Hybrid ROMs for problems in computational fluid dynamics, with special focus on turbulent flows

Saddam Hijazi[1], Giovanni Stabile[1], Andrea Mola[1], Gianluigi Rozza[1]
[1]SISSA mathLab, Trieste, Italy

Introduction

Turbulence is a physical phenomenon which occurs in many engineering fields such as naval engineering (boat wakes), aerospace engineering (flow around the wings) and civil engineering (vortex shedding around circular cylinder in structures). Therefore, simulating turbulent problems in Computational Fluid Dynamics (CFD) is sought. Despite the recent scientific breakthroughs achieved in computer science and high performance computing, simulating CFD problems and in particular turbulent problems using classical discretization methods remains challenging and computationally expensive. This work presents a Reduced Order Model (ROM) which is designed specifically to deal with turbulent flows. The model is a combination of classical projection-based methods and data-driven techniques. It has been validated on the benchmark test case of the flow around a circular cylinder.

The Full Order Model (FOM)

The problem at full order level is described through the unsteady Reynolds Average Navier–Stokes Equations (RANS) with the employment of $k-\omega$ turbulence model. The unsteady RANS equations read as follows

$$\begin{cases} \frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}}\otimes\overline{\boldsymbol{u}}) = \boldsymbol{\nabla}\cdot\left[-\overline{p}\mathbf{I} + (\nu+\nu_t)\left(\boldsymbol{\nabla}\overline{\boldsymbol{u}} + (\boldsymbol{\nabla}\overline{\boldsymbol{u}})^T\right)\right] & \text{in } \Omega\times[0,T],\\ \boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}} = 0 & \text{in } \Omega\times[0,T],\\ \overline{\boldsymbol{u}}(t,\boldsymbol{x}) = \boldsymbol{U}_{in} & \text{on } \Gamma_{In}\times[0,T],\\ \overline{\boldsymbol{u}}(t,\boldsymbol{x}) = \boldsymbol{0} & \text{on } \Gamma_0\times[0,T],\\ (\nu\boldsymbol{\nabla}\overline{\boldsymbol{u}} - \overline{p}\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_{Out}\times[0,T],\\ \overline{\boldsymbol{u}}(0,\boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{x}) & \text{in } (\Omega,0),\\ \nu_t = F(k,\omega), & \text{in } \Omega,\\ \text{Transport-Diffusion equation for } k, & \\ \text{Transport-Diffusion equation for } \omega, & \end{cases} \tag{1}$$

the FOM is parameterized through the vector parameter $\boldsymbol{\mu}$ and then solved for each $\boldsymbol{\mu}\in\mathcal{P}_M = \{\boldsymbol{\mu}_1,...\boldsymbol{\mu}_M\}\subset\mathcal{P}$, where $\mathcal{P}_M$ is a finite set of samples inside the parameter space $\mathcal{P}$, while the time instants at which the fields are acquired are called $\{t_1,t_2,...,t_{N_T}\}\subset[0,T]$

Fully Projection Based ROMs

The starting point in developing the ROM is the usual decomposition of the fields into a sum of global spatial modes multiplied by temporal coefficients:

$$\overline{\boldsymbol{u}}(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_u} a_i(t;\boldsymbol{\mu})\boldsymbol{\phi}_i(\boldsymbol{x}), \quad \overline{p}(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_p} b_i(t;\boldsymbol{\mu})\chi_i(\boldsymbol{x}). \tag{2}$$

The reduced modes $\boldsymbol{\phi}_i(\boldsymbol{x})$ and $\chi_i(\boldsymbol{x})$ are computed by the Proper Orthogonal Decomposition (POD). The POD technique constructs the reduced basis through the snapshots method. The velocity snapshots matrix $\boldsymbol{\mathcal{S}_u}$ is given by:

$$\boldsymbol{\mathcal{S}_u} = \{\overline{\boldsymbol{u}}(\boldsymbol{x},t_1;\boldsymbol{\mu}_1),...,\overline{\boldsymbol{u}}(\boldsymbol{x},t_{N_T};\boldsymbol{\mu}_M)\} \in \mathbb{R}^{N_u^h\times N_s}. \tag{3}$$

We proceed to the projection step of the momentum equation of (1). This step will give the following dynamical system with the unknowns being the vectors of coefficients $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\dot{\boldsymbol{a}} = \nu\boldsymbol{B}\boldsymbol{a} - \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} - \boldsymbol{H}\boldsymbol{b} + \boldsymbol{G}(\nu_t,\boldsymbol{a}). \tag{4}$$

To close the system we used the supremizer enrichment method which allows to project the continuity equation onto the pressure modes, the resulting system is the following

$$\begin{cases} \boldsymbol{M}\dot{\boldsymbol{a}} = \nu\boldsymbol{B}\boldsymbol{a} - \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} - \boldsymbol{H}\boldsymbol{b} + \boldsymbol{G}(\nu_t,\boldsymbol{a}),\\ \boldsymbol{P}\boldsymbol{a} = \boldsymbol{0}. \end{cases} \tag{5}$$

Hybrid ROM approach

The reduced solutions $\boldsymbol{a}$ and $\boldsymbol{b}$ may be found by solving 5 but this is not possible at this point because of the term $\boldsymbol{G}(\nu_t,\boldsymbol{a})$. To approximate the latter term we propose to extend 2 to ν_t, namely:

$$\nu_t(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_{\nu_t}} g_i(t,\boldsymbol{\mu})\eta_i(\boldsymbol{x}), \tag{6}$$

Inserting 6 into 5 yields the following system:

$$\begin{cases} \boldsymbol{M}\dot{\boldsymbol{a}} = \nu(\boldsymbol{B}+\boldsymbol{B_T})\boldsymbol{a} - \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} + \boldsymbol{g}^T(\boldsymbol{C_{T1}}+\boldsymbol{C_{T2}})\boldsymbol{a} - \boldsymbol{H}\boldsymbol{b},\\ \boldsymbol{P}\boldsymbol{a} = \boldsymbol{0}, \end{cases} \tag{7}$$

where $\boldsymbol{G}(\nu_t,\boldsymbol{a}) \approx \boldsymbol{g}^T(\boldsymbol{C_{T1}}+\boldsymbol{C_{T2}})\boldsymbol{a}$. Interpolation using Radial Basis Functions (RBF) is used to approximate the value of $\boldsymbol{g}$. First one can notice the following:

$$\nu_t = f(\overline{\boldsymbol{u}}) \rightarrow \boldsymbol{g} = \tilde{f}(\boldsymbol{a}) \tag{8}$$

The interpolation using RBF functions is based on the following formula :

$$G_L(\boldsymbol{a}) = \sum_{j=1}^{N_s} w_{L,j}\zeta_{L,j}(\|\boldsymbol{a}-\boldsymbol{a}_{L_2}^j\|_{\mathbb{R}^{N_u}}), \quad \text{for} \quad L = 1,2,...,N_{\nu_t}, \tag{9}$$

where $\boldsymbol{a}\in\mathbb{R}^{N_u}$ is a general reduced velocity vector, N_s is the total number of snapshots, $\zeta_{L,j}$ for $j=1,...,N_s$ are the RBF functions which are chosen to be Gaussian functions, $\zeta_{L,j}$ is centered in $\boldsymbol{a}_{L_2}^j$ and $w_{L,j}$ are appropriate weights of $\zeta_{L,j}$. For the computation of the weights, we start by computing the L^2 coefficients of velocity and viscosity as follows:

$$g_{r,l} = (\boldsymbol{\mathcal{S}}_{\nu_t}^r,\eta_l)_{L^2(\Omega)}, \quad \text{for} \quad r=1,2,...,N_s \quad \text{and} \quad l=1,2,...,N_{\nu_t}. \tag{10}$$

$$\boldsymbol{a}_{L_2}^r = [(\boldsymbol{\mathcal{S}}_{\boldsymbol{u}}^r,\boldsymbol{\phi}_1)_{L^2(\Omega)},...,(\boldsymbol{\mathcal{S}}_{\boldsymbol{u}}^r,\boldsymbol{\phi}_{N_u})_{L^2(\Omega)}], \quad \text{for} \quad r=1,2,...,N_s. \tag{11}$$

Now we can compute the weights by using the following property which essentially comes from the data of the FOM:

$$G_L(\boldsymbol{a}_{L_2}^i) = g_{i,L}, \quad \text{for} \quad i=1,2,...,N_s, \tag{12}$$

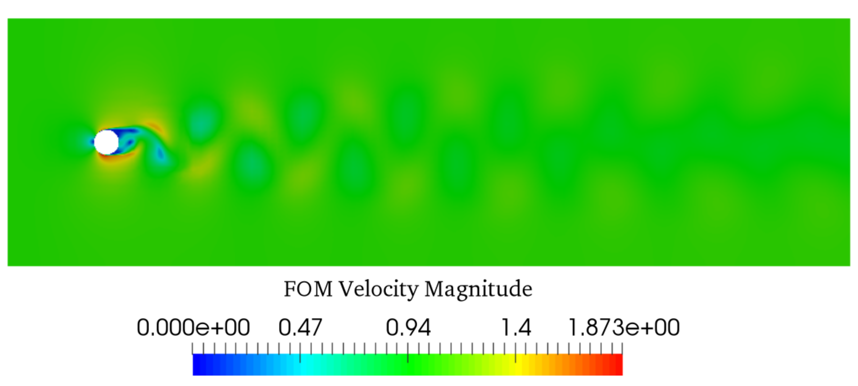
$$\sum_{j=1}^{N_s} w_{L,j}\zeta_{L,j}(\|\boldsymbol{a}_{L_2}^i-\boldsymbol{a}_{L_2}^j\|_{\mathbb{R}^{N_u}}) = g_{i,L}, \quad \text{for} \quad i=1,2,...,N_s. \tag{13}$$

In the online stage one can solve for $\boldsymbol{g}(\boldsymbol{a}^*) = [g_i(\boldsymbol{a}^*)]_{i=1}^{N_{\nu_t}}$

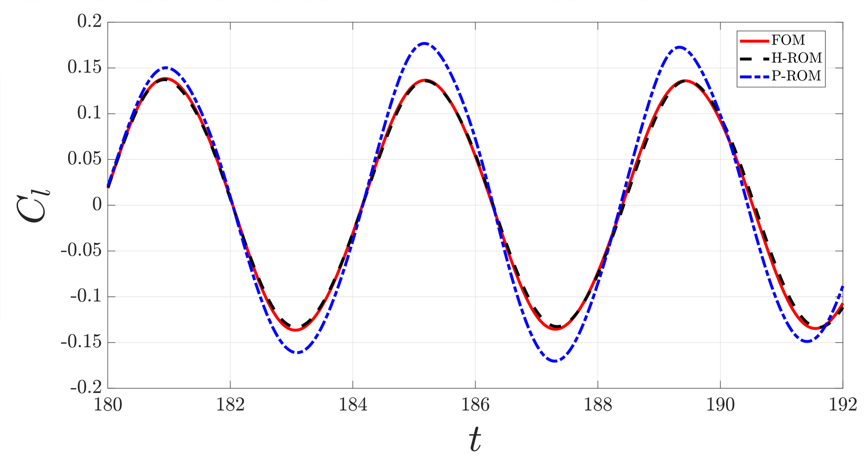
$$g_i(\boldsymbol{a}^*) \approx G_i(\boldsymbol{a}^*) = \sum_{j=1}^{N_s} w_{i,j}\zeta_{i,j}(\|\boldsymbol{a}^*-\boldsymbol{a}_{L_2}^j\|_{\mathbb{R}^{N_u}}), \quad \text{for} \quad i=1,2,...,N_{\nu_t}. \tag{14}$$

Results

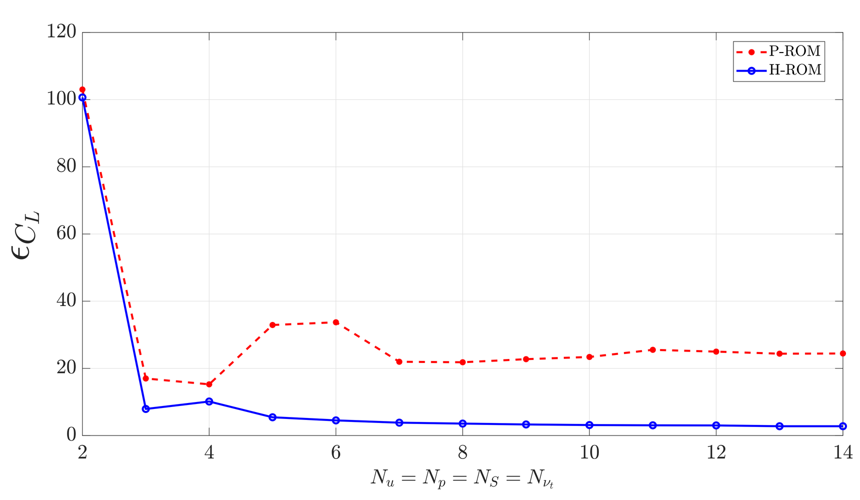
The presented results are for the benchmark case of the flow around a circular cylinder in unsteady state setting. The case is without parameterization so the reduction is done on time (both reproduction of the snapshots and extrapolating in time). The Reynolds number is equal to 10000. The results shown are for an important performance indication parameter which is the lift coefficient C_l resulted from the lift forces acting on the cylinder. The FOM results are compared to those of both the Hybrid ROM (H-ROM) and the fully Projection ROM (P-ROM) which is based on solving 5 with neglecting the term $\boldsymbol{G}$. A quantitative convergence analysis is shown for the decay of the error with the increase of the number of modes used in the H-ROM.



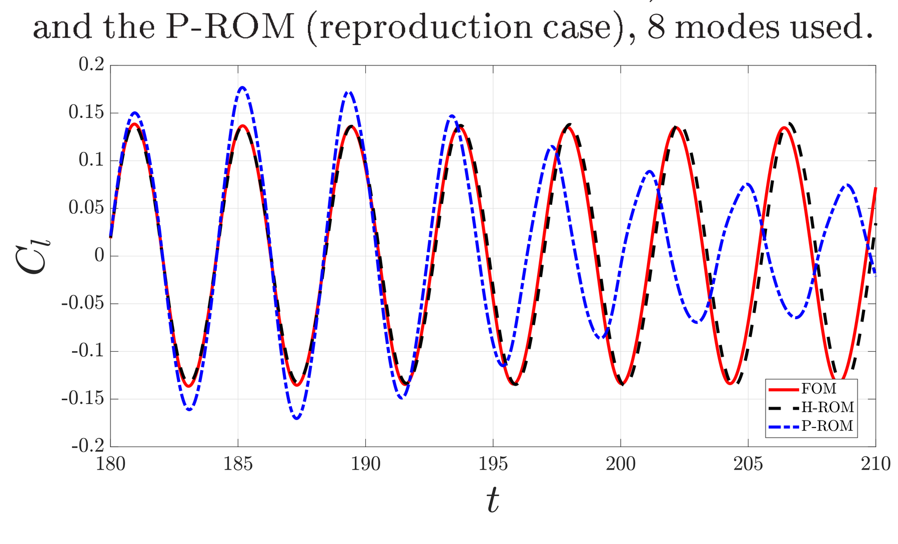
Velociy magnitude FOM field.



Lift coefficients curves for the FOM, the H-ROM and the P-ROM (reproduction case), 8 modes used.



Lift coefficients L^2 error curve (reproduction case).



Lift coefficients curves for the FOM, the H-ROM and the P-ROM (extrapolation case), 8 modes used.

Acknowledgements

This work has been supported by the project AROMA-CFD (GA 681447 - "Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics") funded by the European Research Council.

References

[1] J. S. Hesthaven, G. Rozza, and B. Stamm. Certified reduced basis methods for parametrized partial differential equations. *Springer Briefs in Mathematics*, 2015.

[2] S. Hijazi, G. Stabile, M. A., and G. Rozza. Data-Driven POD–Galerkin reduced order model for turbulent flows. *Submitted*, 2019.

[3] G. Stabile, S. Hijazi, A. Mola, S. Lorenzi, and G. Rozza. POD-Galerkin reduced order methods for CFD using Finite Volume Discretisation: vortex shedding around a circular cylinder. *Communications in Applied and Industrial Mathematics*, 8(1):210–236, dec 2017.

[4] G. Stabile and G. Rozza. Finite volume POD-Galerkin stabilised reduced order methods for the parametrised incompressible Navier–Stokes equations. *Computers & Fluids*, 173:273–284, sep 2018.

Get the pdf