SFB 1294 Data Assimilation

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Reduced basis methods for random PDEs and control problems governed by PDEs

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Background

Example

Consider a certain physical system in steady state, formulated as a PDE.

Poisson's equation on domain $\Omega = [0,1]^2$:

$$\mu \Delta y = u,$$

$u \in U = L^2(\Omega)$	forcing term,
$\mu \in \mathbb{M} \subset \mathbb{R}$	parameter following some distribution,
$y \in Y = H^1_0(\Omega)$	the solution profile depending on μ, i.e. $y = y(\mu)$.

Reformulation in weak form:

$$A(\mu)y(\mu) = u(\mu), \quad y(\mu) \in Y, \qquad (1)$$

with suitable μ-dependent operators $A(\mu) \in \mathcal{B}(Y, Y^*)$, $u(\mu) \in Y^*$.

Accurate but slow solvers for $y(\mu)$ available. Problem:
what if we need $y(\mu)$ for many values of μ in little time?

Central question

Given μ, can we build an approximation $y_N(\mu)$ of $y(\mu)$ s.t.:

- $\|y(\mu) - y_N(\mu)\|_Y \leq \varepsilon$,
- computation time of $y_N(\mu)$ is fast?

Solution: Reduced Order Method

The Reduced Order Method (ROM) exploits inherent smoothness of $\mu \mapsto y(\mu)$. Two step approach:

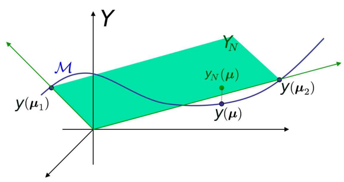
1. *(offline)* Construct $Y_N \subset Y$, $\dim(Y_N) = N$, based on *snapshot solutions* $y(\mu_1), \ldots, y(\mu_M)$, for $M > N$.
2. *(online)* "Galerkin" Projection of $y(\mu)$ onto Y_N to get $y_N(\mu)$.

Step 1 can be done via e.g. Proper Orthogonal Decomposition (POD):

$$Y_N = \underset{V \subset Y, \dim(V) = N}{\arg\min} \sum_{i=1}^{M} \|y(\mu_i) - P_V y(\mu_i)\|_Y^2,$$

where P_V is the orthogonal projector onto V.

Illustration and assumptions



Solution manifold $\mathcal{M}$ well described by $Y_N = \text{span}(y(\mu_1), y(\mu_2))$. Projecting $y(\mu)$ onto Y_N gives a good approximation $y_N(\mu)$, for any μ.

Source: [3].

Answer to central question is positive under assumptions, including:

- Online projection is well-posed,
- $\mathcal{M}$ is sufficiently well-behaved,
- Variable separation of A and u for efficient Galerkin projection.

In case of nonlinearities, need to recover variable separation: can use Empirical Interpolation Method (EIM) and derivatives.

Innovation & key ideas

Linear-Quadratic Optimal Control

Suppose we can choose control $u(\mu)$ in (1).

Objective:
for given desired state $y_d(\mu) \in L^2(\Omega)$, find $u(\mu) \in U$ that minimizes

$$\frac{1}{2}\int_\Omega |y(\mu) - y_d(\mu)|^2 \, dx + \frac{\tau}{2}\int_\Omega |u(\mu)|^2 \, dx,$$

subject to (1).

Reformulation via e.g. Lagrange multipliers:

$$\mathcal{A}(\mu)\chi(\mu) = F(\mu), \quad \chi(\mu) \in \mathcal{X} := Y \times U \times P,$$

for suitable $\mathcal{A}(\mu) \in \mathcal{B}(\mathcal{X}, \mathcal{X}^*)$, $F(\mu) \in \mathcal{B}(\mathcal{X}^*)$ and adjoint space P.
Usually $P = Y$ in applications. Note the analogy to (1).

Can apply ROM: reduce Y, U, P separately to Y_N, U_N, P_N. Also:

we found that for well-posedness we do not need $Y_N = P_N$, which previously was enforced by space "aggregation" of Y_N and P_N.

Atlantic current inversion

State $y = (v, \rho)$ follows Quasi-Geostrophic equation on part of Atlantic Ocean:

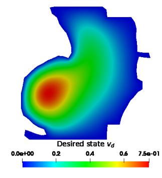
$$\rho = \Delta v \quad \text{in } \Omega, \qquad \rho = 0 \quad \text{on } \partial\Omega,$$

$$\mu_3 \mathcal{F}(v,\rho) + \frac{\partial v}{\partial x_1} + \mu_1 \rho - \mu_2 \Delta \rho = u \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

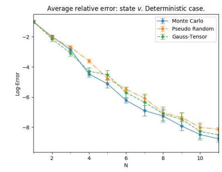
$$\frac{\partial v}{\partial x_1}\frac{\partial \rho}{\partial x_2} - \frac{\partial v}{\partial x_2}\frac{\partial \rho}{\partial x_1} = \mathcal{F}(v,\rho) \quad v, \rho \in Y, \qquad Y = (H^1_0(\Omega))^2.$$

Given μ, find wind action $u(\mu)$ that generates observed streamline v_d.

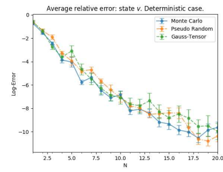
Linear Case ($\mu_3 = 0$).



Desired state

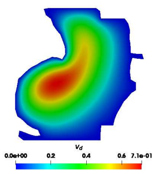


With aggregation

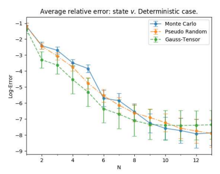


Without aggregation

Nonlinear case: use Newton-Iteration scheme to solve state equation.



Desired state



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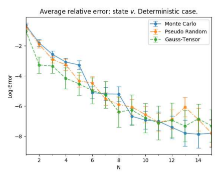

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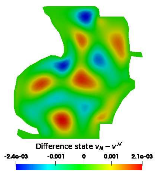
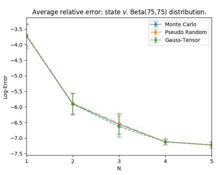
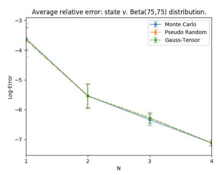


Pointwise error state



With aggregation



Without aggregation

Note that reduction without aggregation exhibits exponential decay of relative errors. Thus, further computational savings can be achieved.

References

[1] Carere, G., Strazzullo, M., Ballarin, F., Rozza, G., Stevenson, R. A weighted POD-reduction approach for parametrized PDE-constrained optimal control problems with random inputs and applications to environmental sciences. *Computers and Mathematics with applications* 102:261-276, 2021.

[2] Strazzullo, M. and Ballarin, F. and Mosetti, R. and Rozza, G., Model Reduction for Parametrized Optimal Control Problems in Environmental Marine Sciences and Engineering. *SIAM Journal on Scientific Computing* 40:B1055-B1079, 2017.

[3] Haasdonk, B., Reduced basis methods for parametrized PDEs - a tutorial introduction for stationary and instationary problems. *Model Reduction and Approximation*, volume 15, chapter 2, pages 65-136, SIAM Publications, Philadelphia, 2017.

Acknowledgements

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