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On Hamiltonian perturbations of hyperbolic PDEs

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Class of 1+1 evolutionary systems

$$w^i_t + A^i_j(w) w^j_x + \varepsilon \left(B^i_j(w) w^j_{xx} + \frac{1}{2} C^i_{jk}(w) w^j_x w^k_x \right) + O(\varepsilon^2) = 0$$

$i = 1, \dots, n$

ε small parameter

ε-expansion: Coefficient of ε^m is a polynomial in w_x, w_{xx}, ..., $w^{(m+1)}$ of the degree $m+1$

$$\deg w^{(k)} = k, \quad k \geq 1$$

Perturbations of **hyperbolic system**

$$v_t^i + A_j^i(v) v_x^j = 0, \quad i = 1, \dots, n$$

eigenvalues of $\left(A_j^i(v)\right)$ are **real and distinct** for any $v = (v^1, \dots, v^n) \in$ ball $\subset \mathbb{R}^n$.

Particular class: **systems of conservation laws**

$$v_t^i + \partial_x \phi^i(v) = 0, \quad i = 1, \dots .$$

Main goal: study of **Hamiltonian perturbations** of hyperbolic systems

Example 1 (Weakly dispersive) KdV

$$w_t + w\, w_x + \frac{\varepsilon^2}{12} w_{xxx} = 0$$

Example 2 Toda lattice

$$\ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}.$$

Continuous version:

$$u_n := q_{n+1} - q_n = u(n\varepsilon), \quad v_n := \dot{q}_n = v(n\varepsilon), \ \ t \mapsto \varepsilon\, t$$

$$u_t = \frac{v(x+\varepsilon) - v(x)}{\varepsilon} = v_x + \frac{1}{2}\varepsilon\, v_{xx} + O(\varepsilon^2)$$

$$v_t = \frac{e^{u(x+\varepsilon)} - e^{u(x)}}{\varepsilon} = e^u u_x + \frac{1}{2}\varepsilon\, (e^u)_{xx} + O(\varepsilon^2)$$

Example 3 Camassa - Holm equation

$$w_t = \left(1 - \varepsilon^2 \partial_x^2\right)^{-1} \left\{ \frac{3}{2} w\, w_x - \varepsilon^2 \left[w_x w_{xx} + \frac{1}{2} w\, w_{xxx} \right] \right\}$$

$$= \frac{3}{2} w\, w_x + \varepsilon^2 \left(w\, w_{xxx} + \frac{7}{2} w_x w_{xx} \right) + O(\varepsilon^4)$$

Equivalencies:

$$w^i \mapsto \tilde{w}^i = f_0^i(w) + \sum_{k\geq 1} \varepsilon^k f_k^i(w; w_x, \dots, w^{(k)})$$

$$\deg f_k^i(w; w_x, \dots, w^{(k)}) = k$$

$$\det \left(\frac{\partial f_0^i(w)}{\partial w^j} \right) \neq 0.$$

$f_k^i(w; w_x, \dots, w^{(k)})$ polynomials in derivatives

Questions:

structure of solutions for

- $t < t_{\mathrm{C}}$
- $t \sim t_{\mathrm{C}}$
- $t > t_{\mathrm{C}}$

Step 1: small t, **quasitriviality**. Locality of perturbative expansion

Example 1

Riemann wave $\mapsto$ KdV

$$v_t + v\, v_x = 0 \quad \mapsto \quad w_t + w\, w_x + \frac{\varepsilon^2}{12} w_{xxx} = 0$$

The substitution

$$w = v + \frac{\epsilon^2}{24}\partial_x^2 (\log v_x) \\ +\epsilon^4 \partial_x^2 \left(\frac{v^{IV}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4}\right) + O(\epsilon^6).$$

Baikov, Gazizov, Ibragimov, 1989 So, for small t the solution to the Cauchy problem with smooth monotone initial data behaves like

$$w(x,t) = v(x,t) + O(\varepsilon^2)$$
$$v(x,t) \quad \text{defined by } x = v\, t - f(v)$$

Universality for Riemann wave equation: near the point of gradient catastrophe any solution locally behaves as (A_2 singularity at $(x,t) = (0,0)$)

$$x = v\, t - \frac{v^3}{6}$$

Example 3

Riemann wave $\mapsto$ Camassa-Holm

$$v_t = \frac{3}{2} v\, v_x \ \mapsto\ w_t = (1-\varepsilon^2\partial_x^2)^{-1}\left(\frac{3}{2} w\, w_x - \varepsilon^2\left[w_x w_{xx} + \frac{1}{2} w\, w_{xxx}\right]\right)$$

$$\begin{aligned} w = v &+ \epsilon^2\,\partial_x\left(\frac{v\,v_{xx}}{3\,v_x} - \frac{v_x}{6}\right) \\ &+\epsilon^4\partial_x\left(\frac{7\,v_{xx}^2}{45\,v_x} + \frac{45\,v\,v_{xx}^3}{16\,v_x^3} - \frac{45\,v^2\,v_{xx}^4}{32\,v_x^5} - \frac{v_{xxx}}{8} - \frac{59\,v\,v_{xx}\,v_{xxx}}{90\,v_x^2}\right. \\ &\quad + \frac{37\,v^2\,v_{xx}^2\,v_{xxx}}{30\,v_x^4} - \frac{7\,v^2\,v_{xxx}^2}{30\,v_x^3} + \frac{5\,v\,v^{IV}}{18\,v_x} \\ &\left. - \frac{31\,v^2\,v_{xx}\,v^{IV}}{90\,v_x^3} + \frac{v^2\,v^V}{18\,v_x^2}\right) + O(\varepsilon^6) \end{aligned}$$

Lorenzoni, 2002

General results: (B.D., S.-Q.Liu, Y.Zhang)

- Any **Hamiltonian** PDE $\rightarrow$ system of conservation laws

$$w_t^i + \partial_x \psi^i(w; w_x, \dots; \varepsilon) = 0, \quad i = 1, \dots n$$

- Any **bihamiltonian** PDE is quasitrivial. The solution can be reduced to solving *linear* systems

• Explicit construction: under assumptions of existence of a **tau-function** and **linear action of the Virasoro symmetries** onto the tau-function

$$\tau \mapsto \tau + \delta\, L_m \tau + O(\delta^2), \quad m \geq -1$$

All solutions regular in ε obtained from the **vacuum solution**

$$L_m \tau = 0, \quad m \geq -1$$

by shifts along the times of the hierarchy (**completeness needed!**).

Parametrized by Frobenius manifolds $\Rightarrow$ $n(n-1)/2$ parametric family of integrable hierarchies (integrable hierarchies of the topological type)

Example $n = 2$ One-dimensional polytropic gas

equation of state $p = \frac{\kappa}{\kappa+1}\rho^{\kappa+1}$:

$$u_t + \left(\frac{u^2}{2} + \rho^\kappa\right)_x = 0$$

$$\rho_t + (\rho\, u)_x = 0$$

Bihamiltonian structure (Olver, 1980)

$$\{u(x), u(y)\}_\lambda^{[0]} = 2\rho^{\kappa-1}(x)\,\delta'(x-y) + \left(\rho^{\kappa-1}\right)_x \delta(x-y),$$
$$\{u(x), \rho(y)\}_\lambda^{[0]} = (u(x) - \lambda)\,\delta'(x-y) + \frac{1}{\kappa}\, v'(x)\,\delta(x-y),$$
$$\{\rho(x), \rho(y)\}_\lambda^{[0]} = \frac{1}{\kappa}\left(2\,\rho(x)\,\delta'(x-y) + \rho'(x)\,\delta(x-y)\right)$$

Integrable bihamiltonian perturbation

$$\frac{\partial u}{\partial t} + \partial_x \left\{ \frac{u^2}{2} + \rho^\kappa + \epsilon^2 \left[\frac{\kappa-2}{8} \rho^{\kappa-3}\, \rho_x^2 + \frac{\kappa}{12}\, \rho^{\kappa-2} \rho_{xx} \right] \right.$$

$$+\epsilon^4 (\kappa-2)(\kappa-3) \left[a_1\, \rho^{-4}\, u_x^2\, \rho_x^2 + a_2\, \rho^{\kappa-6}\, \rho_x^4 + \right.$$

$$a_3\, \rho^{-3}\, u_{xx}\, u_x\, \rho_x + a_4\, \rho^{-2}\, u_{xx}^2 + a_5\, \rho^{-3}\, u_x^2\, \rho_{xx} + a_6\, \rho^{\kappa-5}\, \rho_x^2\, \rho_{xx}$$

$$\left. + a_7\, \rho^{\kappa-4}\, \rho_{xx}^2 + a_8\, \rho^{-2}\, u_x\, u_{xxx} + a_9\, \rho^{\kappa-4}\, \rho_x\, \rho_{xxx} \right]$$

$$\left. + \epsilon^4 \frac{\kappa\, (\kappa^2-1)\, (\kappa^2-4)}{360} \rho^{\kappa-3}\, \rho_{xxxx} \right\} = \mathcal{O}(\epsilon^6),$$

$$\frac{\partial \rho}{\partial t} + \partial_x \left\{ \rho\, u + \epsilon^2 \left(\frac{(2-\kappa)(\kappa-3)}{12\, \kappa\, \rho}\, u_x\, \rho_x + \frac{1}{6}\, u_{xx} \right) \right.$$

$$+\epsilon^4 (\kappa-2)(\kappa-3) \left[b_1\, \rho^{-4}\, u_x\, \rho_x^3 + b_2\, \rho^{-3}\, \rho_x^2\, u_{xx} \right.$$

$$+ b_3\, \rho^{-3}\, u_x\, \rho_x\, \rho_{xx} + b_4\, \rho^{-2}\, u_{xx}\, \rho_{xx} + b_5\, \rho^{-2}\, u_{xxx}\, \rho_x$$

$$\left. \left. + b_6\, \rho^{-2}\, u_x\, \rho_{xxx} + b_7\, \rho^{-1}\, u_{xxxx} \right] \right\} = \mathcal{O}(\epsilon^6)$$

The coefficients are given by

$$a_1 = \frac{36 + 144\,\kappa - 59\,\kappa^2 + 19\,\kappa^3}{5760\,\kappa^3},$$

$$a_2 = \frac{60 + 176\,\kappa + 433\,\kappa^2 - 182\,\kappa^3 + 17\,\kappa^4}{5760\,\kappa^3}$$

$$a_3 = \frac{6 - 19\,\kappa - 11\,\kappa^2 - 4\,\kappa^3}{1440\,\kappa^3}, \quad a_4 = \frac{-6 - 5\,\kappa + 13\,\kappa^2}{1440\,\kappa^3},$$

$$a_5 = \frac{-42 + 13\,\kappa - 7\,\kappa^2}{2880\,\kappa^2}$$

$$a_6 = \frac{-36 - 72\,\kappa - 245\,\kappa^2 - 61\,\kappa^3 + 30\,\kappa^4}{2880\,\kappa^2},$$

$$a_7 = \frac{6 + 5\,\kappa + 15\,\kappa^2 + 5\,\kappa^3 + 5\,\kappa^4}{1440\,\kappa^2}$$

$$a_8 = \frac{1}{120\,\kappa}, \quad a_9 = \frac{2 + 5\,\kappa}{240}$$

$$b_1 = \frac{108 + 192\,\kappa - 97\,\kappa^2 + 17\,\kappa^3}{2880\,\kappa^3},$$

$$b_2 = \frac{-18 - 75\,\kappa + 47\,\kappa^2 - 10\,\kappa^3}{1440\,\kappa^3}$$

$$b_3 = -\frac{6 + 17\,\kappa - 5\,\kappa^2 + 2\,\kappa^3}{288\,\kappa^3}, \; b_4 = \frac{6 - 4\,\kappa + \kappa^2}{180\,\kappa^2},$$

$$b_5 = \frac{6 + \kappa + \kappa^2}{720\,\kappa^2}, \; b_6 = \frac{6 + \kappa + \kappa^2}{720\,\kappa^2}, \; b_7 = -\frac{1}{360\,\kappa}$$

Step 2: Critical behavior
For KdV

$$w = v + \frac{\epsilon^2}{24}\partial_x^2 (\log v_x)$$
$$+\epsilon^4 \partial_x^2 \left(\frac{v^{IV}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4} \right) + O(\epsilon^6).$$

Near critical point

$$v_x \sim 1/\varepsilon, \quad v_{xx} \sim 1/\varepsilon^2, \dots, v^{(m)} \sim \varepsilon^{-m}$$

All terms of the **same** order.
Resummation needed

Problem 1: Prove that near critical point the solution behaves as (universality)

$$u \sim \varepsilon^{\frac{2}{7}} U\left(\frac{x}{\varepsilon^{6/7}}, \frac{t}{\varepsilon^{4/7}}\right) + \mathcal{O}\left(\varepsilon^{\frac{4}{7}}\right)$$

where $U(X,T)$ is the unique **smooth** solution to the ODE

$$X = T\,U - \left[\frac{U^3}{6} + \frac{1}{24}(U'^2 + 2U\,U'') + \frac{U^{IV}}{240}\right]$$

depending on the parameter T.
Prove **existence** (cf. Brezin, Marinari, Parisi 1992) of such a solution (see also Kudashev, Suleimanov)

Step 3: After phase transition: oscillatory behavior. Gurevich, Pitaevski 1973, Whitham asymptotics (leading term)

Problem 2 Determine full asymptotic behavior of **averaged** quantities, $\varepsilon \to 0$

Example Hermitean matrix integrals

$$Z_N(\lambda;\epsilon) = \frac{1}{\mathrm{Vol}(U_N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\, V(A)}\, dA$$

$$V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3} \lambda_k A^k$$

as function of $N = x/\epsilon$, λ
is a tau-function of Toda lattice

Tau-function

$$u = \log \frac{\tau(x+\epsilon)\tau(x-\epsilon)}{\tau^2(x)}$$

$$v = \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(x+\epsilon)}{\tau(x)}.$$

Large N $\sim$ small ϵ expansion of

$$\tau(x, \mathbf{t}; \epsilon) = Z_N(\lambda; \epsilon)$$

$$x = \frac{N}{\epsilon}, \quad t_k = (k+1)!\lambda_{k+1}$$

has the form

$$\log \tau = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{t})$$

so the solution u, v admits regular expansion

$$u = \sum_{k \geq 0} \epsilon^k u_k(x, \mathbf{t})$$
$$v = \sum_{k \geq 0} \epsilon^k v_k(x, \mathbf{t})$$

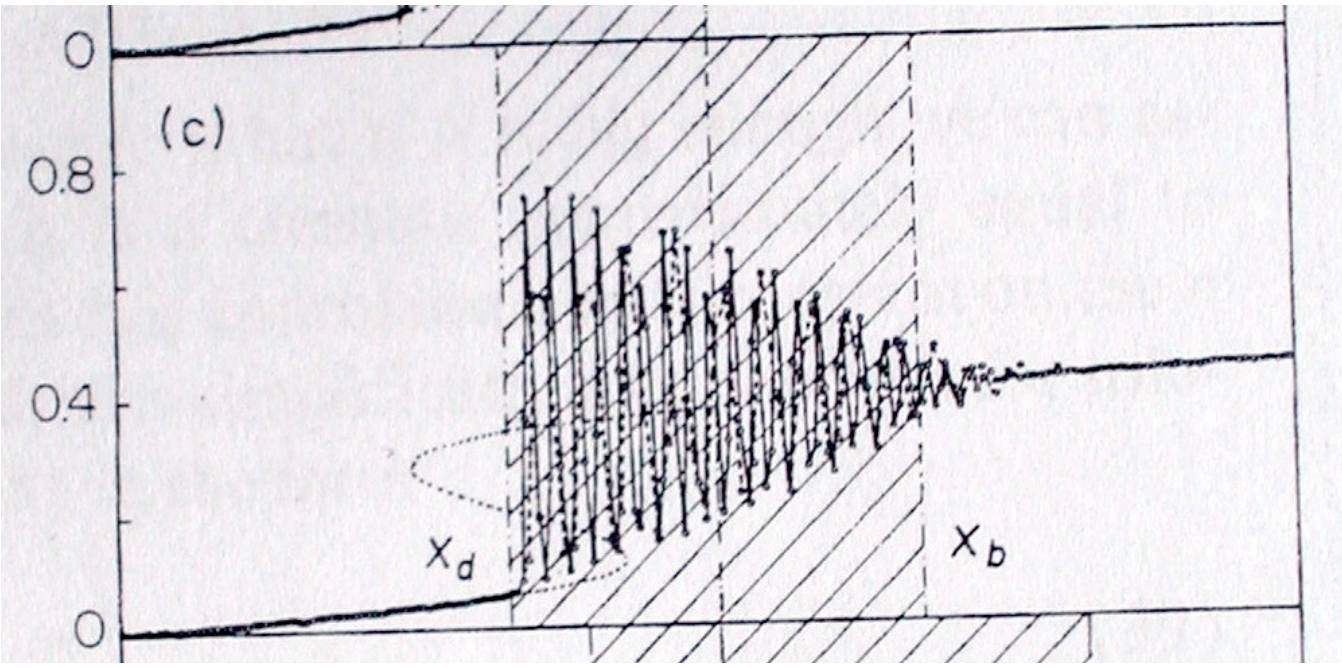
For small λ the ϵ-expansion can be obtained by applying the saddle point method to

$$Z_N = \frac{1}{\mathrm{Vol}(U_N)} \int e^{-\frac{1}{\epsilon}\mathrm{Tr}V(A)} dA$$

$\Rightarrow \mathcal{F}_g(x, \mathbf{t}) =$ generating function of numbers of fat graphs on genus g Riemann surfaces

Corresponds to the one-cut asymptotic distribution of the eigenvalues of the large size Hermitean random matrix A

Multicut case: gaps in the asymptotic distribution of eigenvalues of random matrices $\Rightarrow$ singular behaviour of the correlation functions (terms $\sim e^{\frac{iat}{\epsilon}}$ arise)



(from Jurkiewicz, Phys. Lett. B, 1991)

Smoothed correlation functions: average out the singular terms

Question: Which integrable PDEs describe the large N expansion of *smoothed* correlation functions?

Example 2 $n = 2$,

$$F(u,v) = \frac{1}{2} u\, v^2 + e^u$$

The Frobenius manifold

$$M^2 = \left\{ \lambda(p) = e^p + v + e^{u-p} \right\}$$

=symbol of the difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}$$

Extended Toda hierarchy
(G.Carlet, B.D., Y.Zhang)

$$\epsilon \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[(L^{k+1})_+, L \right]$$

$$\epsilon \frac{\partial L}{\partial s_k} = \frac{2}{k!} \left[\left(L^k (\log L - c_k) \right)_+, L \right]$$

$$c_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$$

$s_0 = x$, other times $s_1, s_2, \ldots$ are new.

Remark Interchanging time/space variables $x = s_0 \leftrightarrow t_0 = \tilde{x}$ transforms Toda $\leftrightarrow$ NLS

Back to matrix models (B.D., T.Grava, in progress)

Claim Substituting

$$\tau_{\text{Toda}}^{\text{vac}}(t_0, t_1, t_2, \dots; s_0, s_1, s_2, \dots; \epsilon)$$

$$t_0 = 0,\ t_1 = -1,\ t_k = (k+1)!\lambda_{k+1},\ ,k \geq 2$$

$$s_0 = x,\ s_k = 0,\ k \geq 1$$

one obtains

$$F := \log \tau_{\text{Toda}}^{\text{vac}}(0, -1, 3!\lambda_3, 4!\lambda_4, \dots; x, 0, \dots; \epsilon)$$

$$= \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \sum_{g\geq 2}\left(\frac{\epsilon}{x}\right)^{2g-2}\frac{B_{2g}}{2g(2g-2)}$$

$$+ \sum_{g\geq 0}\epsilon^{2g-2}F_g(x; \lambda_3, \lambda_4, \dots)$$

$$F_g(x; \lambda_3, \lambda_4, \ldots)$$

$$= \sum_n \sum_{k_1,\ldots,k_n} a_g(k_1, \ldots, k_n)\lambda_{k_1} \ldots \lambda_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2}\right), \qquad |k| = k_1 + \ldots + k_n,$$

and

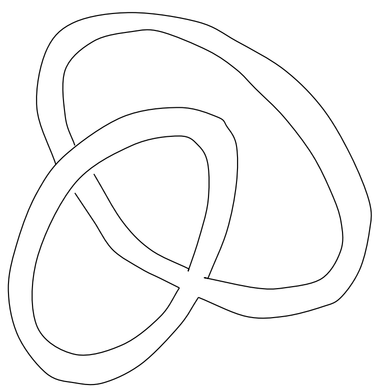
$$a_g(k_1, \ldots, k_n) = \sum_\Gamma \frac{1}{\#\,\mathrm{Sym}\,\Gamma}$$

where

Γ = a connected fat graph of genus g

with n vertices of the valencies $k_1, \ldots, k_n$.

E.g.: genus 1, one vertex, valency 4



$$F=\epsilon^{-2}\left[\frac{1}{2}x^2\left(\log x-\frac{3}{2}\right)+6x^3{\lambda_3}^2+2x^3\lambda_4+216x^4{\lambda_3}^2\lambda_4+18x^4{\lambda_4}^2\right.$$

$$+288x^5{\lambda_4}^3+45x^4\lambda_3\lambda_5+2160x^5\lambda_3\lambda_4\lambda_5+90x^5{\lambda_5}^2+5400x^6\lambda_4{\lambda_5}^2+5x^4\lambda_6$$

$$+1080x^5{\lambda_3}^2\lambda_6+144x^5\lambda_4\lambda_6+4320x^6{\lambda_4}^2\lambda_6+10800x^6\lambda_3\lambda_5\lambda_6+27000x^7{\lambda_5}^2\lambda_6$$

$$\left.+300x^6{\lambda_6}^2+21600x^7\lambda_4{\lambda_6}^2+36000x^8{\lambda_6}^3\right]$$

$$-\frac{1}{12}\log x+\frac{3}{2}x{\lambda_3}^2+x\lambda_4+234x^2{\lambda_3}^2\lambda_4+30x^2{\lambda_4}^2+1056x^3{\lambda_4}^3+60x^2\lambda_3\lambda_5$$

$$+6480x^3\lambda_3\lambda_4\lambda_5+300x^3{\lambda_5}^2+32400x^4\lambda_4{\lambda_5}^2+10x^2\lambda_6+3330x^3{\lambda_3}^2\lambda_6$$

$$+600x^3\lambda_4\lambda_6+31680x^4{\lambda_4}^2\lambda_6+66600x^4\lambda_3\lambda_5\lambda_6+283500x^5{\lambda_5}^2\lambda_6$$

$$+2400x^4{\lambda_6}^2+270000x^5\lambda_4{\lambda_6}^2+696000x^6{\lambda_6}^3$$

$$+\epsilon^2\left[-\frac{1}{240x^2}+240x{\lambda_4}^3+1440x\lambda_3\lambda_4\lambda_5+\frac{1}{2}165x{\lambda_5}^2+28350x^2\lambda_4{\lambda_5}^2\right.$$

$$+675x{\lambda_3}^2\lambda_6+156x\lambda_4\lambda_6+28080x^2{\lambda_4}^2\lambda_6+56160x^2\lambda_3\lambda_5\lambda_6+580950x^3{\lambda_5}^2\lambda_6$$

$$\left.+2385x^2{\lambda_6}^2+580680x^3\lambda_4{\lambda_6}^2+2881800x^4{\lambda_6}^3\right]+\ldots$$

Proof uses **Toda equations** and the **large N expansion** for the Hermitean matrix integral ('t Hooft; D.Bessis, C.Itzykson, J.-B.Zuber)

$$Z_N(\lambda;\epsilon) = \frac{1}{\mathrm{Vol}(U_N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\, V(A)}\, dA$$

$$V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3} \lambda_k A^k$$

where one has to replace

$$N \mapsto \frac{x}{\epsilon}$$

Remark This is the topological solution for the (extended) nonlinear Schrödinger hierarchy

Multicut case: G gaps in the spectrum of random matrices

Claim: The full large N expansion of the smoothed correlation functions is given via the topological tau function associated with the Frobenius structure M^n, $n = 2G + 2$ on the Hurwitz space of hyperelliptic curves

$$\mu^2 = \prod_{i=1}^{2G+2} (\lambda - u_i)$$

Recall the general construction: Frobenius structure on the Hurwitz space M^n = moduli of branched coverings

$$\lambda : \Sigma_G \to \mathbf{P}^1$$

fixed degree, genus G, ramification type at infinity, basis of a- and b-cycles (n = number of branch points $\lambda = u_i$ for generic covering).

Must choose a primary differential dp (say, holomorphic differential with constant a-periods)

Then, for any two vector fields ∂_1, ∂_2 on M^n the inner product

$$\langle \partial_1, \partial_2 \rangle = \sum_{i=1}^{n} \mathrm{res}_{\lambda=u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)}{d\lambda}$$

for any three vector fields ∂_1, ∂_2, ∂_3 on M^n

$$\langle \partial_1 \cdot \partial_2, \partial_3 \rangle = -\sum_{i=1}^{n} \mathrm{res}_{\lambda=u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)\partial_3(\lambda dp)}{d\lambda\, dp}$$

Example $G = 1$ (two-cut case). Here $n = 4$. Flat coordinates on the Hurwitz space of elliptic double coverings with 4 branch points are u, v, w, τ. Can describe by the superpotential (= symbol of Lax operator)

$$\lambda(p) = v + u\left(\log\frac{\theta_1(p-w|\tau)}{\theta_1(p+w|\tau)}\right)'$$

The Frobenius structure given by

$$F = \frac{i}{4\pi}\tau\, v^2 - 2\,u\,v\,w + u^2\log\left[\frac{1}{\pi\,u}\frac{\theta_1(2w|\tau)}{\theta_1'(0|\tau)}\right]$$

Recall

$$\log\left[\frac{\theta_1(x|\tau)}{\pi\,\theta_1'(0|\tau)}\right] = \log\sin\pi\,x + 4\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\frac{\sin^2\pi\,m\,x}{m}$$

$$q = e^{i\,\pi\,\tau}$$

Corresponding integrable hierarchy of the topological type for the functions u, v, w, τ, four infinite chains of times $t^{u,p}$, $t^{v,p}$, $t^{w,p}$, $t^{\tau,p}$. Then

$$Z \sim \tau^{\text{vac}}$$

with $t^{w,1} \mapsto t^{w,1} - 1$, $t^{w,0} = 0$, $t^{w,k} = (k+1)!\lambda_{k+1}$

$$t^{u,0} = x$$

other couplings = 0.

Problem 3 Critical and after critical behaviour for Camassa - Holm?