study the generating function

$$W(J) = \ln\int d\mathbf{M}e^{-N\mathrm{Tr}\left[V_{2W}(\mathbf{M}) + J|\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}|\right]}$$

- To calculate the order parameter:

$$\left.\frac{dW(J)}{dJ}\right|_{J=0} = \langle \mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}\rangle \qquad \boxed{\begin{array}{c}\lim_{N\to\infty}\lim_{J\to 0}\frac{dW}{dJ} \neq 0 \\ \lim_{J\to 0}\lim_{N\to\infty}\frac{dW}{dJ} = 0\end{array}}$$

- Remark: order parameter vanishes for symmetry broken

  $\Rightarrow$ U(N) symmetry broken into U(N/2) x U(N/2)

- Corrections to SSB as $e^{-N\,J\,\left|\lambda_j^{(1)} - \lambda_l^{(2)}\right|}$: contributions from instantons exchanging two eigenvalues between the wells

  $\rightarrow$ instantons progressively restore the broken symmetries, but are suppressed for large N (and large distances)

Technical Aspects

- Itzykson-Zuber formula to integrate over U(N)

$$\mathcal{Z} = \int d\mathbf{U} e^{\mathrm{Tr}\mathbf{AUBU}^\dagger} = \frac{\det\left[e^{a_j b_l}\right]}{\Delta(\{a\})\Delta(\{b\})}, \qquad \Delta(\{a\}) = \prod_{j<l}(a_j - a_l)$$

- With degeneracies needs to be regularized
- Double well case:

  $\mathbf{B}$ has two sets of N/2 degenerate eigenvalues$\pm b$

$$\mathcal{Z} = \frac{\det\begin{bmatrix} a_l^{j-1}e^{b\,a_l} \\ a_l^{j-1}e^{-b\,a_l}\end{bmatrix}}{(2b)^{N^2/4}\left(\prod_{n=1}^{N/2-1} n!\right)^2 \Delta(\{a\})} = \frac{\sum_{\alpha,\alpha'=a} e^{b\sum(\alpha_j - \alpha'_j)}\,\Delta(\{\alpha\})\,\Delta(\{\alpha'\})}{(2b)^{N^2/4}\left(\prod_{n=1}^{N/2-1} n!\right)^2 \Delta(\{a\})}$$

- Sum over assignments of eigenvalues of $\mathbf{A}$ into two sets $\rightarrow$ reduced Van der Monde!

Conclusions & Outlook

- A gap in the eigenvalue distribution induces a spontaneous breaking of U(N) symmetry
- 3 arguments provided: ✓ Plausibility by geometric reasoning,
  ✓ Explicit analytical construction with symmetry breaking term,
  ✓ Numerical experiment to study finite size behavior.
- Eigenvectors corresponding to distant eigenvalues cannot mix: breaking of ergodicity in invariant matrix models
- At finite N: suppression of off-diagonal block of unitary matrices/suppression of spillage of eigenvectors out of localization basin
- Applications:
  - ❑ Characterization of critical behavior at the birth of a cut as a phase transition to lower symmetry
  - ❑ Invariant Matrix models to describe Anderson Metal/Insulator transition (Weakly Confined Matrix Models)
  - ❑ Overlaps and IPR alone cannot detect localization: new approach based on response to perturbation
  - ❑ Matrix models from localization limit of string theories (ABJM): new SSB mechanism for fundamental physics and holographic applications (AdS/CFT, AdS/CMT, QGP…)
  - ❑ Opens matrix models techniques to the study of a whole new set of problems related to eigenvectors

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