Optimisation-Based FEM/ROM Couplings in CFD: Exploring Diverse Scenarios

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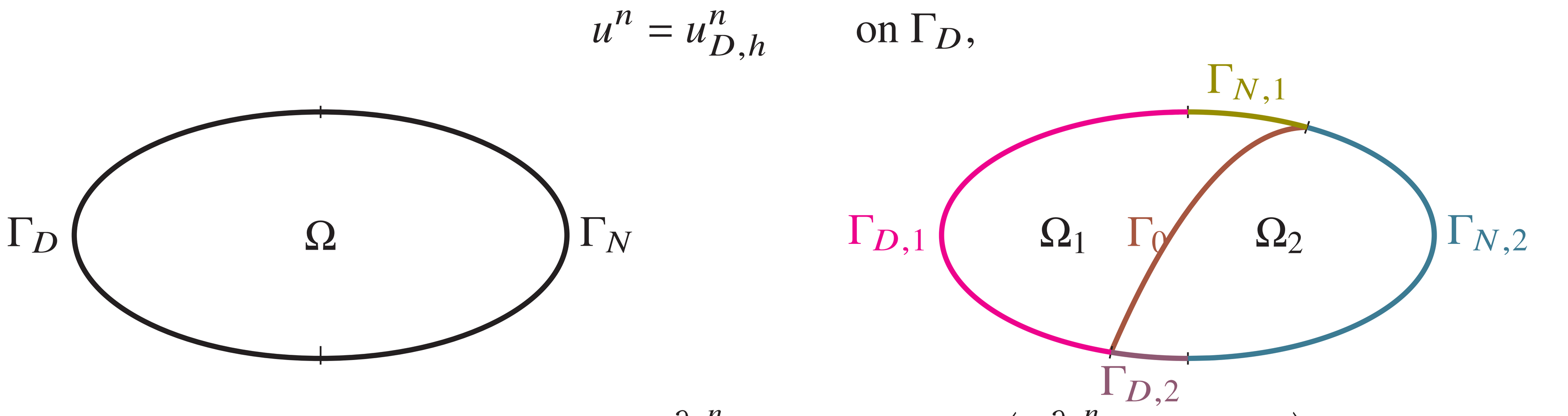
Introduction

This work aims to present an optimisation–based framework for coupling different discretisation models, such as Finite Element (FEM) and ROM for separate subcomponents. In particular, we consider an optimisation–based DD model on two non–overlapping subdomains where the coupling on the common interface is performed by introducing a control variable representing a normal flux. Gradient-based optimisation algorithms are used to construct an iterative procedure to fully decouple the subdomain state solutions as well as to locally generate ROMs on each subdomain. Then, we consider FEM or ROM discretisation models for each of the DD problem components, namely, the triplet state1–state2–control. We perform numerical tests on the backward–facing step Navier-Stokes problem to investigate the efficacy of the presented couplings in terms of optimisation iterations and relative errors.

1 - Monolithic vs. Domain Decomposition (DD) FEM Formulation

We start with introducing high–fidelity monolithic model based on FEM discretisation in space and implicit Euler scheme in time of the non–stationary incompressible Navier–Stokes equations:

$$\frac{m(u_h^n - u_h^{n-1}, v_h)}{\Delta t} + a(u_h^n, v_h) + c(u_h^n, u_h^n, v) + b(v_h, p_h^n) = (f^n, v_h)_\Omega \quad \forall v_i \in V_{0,h},$$
$$b(u_h^n, q_h) = 0 \quad \forall q_h \in Q_h,$$
$$u^n = u_{D,h}^n \quad \text{on } \Gamma_D,$$



The DD conditions: $u_{1,h}^n = u_{2,h}^n$ and $\nu \frac{\partial u_{1,h}^n}{\partial \mathbf{n}_1} - p_{1,h}^n \mathbf{n}_1 = -\left(\nu \frac{\partial u_{2,h}^n}{\partial \mathbf{n}_2} - p_{2,h}^n \mathbf{n}_2\right)$ in X_h.

The discretised optimisation–based DD formulation reads as follows: for $n \geq 1$

minimise over $g_h \in X_h$ *the functional*

$$\mathcal{J}(u_{1,h}^n, u_{2,h}^n; g_h) = \frac{1}{2}\int_{\Gamma_0} \left| u_{1,h}^n - u_{2,h}^n \right|^2 d\Gamma$$

subject to the variational problem:
for $i = 1, 2$ *find* $u_{i,h} \in V_{i,h}$ *and* $p_{i,h} \in Q_{i,h}$ *satisfying*

$$\frac{m_i(u_{i,h}^n - u_{i,h}^{n-1}, v_{i,h})}{\Delta t} + a_i(u_{i,h}^n, v_{i,h}) + c_i(u_{i,h}^n, u_{i,h}^n, v_i) \qquad \forall v_i \in V_{i,0,h},$$
$$+ b_i(v_{i,h}, p_{i,h}^n) = (f_i^n, v_{i,h})_{\Omega_i} + \left((-1)^{i+1} g_h, v_{i,h}\right)_{\Gamma_0}$$
$$b_i(u_{i,h}^n, q_{i,h}) = 0, \qquad \forall q_{i,h} \in Q_{i,h}$$
$$u_i^n = u_{i,D,h}^n \qquad \text{on } \Gamma_{i,D},$$

2 - Adjoint system and the gradient expression

Adjoint problem:

$$\frac{m_i(\eta_{i,h}, \xi_{i,h})}{\Delta t} + a_i(\eta_{i,h}, \xi_{i,h}) + c_i(\eta_{i,h}, u_{i,h}^n, \xi_i) + c_i(u_{i,h}^n, \eta_{i,h}, \xi_{i,h})$$
$$+ b_i(\eta_{i,h}, \lambda_{i,h}) = ((-1)^{i+1} \eta_{i,h}, u_{1,h}^n - u_{2,h}^n)_{\Gamma_0} \quad \forall \eta_{i,h} \in V_{i,0,h},$$
$$b_i(\xi_{i,h}, \mu_{i,h}) = 0 \qquad \forall \mu_{i,h} \in Q_{i,h}.$$

Optimality system:

$$\frac{d\mathcal{J}}{dg}(u_{1,h}^n, u_{2,h}^n; g_h) = \xi_{1,h}\big|_{\Gamma_0} - \xi_{2,h}\big|_{\Gamma_0},$$

3 - ROM setting

- POD–compression in terms of time and physical parameters
- POD–Galerkin ROM
- POD–NN ROM

4 - FEM–ROM couplings

- FEM–FEM–FEM (FFF) coupling
- FEM–ROM–FEM (FRF) coupling

$$\mathcal{J}(u_{1,h}^n, u_{2,N}^n; g_h) = \frac{1}{2}\int_{\Gamma_0} \left| u_{1,h}^n - (\Pi_2^n(\mu))^T u_{2,N}^n \right|^2 d\Gamma$$

$$\frac{d\mathcal{J}}{dg}(u_{1,h}^n, u_{2,N}^n; g_h) = \xi_{1,h}\big|_{\Gamma_0} - [\Pi_{2,0}^T \xi_{2,N}]\big|_{\Gamma_0}$$

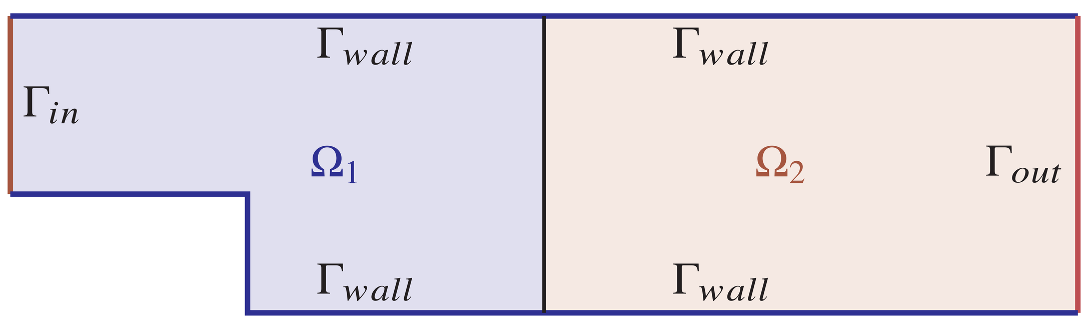
- FEM–ROM–ROM (FRR) coupling

$$\mathcal{J}(u_{1,h}^n, u_{2,N}^n; g_N) = \frac{1}{2}\int_{\Gamma_0} \left| \Pi_1^n(\mu) u_{1,h}^n - u_{2,N}^n \right|^2 d\Gamma$$

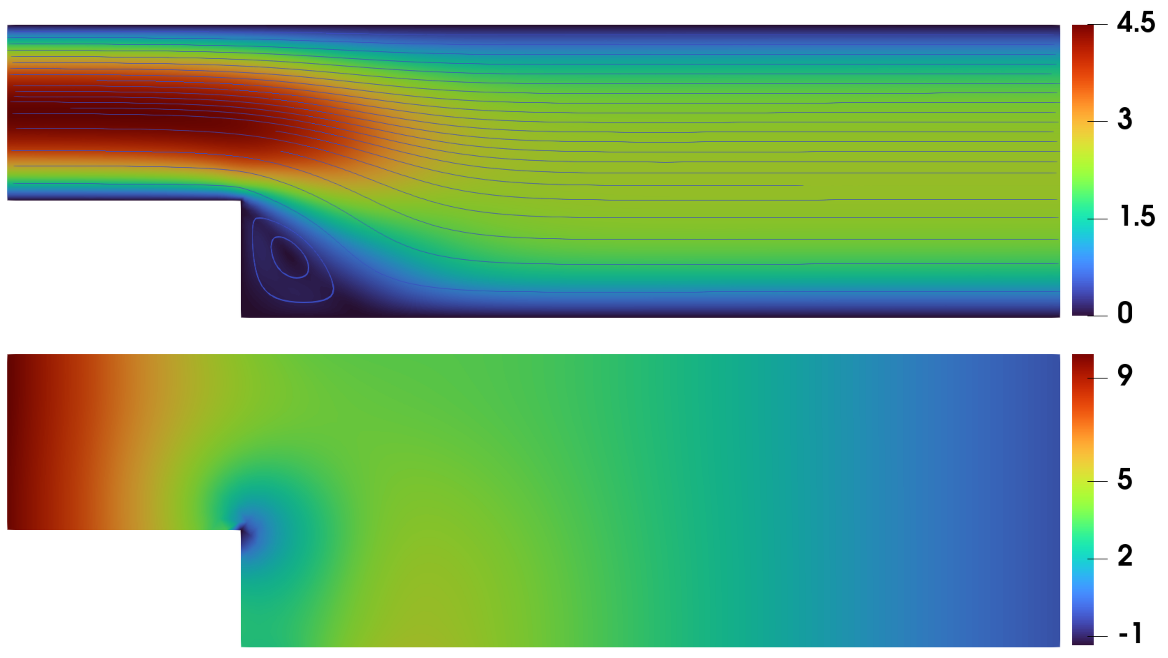
$$\frac{d\mathcal{J}}{dg}(u_{1,h}^n, u_{2,N}^n; g_N) = [\Pi_{1,0}\xi_{1,h}]\big|_{\Gamma_0} - [\xi_{2,N}]\big|_{\Gamma_0},$$

- ROM–ROM–ROM (RRR) coupling

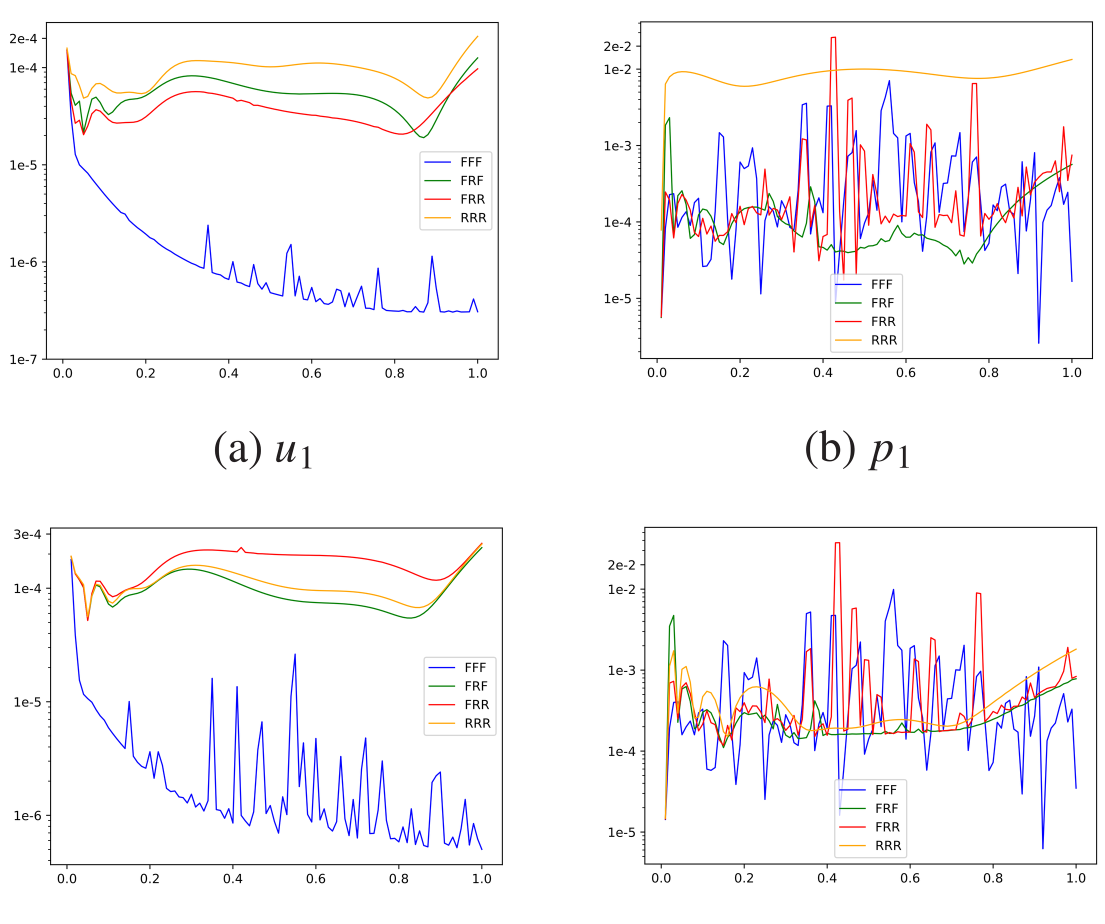
5 - Numerical results



Domain Decomposition

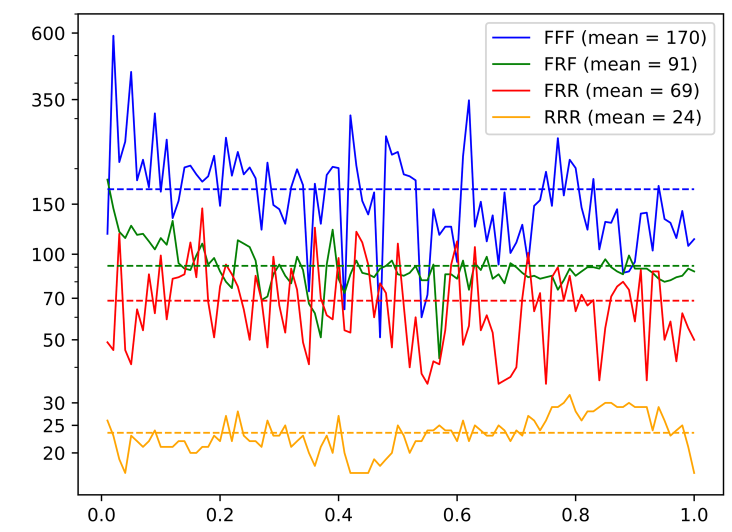


FEM solutions at the final time $t = 1$ (velocity and pressure)

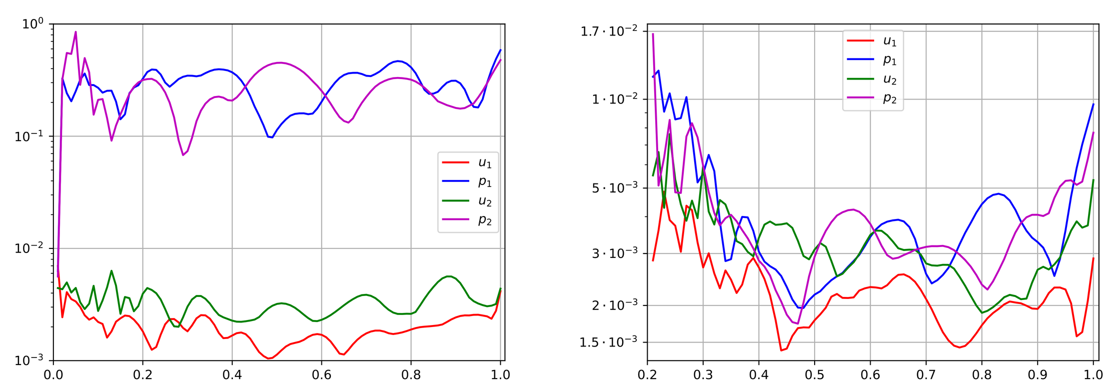


(a) u_1 (b) p_1 (c) u_2 (d) p_2

Relative errors w.r.t the monolithic solution



Iterations numbers



(a) Entire time interval (b) Restricted time interval

POD–NN coupling

6 - Computational science and engineering softwares

EZyRB
`https://mathlab.sissa.it/ezyrb`

EZyRB is a Python package that performs a data-driven model order reduction for parametrized problems exploiting the recent approaches.

RBniCS
`github.com/mathLab/rbnics`

The RBniCS Project contains an implementation in FEniCS of several reduced order modelling techniques for parametrized problems.

References

[1] A. de Castro, P. Kuberry, I. Tezaur, and P. Bochev. *A Novel Partitioned Approach for Reduced Order Model—Finite Element Model (ROM-FEM) and ROM-ROM Coupling*, pages 475–489.

[2] I. Prusak, M. Nonino, D. Torlo, F. Ballarin, and G. Rozza. An optimisation–based domain–decomposition reduced order model for the incompressible navier-stokes equations. *Computers & Mathematics with Applications*, 151:172–189, 2023.

[3] I. Prusak, D. Torlo, M. Nonino, and G. Rozza. Optimisation–based coupling of finite element model and reduced order model for computational fluid dynamics, 2024.

[4] I. Prusak, D. Torlo, M. Nonino, and G. Rozza. An optimisation-based domain-decomposition reduced order model for parameter-dependent non-stationary fluid dynamics problems, 2024.