Integrable systems and applications

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1 Introduction

The modern theory of integrable systems started with the discovery in 1967 that the Korteweg de Vries (KdV) equation is integrable [7]. Such equation is an evolutionary partial differential equation and corresponds to an integrable system with infinite degree of freedom. Before 1967 it was believed that integrability as opposed to chaotic behaviour was a rare phenomena, restricted to particular examples. Indeed there were few examples of known integrable systems and results concerning integrability:

- two-body problem in celestial mechanics (Kepler, Newton 1600-1687);
- geodesics on ellipsoids and separation of variables in Hamilton-Jacobi equation (Jacobi 1837);
- Liouville theorem about the integrability by quadratures of an integrable systems (Liouville 1838);

- harmonic oscillator on the unit sphere (Neumann 1859);
- Clebsh system (rigid body) 1871;
- Lagrange, Euler and Kovalevskaya (1888) tops;
- Noether theorem about the relation between symmetries and integrals of motion of a mechanical system (Emmy Noether 1915);
- global version of Liouville theorem (Arnold 1963).

In 1967 Gardner, Green, Kruskal and Miura realized that the spectrum of the Schrödinger equation $-\frac{d^2}{dx^2}+u(x,t)$ does not change with time if the potential $u(x,t)$ evolves according to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

where $u = u(x,t)$ is a scalar function of $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ and $u_t = \frac{\partial}{\partial t}u(x,t)$ and $u_x = \frac{\partial}{\partial x}u(x,t)$. With this observation it was realized that the KdV equation can be integrated by inverse scattering that can be thought of as a nonlinear analogue of the Fourier transform used to solve linear partial differential equations. Around 1974 there were finite-dimensional versions of the inverse scattering transform that were applied to solve finite dimensional integrable systems like the Toda lattice or the Calogero-Moser systems (Flaschka [5], Manakov [14], Moser [15]). The main goal of these notes is to study integrable sytems with finite and infinite degree of freedoms. We will first study the inverse scattering transform for the open finite Toda lattice. Then we will consider inverse scattering for the KdV equation with rapidly decreasing initial data and periodic initial data. In this latter case, when the periodic initial data is "finite gap", namely when the spectrum of the Hill's equation has only a finite number of open gaps, the evolution in time of the KdV solution $u(x,t)$ corresponds to a linear flow on a finite-dimensional tori.

2 A short review of the classical theory of finite-dimensional integrable systems

We review the basic definitions in the theory of finite-dimensional integrable systems.

2.1 Poisson manifolds

We start with the definition of Poisson bracket.

Definition 2.1 A manifold P is said to be a Poisson manifold if P is endowed with a Poisson bracket, that is a Lie algebra structure defined on the space $\mathcal{C}^\infty(P)$ of smooth functions over P

$$\begin{aligned} \mathcal{C}^\infty(P)\times\mathcal{C}^\infty(P) &\to \mathcal{C}^\infty(P) \\ (f,g) &\mapsto \{f,g\} \end{aligned} \tag{2.1}$$

so that $\forall f,g,h\in\mathcal{C}^\infty(P)$ the bracket $\{\,.\,,\,.\,\}$

- is antisymmetric:
$$\{g,f\}=-\{f,g\}, \tag{2.2}$$
- bilinear
$$\begin{aligned} \{af+bh,g\} &= a\{f,g\}+b\{h,g\}, \\ \{f,ag+bh\} &= a\{f,g\}+b\{f,h\}, \quad a,\,b\in\mathbb{R} \end{aligned} \tag{2.3}$$
- satisfies Jacobi identity
$$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0; \tag{2.4}$$
- it satisfies Leibnitz identity with respect to the product of function
$$\{f\,g,h\}=g\,\{f,h\}+f\,\{g,h\}. \tag{2.5}$$

A Poisson bracket defines an anti-homomorphism from the space $\mathcal{C}^\infty(P)$ to the space of vector fields over P:

$$\begin{aligned} \mathcal{C}^\infty(P) &\to vect(P) \\ f &\to X_f=\{.\,,f\} \end{aligned}$$

so that

$$[X_f,X_g]=-X_{\{f,g\}},$$

where $[.,.]$ is the commutator of vector fields also known as Lie bracket: $L_XY:=[X,Y]$. In order to write the definition 2.1 in local coordinates $x=(x^1,\ldots,x^N)$ let us introduce the matrix

$$\pi^{ij}(x):=\{x^i,x^j\},\quad i,\,j=1,\ldots,N=\dim P. \tag{2.6}$$

Theorem 2.2 *[3] 1) Given a Poisson manifold P, and a system of local coordinates over P, then the matrix $\pi^{ij}(x)$ defined in (2.6) is antisymmetric and satisfies*

$$\frac{\partial\pi^{ij}(x)}{\partial x^s}\pi^{sk}(x)+\frac{\partial\pi^{ki}(x)}{\partial x^s}\pi^{sj}(x)+\frac{\partial\pi^{jk}(x)}{\partial x^s}\pi^{si}(x)=0,\quad 1\le i<j<k\le N. \tag{2.7}$$

Furthermore the Poisson bracket of two smooth functions is calculated according to

$$\{f,g\} = \pi^{ij}(x)\frac{\partial f(x)}{\partial x^i}\frac{\partial g(x)}{\partial x^j}. \tag{2.8}$$

2) Given a change of coordinates

$$\tilde{x}^k = \tilde{x}^k(x), \quad k = 1,\dots,N,$$

then the matrices $\pi^{ij}(x) = \{x^i, x^j\}$ *e* $\tilde{\pi}^{kl}(\tilde{x}) = \{\tilde{x}^k, \tilde{x}^l\}$ *satisfy the rule of transformation of a tensor of type (2,0):*

$$\tilde{\pi}^{kl}(\tilde{x}) = \pi^{ij}(x)\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}. \tag{2.9}$$

3) Viceversa, given a smooth manifold P *and an antisymmetric tensor (2,0)* $\pi^{ij}(x)$ *such that (2.7) is satisfied, then (2.8) defines over* P *a Poisson bracket.*

Proof. The matrix $\pi^{ij}(x)$ is clearly antisymmetric. In order to derive (2.8) one observe that for a fixed function f, the application

$$\begin{aligned} \mathcal{C}^\infty(P) &\to \mathcal{C}^\infty(P) \\ g &\to \{g,f\} \end{aligned}$$

is linear and satisfies Leibnitz rule (2.5), therefore it is a linear differential operator of first order, namely

$$\{g,f\} = X_f g,$$

for a vector field

$$X_f = X_f^j \frac{\partial}{\partial x^j},$$

where we are taking the sum over repeated indices. In order to determine the components of the vector field X_f one considers

$$X_f^j = X_f\, x^j = \{x^j, f\}.$$

Now let us fix x_j and consider the linear map

$$f \mapsto \{f, x^j\} = X_{x^j} f = X_{x^j}^k \frac{\partial}{\partial x^k} f.$$

Since $X_{x^j}^k = \pi^{kj}$ by (2.6), it follows from the above relations that

$$X_f^j = \pi^{jk}\frac{\partial}{\partial x^k} f,$$

so that

$$\{g, f\} = X_f g = X_f^j \frac{\partial g}{\partial x^j} = \pi^{jk} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^j}.$$

Using the same rule for the change of coordinates $\tilde{x}^j = \tilde{x}^j(x)$ and $\tilde{x}^k = \tilde{x}^k(x)$ one obtains the tensor rule (2.9). Equation (2.7) follows from Jacobi identity.

To prove the sufficiency of the theorem one observe that given a $(2,0)$ antisymmetric tensor $\pi^{ij}(x)$, the map

$$(f, g) \mapsto \{f, g\} := \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

is bilinear, antisymmetric and satisfies Leibnitz rule. Furthermore it does not depend on the choice of local coordinates

$$\tilde{\pi}^{kl} \frac{\partial f}{\partial \tilde{x}^k} \frac{\partial g}{\partial \tilde{x}^l} = \pi^{st} \frac{\partial \tilde{x}^k}{\partial x^s} \frac{\partial \tilde{x}^l}{\partial x^t} \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial x^j}{\partial \tilde{x}^l} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \pi^{st} \delta_s^i \delta_t^j \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

In order to show validity of the Jacobi identity it is sufficient to observe that for any functions f, g, h and any antisymmetric tensor $\pi^{ij}(x)$ the following identity is satisfied:

$$\begin{aligned} &\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} \\ &= \left[\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} \right] \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k}. \end{aligned}$$

so that the Jacobi identity follows from (2.7). □

The Poisson bracket is said to be non degenerate if the rank $\pi = dim(P)$. Clearly the antisymmetry implies that only even dimensional manifolds can have a non-degenerate Poisson bracket.

Definition 2.3 Given a Poisson bracket the set of functions that commutes with any other functions of $\mathcal{C}^\infty(P)$, namely

$$\{h \in \mathcal{C}^\infty(P) \,|\, \{h, f\} = 0, \forall f \in \mathcal{C}^\infty(P)\}$$

are called *Casimirs* of the Poisson bracket.

For a nondegenerate Poisson bracket, the only Casimirs are the constant functions.

Definition 2.4 A $2n$-dimensional P manifold is called symplectic manifold if it is endowed with a close non degenerate 2-form ω.

In local coordinates one has

$$\omega = \sum_{i,j=1}^{n} \omega_{ij} dx^i \wedge dx^j$$

where $\wedge$ stands for the exterior product. We recall that the form ω is closed if $d\omega = \sum_{ijk=1}^{n} \frac{\partial}{\partial x^k}\omega_{ij} dx^k \wedge dx^i \wedge dx^j = 0$, which implies that

$$\frac{\partial}{\partial x^k}\omega_{ij} + \frac{\partial}{\partial x^i}\omega_{jk} + \frac{\partial}{\partial x^j}\omega_{ki} = 0, \quad i \neq j \neq k.$$

Lemma 2.5 *A Poisson manifold $\{P, \pi\}$ with non degenerate Poisson bracket π, is a symplectic manifold, with $\omega_{ij} = (\pi^{ij})^{-1}$.*

For a symplectic manifold (P, ω) one has the identies

$$\omega(X_f, .) = -df$$

and

$$\{f, g\} = \omega(X_g, X_f) = X_g(f) = \langle df, X_g \rangle$$

where $\langle ., . \rangle$ is the pairing between one form and vectors, i.e. for a one form $\alpha = \alpha_i dx^i$ and a vector $v = v^i \frac{\partial}{\partial x^i}$ then $\langle \alpha, v \rangle = \alpha_i v^i$. In order to verify the above second identity let X_f^i and X_g^j be the coordinates of the vector fields X_f and X_g respectively, then one has

$$\omega(X_g, X_f) = \sum_{i,j} \omega_{ij} X_g^i X_f^j = \sum_{i,j} \sum_{k,l} \omega_{ij} \pi^{il} \frac{\partial g}{\partial x^l} \pi^{jk} \frac{\partial f}{\partial x^k} = \sum_{il} \pi^{il} \frac{\partial g}{\partial x^l} \frac{\partial f}{\partial x^i} = \{f, g\}.$$

The *classical Darboux theorem* says that in the neighbourhood of every point of a symplectic manifold (P, ω), $\dim P = 2n$, there is a local systems of co-ordinates $(q^1, \ldots q^n, p_1, \ldots, p_n)$ called Darboux coordinates or canonical coordinates such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \,. \tag{2.10}$$

In such coordinates the Poisson bracket takes the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

and the Hamiltonian vector field X_f takes the form

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

and the Poisson tensor π is

$$\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The existence of Darboux coordinates is related to the vanishing of the second group of the so called Poisson cohomology $H^*(P,\pi)$. If the Poisson bracket is non-degenerate, the Poisson cohomology coincides with the de-Rham cohomology and Darboux theorem is equivalent to the vanishing of second de-Rham cohomology group in an open set. In order to have global Darboux coordinates one needs the vanishing of the Poisson cohomology group $H^2(P,\pi)$. There are many tools for computing de Rham cohomology groups, and these groups have probably been computed for most familiar manifolds. However, when π is not symplectic, then $H^*(P,\pi)$ does not vanish even locally [16] and it is is much more difficult to compute than the de Rham cohomology. There are few Poisson (non-symplectic) manifolds for which Poisson cohomology has been computed [8]. The Poisson cohomology $H^*(P,\pi)$ can have infinite dimension even when P is compact, and the problem of determining whether $H^*(P,\pi)$ is finite dimensional or not is already a difficult open problem for most Poisson structures that we know of. In the case of linear Poisson structures, Poisson cohomology is intimately related to Lie algebra cohomology, also known as Chevalley - Eilenberg cohomology, [17].

Given a Poisson manifold (P,π), $\dim P = N$, and a function $H \in \mathcal{C}^\infty(P)$, an Hamiltonian system in local coordinates $(x^1,\dots,x^N)$ is a set of N first order ODEs defined by

$$\dot{x}^i = \{x^i, H\},$$

with initial condition $x^i(t=0) = x^i_0$. For a symplectic manifold (P,ω), $\mathrm{dim}P = 2n$, the Hamilton equations in Darboux coordinates takes the form

$$\begin{aligned} \dot{q}^i &= \{q^i, H\} = \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= \{p_i, H\} = -\frac{\partial H}{\partial q^i}, \quad i = 1,\dots,n \end{aligned} \tag{2.11}$$

with initial conditions $q^i(t=0) = q^i_0$, $p_i(t=0) = p^0_i$.

Definition 2.6 A function $F \in \mathcal{C}^\infty(P)$ is said to be a conserved quantity for the Hamiltonian system (2.11) if

$$\frac{dF}{dt} = \{F, H\} = 0.$$

Namely conserved quantities Poisson commute with the Hamiltonian. We remark that if $F_1,\dots,F_m$ are conserved quantities, then any function of $g = g(F_1,\dots,F_m)$ is a conserved quantity.

Let $\Phi_F(x,s)$ be the Hamiltonian flow associated with $F \in \mathcal{C}^\infty(P)$ and $\Phi_H(x,t)$ the Hamiltonian flow associated with the Hamiltonian H. We use the notation $x(t,s) := \phi_F(\phi_H(x,t),s)$ to indicate the integral curve obtained by first applying the Hamiltonian vector field X_H and then the Hamiltonian vector field X_F. Viceversa we use the notation $x(s,t) := \phi_H(\phi_F(x,s),t)$ to indicate the integral curve obtained by first applying the Hamiltonian vector field X_F and then the Hamiltonian vector field X_H. A natural question is to ask when $x(t,s) = x(s,t)$.

Lemma 2.7 *Let $(P, \{\,.\,,\,.\,\})$ be a nondegenerate Poisson bracket. Consider the Hamiltonians $F, H \in \mathcal{C}^\infty(P)$, in involution $\{F,H\} = 0$ and their Hamiltonian flows*

$$\frac{dx^i}{dt} = \{x^i, H\}, \quad i = 1, \dots, N \tag{2.12}$$

$$\frac{dx^i}{ds} = \{x^i, F\}, \quad i = 1, \dots, N. \tag{2.13}$$

Then

$$\frac{d}{ds}\frac{dx^i}{dt} = \frac{d}{dt}\frac{dx^i}{ds}.$$

Proof. Take the derivative with respect to s of equation (2.12) and with respect to t of equation (2.13). One has

$$\frac{d}{ds}\frac{dx^i}{dt} = \frac{d}{ds}\{x^i, H\} = \{\frac{d}{ds}x^i, H\} + \{x^i, \frac{d}{ds}H\} = \{\{x^i, F\}, H\}, \quad i = 1, \dots, N$$

$$\frac{d}{dt}\frac{dx^i}{ds} = \frac{d}{dt}\{x^i, F\} = \{\frac{d}{dt}x^i, H\} + \{x^i, \frac{d}{dt}H\} = \{\{x^i, H\}, F\}, \quad i = 1, \dots, N$$

Subtracting the two terms and applying Jacobi identity one arrives to

$$\frac{d}{ds}\frac{dx^i}{dt} - \frac{d}{dt}\frac{dx^i}{ds} = \{\{x^i, F\}, H\} + \{\{H, x^i\}, F\} = \{\{H, F\}, x^i\} = 0, \quad i = 1, \dots, N.$$

which is equal to zero by the commutativity of F and H. $\square$

In order to introduce Liouville theorem, we first define the concept of Lagrangian sub manifold and integrable system.

Definition 2.8 Let P be a symplectic manifold of dimension $2n$. A a sub-manifold $G \subset P$ is called a Lagrangian submanifold if $\dim\Lambda = n$ and the symplectic form is identically zero on vectors tangent to G, namely

$$\omega(X, Y) = 0, \quad \forall X, Y \in TG.$$

Definition 2.9 A Hamiltonian system defined on a $2n$ dimensional Poisson manifold P with non degenerate Poisson bracket and with Hamiltonian $H \in \mathcal{C}^\infty(P)$ is called completely integrable if there are n independent conserved quantities $H = H_1, \ldots, H_n$ in involution, namely

$$\{H_j, H_k\} = 0, \quad j, k = 1, \ldots, n \tag{2.14}$$

and the gradients $\nabla H_1, \ldots \nabla H_n$ are linearly independent.

Let us consider the level surface

$$M_E = \{(p, q) \in P \,|\, H_1(p, q) = E_1, \quad H_2(p, q) = E_2, \quad H_n(p, q) = E_n\} \tag{2.15}$$

for some constants $E = (E_1, \ldots, E_n)$.

Theorem 2.10 *[Liouville, see e.g. [3]] Consider a completely integrable Hamiltonian system on a non degenerate Poisson manifold P of dimension $2n$ and with canonical coordinates (q, p). Let us suppose that the Hamiltonians $H_1(p, q)$, ..., $H_n(p, q)$ are linearly independent on the level surface M_E (2.15) for a given $E = (E_1, \ldots, E_n)$. The Hamiltonian flows on M_E are integrable by quadratures.*

Proof. By definition the system posses n independent conserved quantities $H_1 = H$, $H_2, \ldots H_n$. Without loosing generality, we assume that (q, p) are canonical coordinates with respect to the symplectic form ω and the Poisson bracket $\{.,.\}$.

The gradients

$$\nabla H_j = \left(\frac{\partial H_j}{\partial q^1}, \ldots, \frac{\partial H_j}{\partial q^n}, \frac{\partial H_j}{\partial p_1}, \ldots, \frac{\partial H_j}{\partial p_n}\right)$$

are orthogonal to the surface M_E. Since the vector fields X_{H_j} are orthogonal to ∇H_k because $\{H_j, H_k\} = 0$, it follows that the vector fields X_{H_j} are tangent to the level surface M_E. Furthermore, since the Hamiltonian H_j are linearly independent, it follows that the vector fields X_{H_j}, $j = 1, \ldots, n$ generate all the tangent space TM_E. Therefore the symplectic form is identically zero on the tangent space to M_E, namely $\omega|_{TM_E} \equiv 0$ because

$$\omega(X_{H_j}, X_{H_k}) = \{H_k, H_j\} = 0.$$

This is equivalent to say that M_E is a Lagrangian submanifold. We also observe that since ∇H_j, $j = 1, \ldots, n$ are linearly independent, it is possible to assume, without loosing in generality that

$$\det \frac{\partial H_j}{\partial p_k} \neq 0.$$

Then by the implicit function theorem we can define

$$p_k = p_k(q, E).$$

Putting together the last two observations, we have for fixed $E = (E_1, \ldots, E_n)$

$$0 = \omega|_{TM_E} = \sum_i dp_i(q, E) \wedge dq^i = \sum_{ij} \frac{\partial p_i}{\partial q^j} dq^j \wedge dq^i$$

which implies

$$\frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} = 0, \quad i \neq j.$$

The above identity implies that the one form $W = p_i dq^i$ is exact, and therefore there exists a function $S = S(q, E)$ so that $W = dS$. The function S is the generating function of a canonical transformation. Recall that a change of coordinates $x \to \Phi(x)$ is defined by $2n$ functions. The change of coordinates is a canonical transformation if $\Phi^*\omega = \omega$ where Φ^* is the pullback of Φ. Since $\omega = dW$ and the pullback commutes with differentiation, one has

$$\Phi^*\omega - \omega = \Phi^*(dW - dW) = d(\Phi^*W - W) = 0.$$

Namely the form $d(\Phi^*W - W)$ is exact, so there is locally a function S so that $\Phi^*W - W = S$ and S is called the generating function of the canonical transformation. In other words a canonical change of coordinates is defined by one function. After this digression we come back to the function $S = S(q, E)$ which is the generating function of a canonical transformation $(q, p) \to (\psi, E)$ where

$$p_i = \frac{\partial S}{\partial q^i}, \quad \psi_i = \frac{\partial S}{\partial E_i}.$$

In the canonical coordinates (ψ, E) the Hamiltonian flow with respect to the Hamiltonian $H_1 = H$ takes the form

$$\psi_i = \{\psi_i, H_1\} = \frac{\partial H_1}{\partial E_i} = \delta_{1i}$$
$$E_i = \{E_i, H_1\} = -\frac{\partial H_1}{\partial \psi_i} = 0.$$

So the above equations can be integrated in a trivial way:

$$\psi_1 = t + \psi_1^0, \;\; \psi_i = \psi_i^0, \quad i = 2, \ldots, n \qquad\qquad E_i = E_i^0, \;\; i = 1, \ldots, n$$

where $\psi_i^0$and E_i^0 are constants. Therefore we have shown that the Hamiltonian flow can be integrated by quadratures. Furthermore

$$q = q(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E), \quad p = p(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E).$$

$\square$

In 1968 Arnold observed that if the level surface M_E is compact, the motion takes place on a torus and is quasi-periodic.

Theorem 2.11 (Arnold) *If the level surface M_{E^0} defined in (2.15) is compact and connected then the level surfaces M_E for $|E - E^0|$ sufficiently small are diffeomorphic to a torus*

$$M_E \simeq T^n = \{(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n \,|\, \phi_i \sim \phi_i + 2\pi,\, i = 1, \ldots, n\}, \tag{2.16}$$

and the motion on M_E is quasi-periodic, namely

$$\phi_1(t) = \omega_1(E)\, t + \phi_1^0, \ldots, \phi_n(t) = \omega_n(E)\, t + \phi_n^0 \tag{2.17}$$

where $\omega_1(E)$, $\ldots$, $\omega_n(E)$ depends on E and the phases ϕ_1^0, $\ldots$, ϕ_n^0 are arbitrary.

Proof. To prove the theorem we use a standard lemma (see [3]).

Lemma 2.12 *Let M be a compact connected n-dimensional manifold. If on M there are n linearly independ vector fields $X_1, \ldots, X_n$ such that*

$$[X_i, X_j] = 0, \quad i, j = 1 \ldots, n$$

then $M \simeq T^N$, the n-dimensional torus.

In our case the vector field $X_{H_1}, \ldots, X_{H_n}$ are linearly independent and commuting, so, in the case M_{E^0} is compact and connected, it is also isomorphic to a n-dimensional torus. By continuity, for small values of $|E - E^0|$ the surface M_E is also isomorphic to a torus. The coordinates $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ introduced in the proof of Liouville theorem 2.10 are not angles on the torus. Let us make a change of variable $\phi = \phi(\psi)$ so that the coordinates $\phi = (\phi_1, \ldots, \phi_n)$ are angles on the torus and let $I_1(E), \ldots I_n(E)$ be the canonical variables associated to the angles $(\phi_1, \ldots, \phi_n)$. By definition one has for any Hamiltonian H_m

$$X_{H_m} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial E_j} \frac{\partial}{\partial \psi_j} = \frac{\partial}{\partial \psi_m} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial I_j} \frac{\partial}{\partial \phi_j},$$

since H_m depends only on E and ϕ depends only on ψ. It follows that ϕ_j and ψ_k are related by a linear transformation

$$\phi_j = \sum_m \sigma_{jm} \psi_m, \quad \sigma_{jm} = \sigma_{jm}(E), \quad \det \sigma_{jm} \neq 0.$$

Comparing the above two relations one arrives to

$$\sigma_{jm} = \frac{\partial H_m}{\partial I_j}.$$

Let us verify that (ϕ, I) are indeed canonical variables:

$$\{\phi_j, I_k\} = \{\sum_k \sigma_{jm}\psi_m, I_k\} = \sum_m \sigma_{jm}\{\psi_m, I_k\} = \sum_m \sigma_{jm}\frac{\partial I_k}{\partial E_m} = \sum_m \frac{\partial H_m}{\partial I_j}\frac{\partial I_k}{\partial E_m} = \delta_{jk}.$$

The equation of motions in the variables (ϕ, I) are given by

$$\dot{\phi}_k = \frac{\partial H_1}{\partial I_k} =: \omega_k(E)$$

$$\dot{I}_k = \frac{\partial H_1}{\partial \phi_k} = 0$$

therefore the motion is quasi periodic on the tori. In the variable (p, q), with $p = p(\phi, I)$, $q = q(\phi, I)$, the evolution is given as

$$q = q(\omega_1 t + \phi_1^0, \ldots, \omega_n t + \phi_n^0, I)$$

$$p = p(\omega_1 t + \phi_1^0, \ldots, \omega_n t + \phi_n^0, I),$$

where $(\phi_1^0, \ldots, \phi_n^0)$ are constant phases. □

3 Bi-Hamiltonian geometry and Lax pair

In this subsections we give the basic concepts of bi-Hamiltonian geometry, skipping most of the relevant proofs.

Definition 3.1 Two Poisson tensors π_0 and π_1 on a manifold P are called compatible if

$$c_0\pi_0 + c_1\pi_1$$

is a Poisson tensor for any real c_0 and c_1.

Definition 3.2 A vector field X on a manifold is called a bi-Hamiltonian system if it is Hamiltonian with respect to two compatible Poisson structures π_1 and π_0: $X = X_{H_1}^{\pi_1} = X_{H_0}^{\pi_0}$ or equivalently

$$\{\ .\ , H_1\}_1 = \{\ .\ , H_0\}_0 \tag{3.1}$$

From now on we assume to have a Poisson manifold P of dimension $2n$ with non degenerate Poisson bracket.

Remark 3.3 Bi-Hamiltonian systems admit large sets of first integrals, which make them into integrable Hamiltonian systems. Conversely, a vast majority of known integrable systems turn out to be bi-Hamiltonian. The importance of bi-Hamiltonian systems for the recursive construction of integrals of motion starts with Magri [12] and there is now a very large amount of articles on the subject.

Lemma 3.4 *[12] Let H_0, H_1, ..., be a sequence of functions on Poisson manifold P with compatible Poisson structures π_1 and π_0 satisfying the recursion relation*

$$\{\ .\ , H_{p+1}\}_1 = \{\ .\ , H_p\}_0, \quad p = 0, 1, \ldots \tag{3.2}$$

Then

$$\{H_p, H_q\}_1 = \{H_p, H_q\}_0 = 0, \quad p, q = 0, 1, \ldots$$

Proof. Let $p < q$ and $q - p = 2m$ for some $m > 0$. Using the recursion and antisymmetry of the brackets we obtain

$$\{H_p, H_q\}_1 = \{H_p, H_{q-1}\}_0 = -\{H_{q-1}, H_p\}_0 = -\{H_{q-1}, H_{p+1}\}_1 = \{H_{p+1}, H_{q-1}\}_1.$$

Iterating one arrives to

$$\{H_p, H_q\}_1 = \cdots = \{H_{p+m}, H_{q-m}\}_1 = 0$$

since $p + m = q - m$. In a similar way in the case $q - p = 2m + 1$ one obtains

$$\{H_p, H_q\}_1 = \cdots = \{H_n, H_{n+1}\}_1 = \{H_n, H_n\}_0 = 0$$

where $n = p + m = q - m - 1$. □

We remark that this proof uses only (3.2) and the skew symmetry of π_1 an π_0, while it does not uses the assumption of compatibility of the Poisson structures.

However, the assumption that π_1 and π_0 are compatible Poisson structures is essential in order to guarantee the existence of functions H_k fulfilling the Lenard-Magri recursion relations (3.2) . The question of existence of such functions in the case of an arbitrary bi-Hamiltonian structure is a difficult problem. In the special case π_1 is invertible, one can defined the field $(1,1)$ tensor $N : TP \to TP$

$$N = \pi_0 \pi_1^{-1} \tag{3.3}$$

which is called the recursion operator for the bi-Hamiltonian structure. The recursion procedures is effective, namely it produces first integrals when, for a given vector $X = X(x)$ of the tangent space T_xP, the vectors $N^k X$ for $0 \leq k \leq 2n - 1$ span the whole tangent space T_xP, where we assume that $\dim P = 2n$.

We recall that the torsion of a $(1,1)$ tensor N on a manifold P is the vector values two-form $T(N)$

$$T(N)(X, Y) = [NX, NY] - N([NX, Y] + [X, NY]) + N^2[X, Y]. \tag{3.4}$$

Definition 3.5 A $(1,1)$ tensor with vanishing torsion is called Nijenhuis tensor or Nijenhuis operator.

We state without proving the following lemma

Lemma 3.6 *[10] If (π_1, π_0) are compatible Hamiltonian structures on P and π_1 is invertible, then the recursion operator $N = \pi_0 \pi_1^{-1}$ is a Nijenhuis operator.*

Remark 3.7 If both π_0 and π_1 are invertible then both the torsions $T(N)$ and $T(N^{-1})$ are equal to zero. One observe that the existence of the Lenard-Magri chain implies that

$$dH_{p+1} = N^* dH_p, \quad p \geq 0.$$

where $N^* : T^*P \to T^*P$ is the adjoint operator, namely the $(1,1)$ tensor $\pi_1^{-1}\pi_0$ applied to the exact differential dH_p gives the exact differential differential dH_{p+1}. When one applies a $(1,1)$ tensor to an exact one form, gets a one form that in general is not exact. The condition of getting an exact differential is not a trivial condition. The difficult part of the bi-Hamiltonian geometry is to show the existence of the functions H_p, $p = 0, 1, \ldots.$ satisfying the Lenard-Magri recursion formula (3.2). In the case in which π_0 and π_1 have maximal rank, this problem has a quite explicit solution which we are going to explain below.

Remark 3.8 If X is a bihamiltonian vector field, namely

$$X = \pi_1 dH_1 = \pi_0 dH_0$$

then $L_X \pi_1 = L_X \pi_0 = 0$ where L_{X_1} is the Lie derivative. Indeed one has

$$(L_X \pi)^{ij} = X^k \frac{\partial \pi^{ij}}{\partial x^k} - \frac{\partial X^i}{\partial x^k} \pi^{kj} - \frac{\partial X^j}{\partial x^k} \pi^{ik}$$

and substituting in the above relation $X^k_{H_1} = \pi_1^{km} \dfrac{\partial H_1}{\partial x^m}$ or $X^k_{H_0} = \pi_0^{km} \dfrac{\partial H_0}{\partial x^m}$ and using (2.7) one immediately obtains that the Lie derivative of the Poisson tensors with respect to their Hamiltonian vector fields is equal to zero.

If π_1 is invertible it follows that also $L_X N = 0$. Let us write in components the above condition

$$(L_X N)^i_j = \sum_k \left(X^k \frac{\partial}{\partial x_k} N^i_j + \frac{\partial X^k}{\partial x_j} N^i_k - \frac{\partial X^i}{\partial x_k} N^k_j \right)$$

If we interpret N as a matrix with entries ij given by N^i_j, one can write the r.h.s. of the above equation in matrix form

$$L_X N + [N, J] = 0 \tag{3.5}$$

where now $(L_X N)_{ij}$ stands for the Lie derivative with respect to X of the function N^i_j and the bracket $[.,.]$ stands for the commutator of matrices and the matrix J is the Jacobian

$$(J)_{kl} = \frac{\partial X^k}{\partial x_l}.$$

Introducing the time t associated to the vector field X, one arrives to the equation

$$\frac{dN}{dt} = [J, N]. \tag{3.6}$$

Such equation has the so-called Lax form and the trace of N^k are all conserved quantities (see below). Such quantities are proportional to the Hamiltonians appearing in the Lenard-Magri recursion formula.

When compatible Poisson tensors do not have maximal rank, the geometry is much more reach and also the corresponding theory of integrable systems [2],[16], [17]. However these issues are beyond the scope of these lectures.

3.1 First integrals associated to a Lax pair

One of the most known method to construct first integrals of a Hamiltonian system is through symmetries of the space P. Another powerful method is due to Lax [11] and represents the starting point of the modern theory of integrable systems. Given an ODE

$$\dot{x} = f(x), \quad x = (x^1, \ldots, x^N) \tag{3.7}$$

and two $m \times m$ matrices $L = (L_{ij}(x))$, $A = (A_{ij}(x))$, they constitute a *Lax pair* for the dynamical systems if for every solution $x = x(t)$ of (3.7) the matrices $L = (L_{ij}(x(t)))$ and $A = (A_{ij}(x(t)))$ satisfy the equation

$$\dot{L} = [A, L] := AL - LA \tag{3.8}$$

and the validity of (3.8) for $L = L(x)$, $A = A(x)$ implies (3.7).

Theorem 3.9 *Given a Lax pair for the dynamical system (3.7), then the eigenvalues $\lambda_1(x), \ldots, \lambda_m(x)$ of $L(x)$ are integrals of motion for the dynamical system.*

Proof. The coefficients $a_1(x), \ldots, a_m(x)$ of the characteristic polynomial

$$\det(L - \lambda I) = (-1)^m \left[\lambda^m - a_1(x)\lambda^{m-1} + a_2(x)\lambda^{m-2} + \cdots + (-1)^m a_m(x)\right] \tag{3.9}$$

of the matrix $L = L(x)$ are polynomials in tr L, tr L^2, $\ldots$, tr L^m:

$$a_1 = \text{tr } L, \quad a_2 = \frac{1}{2}\left[(\text{tr } L)^2 - \text{tr } L^2\right], \ a_3 = \ldots$$

Next we show that

$$\mathrm{tr}\ L^k, \quad k = 1,\, 2, \ldots \tag{3.10}$$

are first integral of the dynamical system. Indeed for $k = 1$

$$\frac{d}{dt}\mathrm{tr}\ L = \mathrm{tr}\ \dot{L} = \mathrm{tr}\,(A\,L - L\,A) = 0.$$

more generally

$$\frac{d}{dt}\mathrm{tr}\, L^k = k\mathrm{tr}\,([A, L]\, L^{k-1}) = 0. \tag{3.11}$$

Since the coefficients of the characteristic polynomial $L(x)$ are constants of motion it follows that its eigenvalues are constants of motion. □

Another proof of the theorem, close to Lax's original proof, can be obtained observing that the solution of the equation $\dot{L} = [A, L]$ can be represented in the form

$$L(t) = Q(t)L(t_0)Q^{-1}(t) \tag{3.12}$$

where the evolution of $Q = Q(t)$ is determined from the equation

$$\dot{Q} = A(t)Q \tag{3.13}$$

with initial data

$$Q(t_0) = 1.$$

Then the characteristic polynomials of $L(t_0)$ e $Q(t)L(t_0)Q^{-1}(t)$ are the same and consequently the eigenvalues are the same.

Remark 3.10 Recalling the results of section 3 for a bihamiltonian system (P, π_0, π_1) one can write the Lax equation (3.6). Therefore the traces of the matrix $N = \pi_0\pi_1^{-1}$ are integrals of motion.

Example 3.11 [4] Let us consider in $\mathbb{R}^{2n}$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ the canonical Poisson bracket π_0 and the non degenerate Poisson bracket π_1 given by

$$\pi_1 = \sum_{i=1}^{n-1} e^{(q^i - q^{i+1})}\frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \sum_{i<j} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q^i}. \tag{3.14}$$

The canonical brackets π_0 and π_1 are compatible brackets. The first traces of the recursion operator $N = \pi_1\pi_0^{-1}$ are given by

$$H_0 = \frac{1}{2}\mathrm{tr}\, N = \sum_{i=1}^{n} p_i, \quad H_1 = \frac{1}{4}\mathrm{tr}\, N^2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

$$H_2 = \frac{1}{6}\mathrm{tr}\, N^3 = \frac{1}{3}\sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1}(p_i + p_{i+1})e^{q_i - q_{i+1}},$$

and so on. The Hamiltonian H_1 is the Hamiltonian of the open Toda lattice equation with respect to the Poisson bracket π_0 (see next section). The conserved quantities given by $H_k = \dfrac{1}{2(k+1)} \mathrm{Tr}\, N^{k+1}, \quad 0 \le k \le n-1$ are independent and involution with respect to both Poisson brackets π_0 and π_1 and satisfy the Lenard-Magri recursion

$$\{., H_k\}_0 = \{., H_{k-1}\}_1.$$

4 The Toda system

Let us consider the system of n points $q_1,\ q_2, \ldots, q_n$ on the real line interacting with nearest neighbour interaction potential

$$U(q_1, \ldots, q_n) = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

the so called Toda lattice. The Hamiltonian $H(q,p) \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ takes the form

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}. \tag{4.1}$$

There are two possible boundary conditions:

- open Toda with $q_0 = -\infty$ and $q_{n+1} = +\infty$
- closed Toda with $q_1 = q_{n+1}$.

Here we analyse the open Toda lattice. The Hamilton equations with respect to the canonical Poisson bracket

$$\{q_k, p_j\} = \delta_{kj}, \quad \{q_k, q_j\} = \{p_k, p_j\} = 0, jk = 1, \ldots, n \tag{4.2}$$

are

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} = p_k, \quad k = 1, \ldots, n \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} = \begin{cases} -e^{q_1 - q_2} & \text{if } k = 1 \\ e^{q_{k-1} - q_k} - e^{q_k - q_{k+1}} & \text{if } 2 \le k \le n-1 \\ e^{q_{n-1} - q_n} & \text{if } k = n \end{cases} \end{aligned}$$

Since the Hamiltonian is translation invariant, the total momentum is a conserved quantity together with the Hamiltonian.

Flaschka [5],[6] and Manakov [14] separetely showed that the Toda lattice is a completely integrable system. Let us introduce a new set of dependent variables

$$\begin{aligned} a_k &= \frac{1}{2}\, e^{\frac{q_k - q_{k+1}}{2}}, \quad k = 1, \ldots, n-1 \\ b_k &= -\frac{1}{2}\, p_k, \quad k = 1, \ldots, n, \end{aligned} \tag{4.3}$$

with evolution given by the equations

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k), \quad k = 1, \ldots, n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2), \quad k = 1, \ldots, n, \end{aligned} \tag{4.4}$$

where we use the convention that $a_0 = a_n = 0$. Observe that there are only $2n-1$ variables and this is due the translation invariance of the original system. The equations (4.4) have an Hamiltonian form with Hamiltonian

$$H(a, b) = 2\sum_{i=1}^{n} b_i^2 + 4\sum_{i=1}^{n-1} a_i^2$$

with Poisson bracket define on $(\mathbb{R}^*)^{n-1} \times \mathbb{R}^n$ given by

$$\{a_i, b_j\} = -\frac{1}{4}\delta_{ij} a_i + \frac{1}{4}\delta_{i,j-1} a_i, \quad i = 1, \ldots, n-1, \;\; j = 1, \ldots, n,$$

while all the other entries are equal to zero. We observe that the total momentum $\sum_{k=1}^{n} b_k$ is a Casimir of the above Poisson bracket

Next we introduce the tridiagonal $n \times n$ matrices:

$$
\begin{aligned}
L &= \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{pmatrix} \\
A &= \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ -a_1 & 0 & a_2 & & 0 & 0 \\ 0 & -a_2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & a_{n-1} \\ 0 & & & & -a_{n-1} & 0 \end{pmatrix}
\end{aligned}
\tag{4.5}
$$

where $A = L_+ - L_-$ and we are using the following notation: for a square matrix X we call X_+ the upper triangolar part of X

$$
(X_+)_{ij} = \begin{cases} X_{ij}, & i < j \\ 0, & \text{otherwise} \end{cases}
$$

and in a similar way by X_- the lower triangular part of X

$$
(X_-)_{ij} = \begin{cases} X_{ij}, & i < j \\ 0, & \text{otherwise.} \end{cases}
$$

A straighforward calculation shows that

Lemma 4.1 *The Toda lattice equations (4.4) are equivalent to*

$$
\frac{dL}{dt} = [A, L]. \tag{4.6}
$$

Exercise 4.2 Determine the Lax pair for the closed Toda lattice.

The open Toda lattice equation is sometimes written in the literature in Hessebeg form. Conjugating the matrix L by a diagonal matrix $D = \mathrm{diag}(1, a_1, a_1 a_2, \dots, \prod_{j=1}^{n-1} a_j)$

yelds the matrix $\widehat{L} = DLD^{-1}$

$$\widehat{L} = \begin{pmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ a_1^2 & b_2 & 1 & & 0 & 0 \\ 0 & a_2^2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & 1 \\ 0 & & & & a_{n-1}^2 & b_n \end{pmatrix} \tag{4.7}$$

The Toda equations (4.4) take the form

$$\frac{d\widehat{L}}{dt} = -2[\widehat{A}, \widehat{L}] \tag{4.8}$$

where the matrix $\widehat{A} = \widehat{L}_-$ namely

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1^2 & 0 & 0 & & 0 & 0 \\ 0 & a_2^2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & 0 \\ 0 & & & & a_{n-1}^2 & 0 \end{pmatrix} \tag{4.9}$$

From the results of the previous section, the Lax formulation guarantees the existence of conserved quantities, namely the traces

$$F_j = \text{tr } L^{j+1}, \quad j = 0, \dots, n-1.$$

are conserved quantities. To show the independence of the integrals $F_0, \dots, F_{n-1}$ we observe that

$$F_{j-1} = \sum_{k=1}^{n} b_k^j + \text{lower order polynomials of } a_k \text{ and } b_k.$$

Since the polynomials $b_1^j + b_2^j + \cdots + b_n^j$ for $j = 1, \dots, n$ are linearly independent with respect to the variables $b_1, \dots, b_n$, it follows that the integrals $F_0, \dots, F_{n-1}$ are functionally independent. Next we show that the integrals are in involution. For the purpose we need the following lemma.

Lemma 4.3 *(i) The spectrum of L consists of n distinct real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.*

(ii) *Let* $Lv = \lambda v$ *with* $v = (v_1, \dots v_n)^t$. *Then* $v_1 \neq 0$ *and* $v_n \neq 0$. *Furthermore,* $v_k = v_1 p_k(\lambda)$ *where* $p_k(\lambda)$ *is a polynomial of degree* k *in* λ.

Proof. We will first prove (ii). From the equation $Lv = \lambda v$ one obtains

$$(b_1 - \lambda)v_1 + a_1 v_2 = 0 \tag{4.10}$$

$$a_{k-1}v_{k-1} + (b_k - \lambda)v_k + a_k v_{k+1} = 0, \quad 2 \leq k < n. \tag{4.11}$$

Since $a_1 \neq 0$ clearly $v_1 = 0 \Longrightarrow v_2 = 0$, but then from (4.11) with $k = 2$, since $a_2 \neq 0$, then $v_1 = 0$ and $v_2 = 0$ implies $v_3 = 0$. Hence $v = 0$ if $v_1 = 0$. Therefore $v_1 \neq 0$. In the same way it can be proved that $v_n \neq 0$. From (4.10) and (4.11) it easily follows that v_k is a polynomial of degree k in λ. To prove (i), since L is symmetric, the eigenvalues are real. In order to show that the eigenvalues are distinct, let us suppose that v and $\tilde{v}$ are two eigenvalues corresponding to the same eigenvector λ. Then the linear combination $\alpha v + \beta \tilde{v}$ is also an eigenvector of L with eigenvalue λ. But then one can choose $\alpha \neq 0$ and $\beta \neq 0$ so that $\alpha v_1 + \beta \tilde{v}_1 = 0$ and by (ii) it follows that $\alpha v + \beta \tilde{v}$=0 implying that v and $\tilde{v}$ are dependent. $\square$

Using the above lemma one has

$$\det \frac{\partial F_j}{\partial \lambda_k} = \begin{pmatrix} 1 & 1 & \dots 1 & \\ 2\lambda_1 & 2\lambda_2 & \dots & 2\lambda_n \\ \dots & \dots & \dots & \dots \\ (n-1)\lambda_1 & (n-1)\lambda_2 & \dots & (n-1)\lambda_n \end{pmatrix} = n! \prod_{i<j} (\lambda_i - \lambda_j) \neq 0,$$

because the eigenvalues are all distinct. This shows that we can take the eigenvalues $\lambda_1, \dots, \lambda_n$ as a new set of functionally independent variables. In order to show that the Toda lattice is an integrable system we also need to show that the functions $F_1, \dots, F_n$, or equivalently the eigenvalues $\lambda_1, \dots, \lambda_n$ commute with respect to the canonical Poisson bracket. For the purpose let us consider the equation

$$Lv = \lambda v, \tag{4.12}$$

where v is a normalised eigenvector, $v = (v_1, \dots, v_n)^t$ and $(v, v) = 1$. Then we introduce the discrete Wronkstian

$$W_i(v, w) = a_i(v_i w_{i+1} - v_{i+1} w_i) \tag{4.13}$$

where w is an eigenvector with respect to the eigenvalue μ. We use the convention that $W_0 = W_n = 0$. It is easy to see using the equation (4.12) that the Wronkstian satisfies

$$W_i = (\mu - \lambda)v_i w_i + W_{i-1}. \tag{4.14}$$

Indeed we have from (4.12)

$$(b_i - \lambda)v_i + a_{i-1}v_{i-1} + a_i v_{i+1} = 0, \quad (b_i - \mu)w_i + a_{i-1}w_{i-1} + a_i w_{i+1} = 0.$$

Multiplying the first equation by w_i and the second by v_i and subtracting them, one obtains the statement. We are ready to prove the following.

Proposition 4.4 *The eigenvalues of L commute with respect to the canonical Poisson bracket (4.2).*

Proof. Let us consider the equation (4.12) and its variational derivative

$$\delta L\, v + L\, \delta v = v\, \delta\lambda + \lambda\, \delta v$$

Taking the scalar product with respect to v and using $(v, v) = 1$ one obtains

$$\delta\lambda = (v, \delta L\, v) + (v, (L - \lambda)\delta v) = (v, \delta L\, v) + ((L - \lambda)v, \delta v) = (v, \delta L\, v) \tag{4.15}$$

where we use the fact that the operator L is symmetric.

Let λ and μ be two eigenvalues of L with normalized eigenvectors v an w respectively. Then from (4.15) one has

$$\begin{aligned} \frac{\partial\lambda}{\partial p_i} &= (v, \frac{\partial L}{\partial p_i} v) = -\frac{1}{2} v_i^2 \\ \frac{\partial\lambda}{\partial q_i} &= (v, \frac{\partial L}{\partial q_i} v) = a_i v_i v_{i+1} - a_{i-1} v_i v_{i-1}, \quad i = 1, \ldots, n, \end{aligned} \tag{4.16}$$

where we use the fact that $(v, v) = 1$ and we define $a_0 = 0 = a_n$. The same relations hold for the eigenvalue μ. Then one has

$$\begin{aligned} \{\lambda, \mu\} &= \sum_{i=1}^{n} \left(\frac{\partial\lambda}{\partial q_i} \frac{\partial\mu}{\partial p_i} - \frac{\partial\lambda}{\partial p_i} \frac{\partial\mu}{\partial q_i} \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} \left(v_i w_i (a_i (v_i w_{i+1} - v_{i+1} w_i) + a_{i-1}(w_i v_{i-1} - v_i w_{i-1}) \right). \end{aligned} \tag{4.17}$$

Using the definition of Wronkstian in (4.13) and the identity (4.14) one can reduce the above relation to the form

$$\{\lambda, \mu\} = \frac{1}{2(\mu - \lambda)} \sum_{i=1}^{n} (W_i^2 - W_{i-1}^2) = \frac{W_n^2 - W_0^2}{2(\mu - \lambda)} = 0.$$

$\square$

Summarazing, we have proved the following theorem.

Theorem 4.5 *The Toda Lattice is a completely integrable Hamiltonian system.*

By Liouville theorem it follows that the Toda system can be integrated by quadratures. Let us show how to do this. By the lemma 4.3 it follows that

$$L = U\Lambda U^t \tag{4.18}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with distinct eigenvalues and U is an orthogonal matrix $UU^t = 1$ with entries $U_{ij} = u_{ij}$ the normalized eigenvectors $u_i = (u_{1i}, \ldots u_{ni})^t$ of L. From $UU^t = U^tU = 1$ one has

$$(u_i, u_j) = \delta_{ij}, \quad \sum_{k=1}^{n}(u_{kj})^2 = 1, \quad i, j = 1, \ldots, n.$$

We know the eigenvalues of $L(t)$, since they are constants of motion. In order to know $L(t)$ at time t we need to know the orthogonal matrix $U = U(t)$, with entries $U_{ij} = u_{ij}$. From (4.6) and (4.18) one has that

$$\dot{U} = AU. \tag{4.19}$$

In particular, the dynamics implied by the above equation on the first row u_{1i}, $i = 1, \ldots, n$ of the matrix U is quite simple.

Lemma 4.6 *The time evolution of the first row of the matrix U, namely the entries u_{1i} $i = 1, \ldots, n$ are given by*

$$u_{1i}(t)^2 = \frac{e^{2\lambda_i t}u_{1i}(0)^2}{\sum_{k=1}^{n} e^{2\lambda_k t}u_{1k}(0)^2}, \quad i = 1, \ldots, n. \tag{4.20}$$

Proof. From (4.19) one has

$$\frac{du_{1i}}{dt} = (AU)_{1i} = a_1 u_{i2}$$

and from the relation $Lu_i = \lambda_i u_i$, with $u_i = (u_{1i}, \ldots, u_{ni})^t$, one reduces the above equation to the form

$$\frac{du_{1i}}{dt} = (\lambda_i - b_1)u_{1i}.$$

The solution is given by

$$u_{1i}(t) = E(t)e^{\lambda_i t}u_{1i}(0), \quad E(t) = exp\left(-\int_0^t b_1(\tau)d\tau\right)$$

Using the normalization conditions

$$1 = \sum_{i=1}^{n} u_{1i}(t)^2 = E(t)^2 \sum_{i=1}^{n} e^{2\lambda_i t} u_{1i}(0)^2$$

which implies

$$E(t)^2 = \left(\sum_{i=1}^{n} e^{2\lambda_i t} u_{1i}(0)^2 \right)^{-1}$$

one arrives to the statement of the lemma. □

Introducing the notation

$$\xi_i(t) = u_{1i}(t), \quad i = 1, \ldots, n \tag{4.21}$$

one can see from lemma 4.3 that the orthogonal matrix U can be written in the form

$$U = \begin{pmatrix} \xi_1(t)p_0(\lambda_1, t) & \xi_2(t)p_0(\lambda_2, t) & \ldots & \xi_n(t)p_0(\lambda_n, t) \\ \xi_1(t)p_1(\lambda_1, t) & \xi_2(t)p_1(\lambda_2, t) & \ldots & \xi_n(t)p_1(\lambda_n, t) \\ \vdots & \vdots & & \vdots \\ \xi_1(t)p_{n-1}(\lambda_1, t) & \xi_2(t)p_{n-1}(\lambda_2, t) & \ldots & \xi_n(t)p_{n-1}(\lambda_n, t) \end{pmatrix}$$

where $p_k(\lambda, t)$ is a polynomial of degree k in λ. Since U is an orthogonal matrix, the orthogonality relations on the rows of U take the form

$$\sum_{k=1}^{n} \xi_k^2 p_m(\lambda_k) p_j(\lambda_k) = \delta_{mj}. \tag{4.22}$$

In other words, the polynomials $p_j(\lambda, t)$ are normalized orthogonal polynomials with respect to the discrete weights ξ_k^2 at the points λ_k. To find the orthogonal polynomials from the weights, is a standard procedure, called QR factorisation, which is a decomposition of a matrix into an orthogonal matrix and an upper triangular matrix. There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process. The QR factorisation for solving the Toda lattice equations was developed by Symes [18]. Therefore, from the weights $\xi_1(t), \ldots \xi_n(t)$ at time t one can get the orthogonal matrix $U(t)$. The explicit calculations on the Toda lattice (and the full hierarchy) can also be found in [9]. Here we report the result without stating the proof. Introducing the quantities

$$c_{ij}(t) = \sum_{k=1}^{n} u_{ik}(0) u_{jk}(0) e^{2\lambda_k t}$$

and

$$D_k(t) = \det(\{c_{ij}(t)\}_{i,j=1\dots,k}), \quad D_0(t) = 1,$$

the matrix entries of U are given by

$$u_{ki}(t) = \frac{e^{\lambda_i t}}{\sqrt{D_k(t) D_{k-1}(t)}} \begin{vmatrix} c_{11}(t) & \dots & c_{1k}(t) \\ & \ddots & \\ c_{k-1\,1}(t) & \dots & c_{k-1\,k}(t) \\ u_{1i}(0) & \dots & u_{ki}(0) \end{vmatrix}.$$

Using the identity $L(t) = U(t)L(0)U(t)^t$ we recover the quantities $b_j(t)$ and $a_j(t)$, namely

$$b_j(t) = \sum_{k=1}^{n} \lambda_k u_{jk}^2(t), \quad j = 1, \dots, n, \tag{4.23}$$

$$a_j(t) = \sum_{k=1}^{n} \lambda_k u_{jk}(t) u_{j+1\,k}(t), \quad j = 1, \dots, n-1. \tag{4.24}$$

We are going to derive a different procedure to integrate the Toda lattice due to Moser [15]. Recall that we have denoted by $(\xi_1, \dots \xi_n)$ the first row of the matrix U. Consider the set

$$\text{Spec} = \{\lambda_1 < \lambda_2 < \dots < \lambda_n,\ (\xi_1, \dots, \xi_n),\ \xi_i > 0,\ \sum_{i=1}^{n} \xi_i^2 = 1\}, \tag{4.25}$$

which is the spectral data associated to the matrix $L = L(a, b)$. The matrix L is a Jacobi matrix, namely a tridiagonal symmetric matrix where the lower and upper diagonal entries are positive.

Theorem 4.7 (Moser) *The spectral map*

$$S : L(a, b) \to Spec$$

is a bijection between Jacobi matrices and the set Spec.

Proof. We need to show that for a given set $(\lambda, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_1 < \lambda_2 < \dots < \lambda_n$ and $\xi = (\xi_1, \dots, \xi_n)$ with $(\xi, \xi) = 1$ and $\xi_i > 0$ there is a unique Jacobi matrix with such spectral data. For the purpose define for $j = 0, \dots, n-1$ the $(n-j) \times (n-j)$ matrices

$$\Delta_j(z) = \det \begin{pmatrix} z - b_{j+1} & -a_{j+1} & 0 & 0 & \dots & \dots \\ -a_{j+1} & z - b_{j+2} & -a_{j+2} & 0 & \dots & \dots \\ 0 & -a_{j+2} & z - b_{j+3} & -a_{j+3} & & \\ & \dots & \dots & & & \dots \\ \dots & & 0 & 0 & -a_{n-1} & z - b_n \end{pmatrix}$$

with $\Delta_n(z) := 1$ and $\Delta_{n+1}(z) = 0$ and $\Delta_0(z) = \det(zI - L)$. It is easy to see that Δ_j is a polynomial of degree $n - j$. Furthermore, expanding the determinant along the first column, one obtains the recursion relation

$$\Delta_j(z) = (z - b_{j+1})\Delta_{j+1} - a_{j+1}^2 \Delta_{j+2}. \tag{4.26}$$

Now let us consider the entry $(1,1)$ of the resolvent $R := (zI - L)^{-1}$. Such entry turns out the be equal to

$$R(z)_{11} = \frac{\Delta_1(z)}{\Delta_0(z)}$$

by the form of the inverse of a matrix. On the other hand, one also has

$$R(z)_{11} = (zI - L)^{-1}_{11} = (U(zI - \Lambda)U^t)^{-1}_{11} = (U(zI - \Lambda)^{-1}U^t)_{11} = \sum_{i=1}^{n} \frac{\xi_i^2}{z - \lambda_i}$$

Combining the above two relations and the recursive formula (4.26) one arrives to the continued fraction expansion

$$\sum_{i=1}^{n} \frac{\xi_i^2}{z - \lambda_i} = \frac{1}{\frac{\Delta_0}{\Delta_1}} = \frac{1}{z - b_1 - \frac{a_1^2}{\frac{\Delta_1}{\Delta_2}}} = \frac{1}{z - b_1 - \frac{a_1^2}{z - b_2 - \frac{a_2^2}{\ddots \, \cdots - \frac{a_{n-1}^2}{z - b_n}}}} \tag{4.27}$$

$\square$

For example from the continued fraction expansion one has

$$b_1 = \sum_{i=1}^{n} \lambda_i \xi_i^2, \quad a_{n-1} = \sum_{i<j} (\lambda_i - \lambda_j)\xi_i \xi_j.$$

So the integration of the Toda lattice is obtained by the following diagram:

$$\begin{array}{ccc} \{a_i(0),\, b_i(0)\} & \overset{\text{direct spectral problem}}{\Longrightarrow} & \{\lambda_1, \ldots, \lambda_n,\, \xi_1(0) \ldots, \xi_n(0)\} \\ & & \Downarrow \\ \{a_i(t),\, b_i(t)\} & \overset{\text{inverse spectral problem}}{\Longleftarrow} & \{\lambda_1, \ldots, \lambda_n,\, \xi_1(t) \ldots, \xi_n(t)\}. \end{array}$$

Such procedure is called inverse scattering.

Example 4.8 In the particular case $n = 2$ from the continued fraction expansion

$$\frac{\xi_1^2}{z-\lambda_1} + \frac{\xi_2^2}{z-\lambda_2} = \frac{1}{z - b_1 - \dfrac{a_1^2}{z-b_2}}$$

one can get easily the explicit formulas of the solution

$$b_1(t) = -\frac{1}{2}p_1 = \lambda_1\xi_1(t)^2 + \lambda_2\xi_2(t)^2 = \frac{\lambda_1\xi_1(0)^2e^{2\lambda_1 t} + \lambda_2\xi_2(0)^2e^{2\lambda_2 t}}{\xi_1(0)^2e^{2\lambda_1 t} + \xi_2(0)^2e^{2\lambda_2 t}}$$

$$b_2(t) = -\frac{1}{2}p_2 = \lambda_2\xi_1(t)^2 + \lambda_1\xi_2(t)^2 = \frac{\lambda_2\xi_1(0)^2e^{2\lambda_1 t} + \lambda_1\xi_2(0)^2e^{2\lambda_2 t}}{\xi_1(0)^2e^{2\lambda_1 t} + \xi_2(0)^2e^{2\lambda_2 t}}$$

$$a_1 = \frac{1}{2}e^{\frac{q_1-q_2}{2}} = (\lambda_2-\lambda_1)\xi_1(t)\xi_2(t) = \frac{(\lambda_2-\lambda_1)\xi_1(0)\xi_2(0)e^{(\lambda_1+\lambda_2)t}}{\xi_1(0)^2e^{2\lambda_1 t} + \xi_2(0)^2e^{2\lambda_2 t}}$$

or equivalently

$$q_1 = -\log\left(\xi_1(0)^2e^{2\lambda_1 t} + \xi_2(0)^2e^{2\lambda_2 t}\right)$$

$$q_2 = -2(\lambda_1+\lambda_2)t - 2\log(2(\lambda_1-\lambda_2)\xi_1(0)\xi_2(0)) + \log\left(\xi_1(0)^2e^{2\lambda_1 t} + \xi_2(0)^2e^{2\lambda_2 t}\right).$$

Observe that for $t \to +\infty$ one has

$$a_1(t) \to 0, \quad b_1(t) \to \lambda_2, \quad b_2(t) \to \lambda_1.$$

Exercise 4.9 Prove that for $t \to +\infty$ the Lax matrix becomes diagonal with entries

$$L \to \mathrm{drag}(\lambda_n, \lambda_{n-1}, \ldots, \lambda_2, \lambda_1).$$

4.1 The Hamiltonian flows

Let us consider the Hamiltonians

$$H_k = (-1)^{k+1}\frac{4}{k+1}\mathrm{tr}\ L^{k+1}, \quad k = 0, \ldots, n-1.$$

Observe that $H_1 = H$ is the Hamiltonian of the Toda lattice. The Hamiltonian flows generated by H_k are called Toda lattice hierarchy and they are given by the Hamilton equations

$$\left.\begin{aligned} \frac{\partial q_i}{\partial t_k} &= \frac{\partial H_k}{\partial p_i} \\ \frac{\partial p_i}{\partial t_k} &= -\frac{\partial H_k}{\partial q_i} \end{aligned}\right\}, \quad i = 1, \ldots, n. \tag{4.28}$$

Such flows have a Lax representation given by the following expressions. Let us define the matrices A_k, $k \geq 1$ as

$$A_k = \left(L^k\right)_+ - \left(L^k\right)_-, \quad k = 1, 2, \ldots. \tag{4.29}$$

with the convention that $A_1 = A$.

Exercise 4.10 Show that for $k \geq 1$ the commutator $[A_k, L]$ is a tridiagonal matrix.

Hint: using commutativity of $[L^k, L] = 0$ show that

$$[A_k, L] =, \left[2(L^k)_+ - (L^k)_0, L\right],$$

where $(L^k)_0$ is the diagonal of L^k. Then use the property of the commutator between an upper triangular matrix and a tridiagonal matrix.

Exercise 4.11 Show that the Hamiltonian flows (4.28) are equivalent to the equations

$$\frac{\partial L}{\partial t_k} := [A_k, L], \quad k = 1, \ldots, n. \tag{4.30}$$

Exercise 4.12 Show from the commutativity of the Toda flows that the matrices A_j , A_k satisfy the equations

$$\frac{\partial A_k}{\partial t_l} - \frac{\partial A_l}{\partial t_k} = [A_k, A_l], \quad j, k = 1, \ldots, n. \tag{4.31}$$

Such equations are also called zero curvature representation.

Exercise 4.13 Integrate the Toda hierarchy for any time t_j, $j = 1, \ldots, n$.

Exercise 4.14 Consider the recursion operator $N = \pi_1 \pi_0^{-1}$ where π_0 is the canonical Poisson bracket and π_1 defined in (3.14) is compatible with π_0. Show that

$$H_k = \frac{1}{2(k+1)} \mathrm{Tr} N^{k+1} = (-1)^{k+1} \frac{4}{k+1} \mathrm{Tr} L^{k+1}$$

where L is the Lax matrix (4.5).

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References

[1] A. Cannas da Silva, A. Weinstein, Geometric models for noncommutative algebras, Amer. Math. Soc. 1999 (Berkeley Mathematics Lecture Notes).

[2] B. Dubrovin, Y. Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants. Preprint arXiv:math/0108160.

[3] B. Dubrovin, Istituzioni di Fisica Matematica http://people.sissa.it/$\sim$ dubrovin

[4] R.L. Fernandes, On the master symmetries and bi-Hamiltonian structure of the Toda lattice. J. Phys. A **26** (1993), no. 15, 3797 - 3803.

[5] H. Flaschka, The Toda lattice. I. Existence of integrals, Phys. Rev. B 9, (1974) 1924 - 1925.

[6] Flaschka, H.: On the Toda lattice II, Prog. Theor. Phys.**51** (1974), 703-716.

[7] Gardner C. S.; Green J. M; Kruskal M. D.; Miura R. M. *Phys. Rev. Lett.* **19** (1967), 1095.

[8] V. L. Ginzburg, Momentum mappings and Poisson cohomology, Internat. J. Math. 7 (1996), no. 3, 329 - 358.

[9] Y. Kodama, K. McLaughlin Explicit integration of the full Symmetric Today Hierarchy and the sorting problem, Lett. Math Physics **37**, (1996) 37 - 47.

[10] Y. Kosmann-Schwarzbach, F. Magri, Lax - Nijenhuis operators for integrable systems, *J. Math. Phys.* **37**, (1996), 6173 - 6197.

[11] Lax, Peter D. Integrals of nonlinear equations of evolution and solitary waves. Comm. Pure Appl. Math. **21** (1968) 467 - 490.

[12] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. **19**, (1978) 1156 - 1162.

[13] H. McKean, Compatible brackets in Hamiltonian mechanics, In: "Important development in soliton theory", 344 - 354, A.S. Fokas, V.E. Zakharov eds., Springer Series in Nonlinear dynamics, 1993.

[14] Manakov, S. V.: Complete integrability and stochastization of discrete dynamical systems, Soviet Phys. JETP 40 (1975), 703-716 (Zh. Exp. Teor. Fiz. 67 (1974), 543-555).

[15] Moser, J., Finitely many mass points on the line under the Influence of an exponential potential - An integrable system, in: J. Moser (ed), Dynamical Systems, Theory and Applications, Lecture Notes in Physics 38, Springer-Verlag, Berlin, New York, 1975, pp. 467-497.

[16] A. Lichnerowicz, Les variètès de Poisson et leurs algébres de Lie associées, J. Differential Geometry **12** (1977), 253 - 300.

[17] J.P. Dufour, N.T. Zung, Normal forms of Poisson structures. Lectures on Poisson geometry, 109 - 169, Geom. Topol. Monogr., 17, Geom. Topol. Publ., Coventry, 2011.

[18] Symes, W.,The QR algorithm and scattering for the finite non-periodic Toda lattice, Physica D **4** (1982), 275 - 278.