# Exploration and Comparison of Reduced Order Modelling Technique for Parametrized System

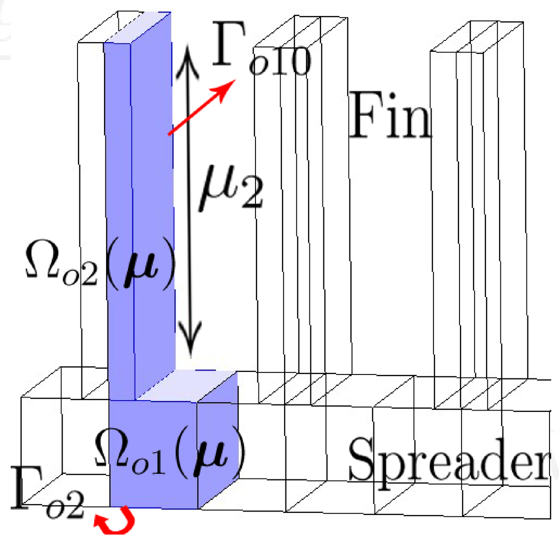
Fabrizio GELSOMINO
Supervisors: Alfio QUARTERONI & Gianluigi ROZZA
CMCS - Chaire de Modélisation et Calcul scientifique - Modelling and Scientific Computing Chair
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

## Abstract

Reduced Basis (RB) method has successfully been used in 2D to solve scalar and vectorial parametrized problems. The *rbMIT* software is a library developed in Matlab to solve this kind of problems. The idea is to interface the rbMIT library (in Matlab) with the COMSOL Multiphysics to extend the library to solve 3D problems. COMSOL is a powerful tool for solving and modelling engineering problems based on PDEs. It permits to create your own 3D geometry and to generate meshes. We present two *Worked Problems*, the *Thermal Fin* and the *Graetz Flow* and we compare two reduced order modelling techniques: the *Greedy-RB* and the *Proper Orthogonal Decomposition* (*POD*).

## The Steady Thermal Fin Problem

The main function of a heat sink is to transfer heat from an object at a higher temperature to another one at a lower temperature with greater heat capacity.




The heat sink comprises of a *base/spreader* which in turn supports a number of *plate fins* exposed to flowing air. Examples of systems that require a heat sink to reduce their temperature are microprocessors and refrigeration. The output is the average temperature on $\Gamma_{o2}$.

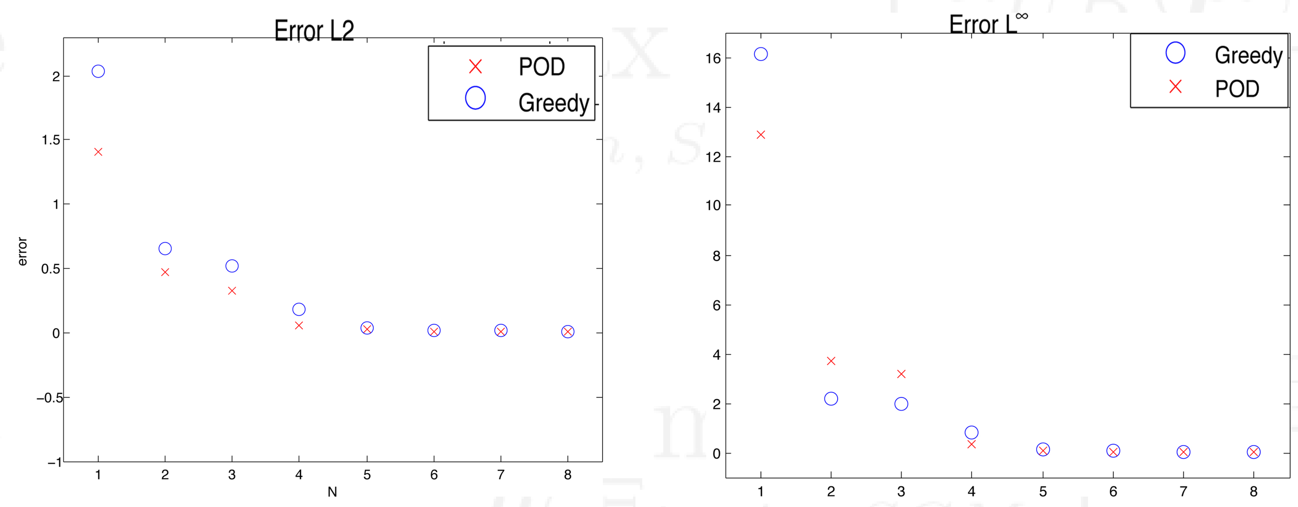
### Parameters

$\mu_1$ is the Biot number  $\mu_2$ is the height of the fin
$\mu_3$ is the conduction ratio

### Equations

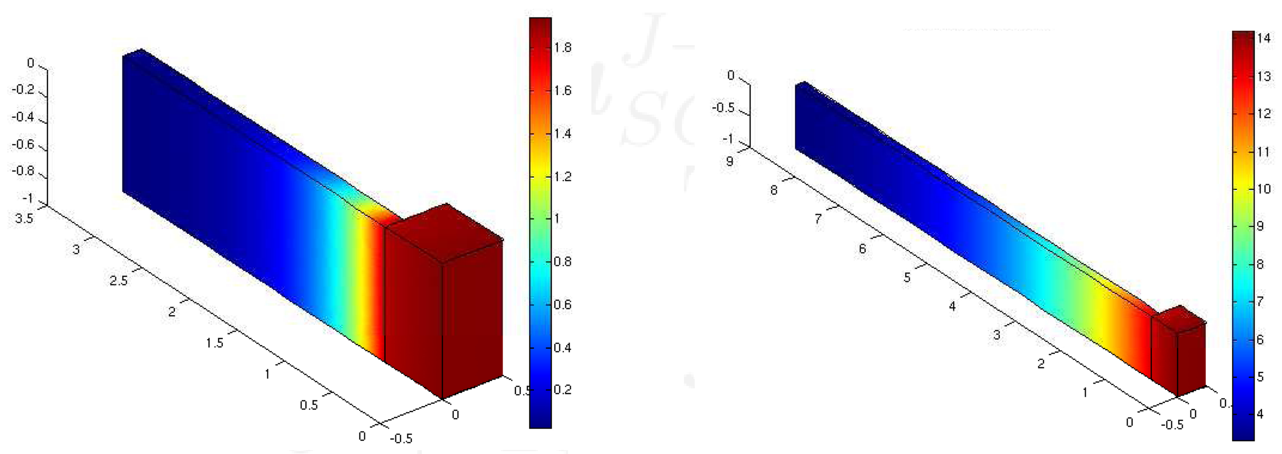
$$\begin{cases} -\mu_3\Delta u_o(\boldsymbol{\mu}) = 0 \text{ in } \Omega_{o1}(\boldsymbol{\mu}) & \text{and } \; -\Delta u_o(\boldsymbol{\mu}) = 0 \text{ in } \Omega_{o2}(\boldsymbol{\mu}) \\ \mu_3\frac{\partial}{\partial \boldsymbol{n}}u_o(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_{o2}, \\ \frac{\partial}{\partial \boldsymbol{n}}u_o(\boldsymbol{\mu}) = 0 & \text{on } (\partial\Omega_{o1}\backslash\Gamma_{o2}) \cup (\partial\Omega_{o2}\backslash\Gamma_{o10}), \\ \frac{\partial}{\partial \boldsymbol{n}}u_o(\boldsymbol{\mu}) + \mu_1 u_o(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{o10} \text{ (Heat transfer by convection)} \end{cases}$$

### Comparison between Greedy-RB and POD




Error between the FE and the RB solution in $L^2$ and $L^\infty$ norm.

### Visualization



Representative solutions for $\boldsymbol{\mu} = (0.5, 2.75, 10)$ (left) and $\boldsymbol{\mu} = (0.01, 8, 1)$ (right).

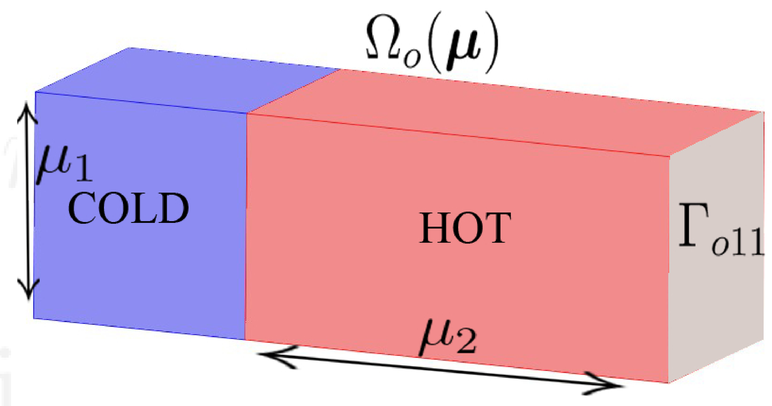
### Computational Time (Thermal Fin)

$$t^{Offline}(\mathcal{N}) = \frac{\text{Time to construct the reduced basis}}{\text{Time to evaluate the output with the FE}} = 1158 \; (\textit{break-even})$$

$$t^{Online}(\mathcal{N}, N) = \frac{\text{Time to evaluate the output with the RB}}{\text{Time to evaluate the output with the FE}} = 0.006.$$

## The Time-Dependent Graetz Flow Problem

This is a classical problem in literature dealing with forced heat convection combined with heat conduction in a duct. The duct is separated in two parts, one with cold walls and the other one with hot walls. The temperature at inlet is imposed and the flow has a known given convective field. The output is the average temperature in the duct.



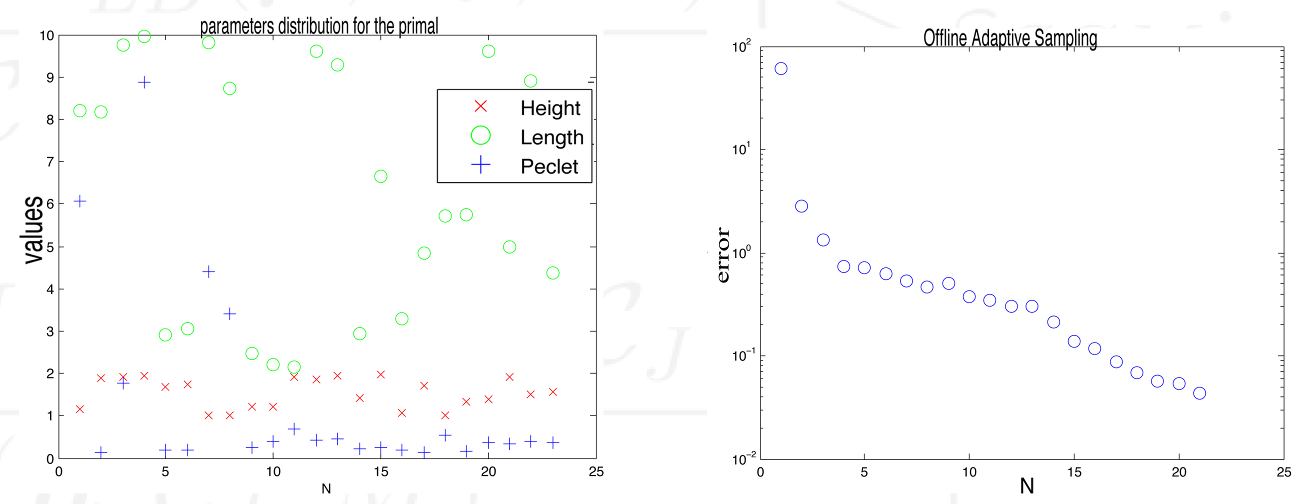

### Parameters

$\mu_1$ is the height of the duct  $\mu_2$ is the length of the hot zone
$\mu_3$ is the Peclet number

### Equations

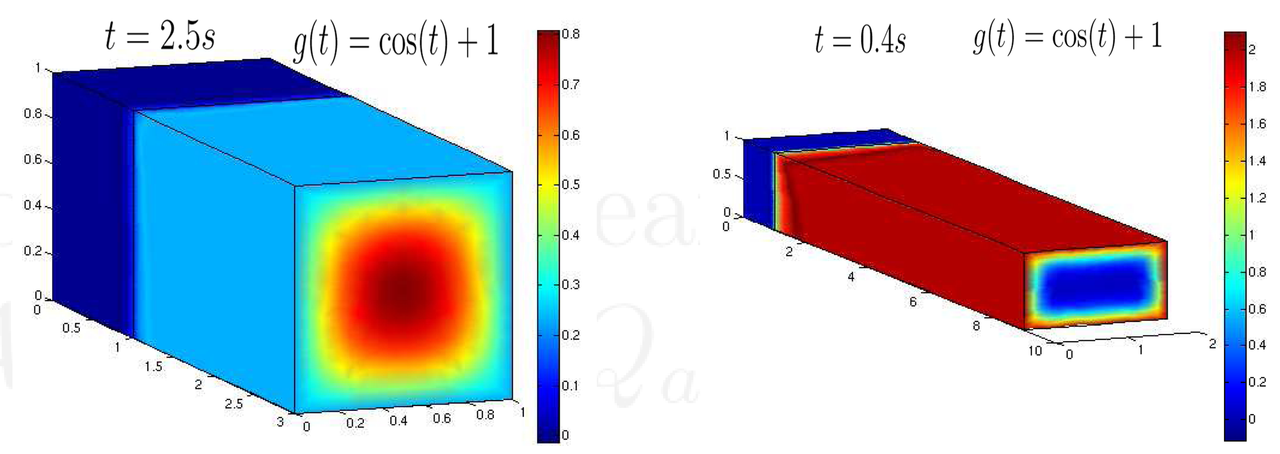
$$\begin{cases} \frac{\partial u_o(t;\boldsymbol{\mu})}{\partial t} - (\mu_3)^{-1}\Delta u_o(t;\boldsymbol{\mu}) + x_{o2}(1-x_{o2})\frac{\partial}{\partial x_1}u_o(t;\boldsymbol{\mu}) = 0 \text{ in } \Omega_o(\boldsymbol{\mu}), \; t\in[0,T] \\ u_o(t;\boldsymbol{\mu}) = 0 & \text{on cold walls, } t \in (0,T] \\ u_o(t;\boldsymbol{\mu}) = g(t) & \text{on hot walls } t \in (0,T] \\ \frac{\partial}{\partial \boldsymbol{n}}u_o(t;\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{o11}, \; t\in(0,T] \\ u_o(t=0;\boldsymbol{\mu}) = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \text{ (Initial condition)} \end{cases}$$

### Chosen parameters with the Greedy-RB




Chosen parameters by the Greedy-RB and convergence.

### Visualization




Representative solutions for $\boldsymbol{\mu} = (1, 2, 10)$ (left) and for $\boldsymbol{\mu} = (2, 8, 10)$ (right).

### Computational Time (Graetz flow)

$$t^{Offline}(\mathcal{N}) = \frac{\text{Time to construct the reduced basis}}{\text{Time to evaluate the output with the FE}} = 32.4 \; (\textit{break-even})$$

$$t^{Online}(\mathcal{N}, N) = \frac{\text{Time to evaluate the output with the RB}}{\text{Time to evaluate the output with the FE}} = 0.005.$$

## Reduced basis method

The reduced basis method is used to evaluate any kind of outputs $s^e(\boldsymbol{\mu})$ (maximal or average temperature, flow rates, heat transfer rates, etc.) which depends on a field variable $u^e(\boldsymbol{\mu})$, solution of a parametric PDE

$$a\left(u^e(\boldsymbol{\mu}), v; \boldsymbol{\mu}\right) = f(v; \boldsymbol{\mu}), \quad \forall v \in X^e,$$

where $X^e$ is a functional space, $a: X^e \times X^e \times \mathcal{D} \longrightarrow \mathbb{R}$ is a continuous (coercive) parametric bilinear form and $f$ is a continuous parametric linear functional (the superscript $^e$ refers to *exact*).
The parameter $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ may represent boundary condition and sources, geometric configuration or physical properties. The set $\mathcal{D}$ is the parameter set and $P$ is the number of parameters:
The idea is to be able to evaluate the output for a great number of parameters at a reduced cost.
For a given finite element space $X^{\mathcal{N}}$ of dimension $\mathcal{N}$, the idea is to construct, for $1 \le N \le N_{max} \in \mathbb{N}$, a $N$-dimensional space $X_N$, with $N << \mathcal{N}$ and to take the Galerkin projection on the space $X^{\mathcal{N}}$. So, we have to choose $N$ parameters $\boldsymbol{\mu_1}, \dots, \boldsymbol{\mu_N}$ and compute $N$ finite element solution $\xi_1^{\mathcal{N}}, \dots, \xi_N^{\mathcal{N}}$ associated to the parameters, called *snapshots*.
After orthonormalization, these solutions will be the base of $X_N$ and then for an arbitrary value $\boldsymbol{\mu}^* \in \mathcal{D}$, we can compute the solution associated to this parameter (denoted $u_N^{\mathcal{N}}(\boldsymbol{\mu}^*)$) taking a good linear combinations of $\xi_k^{\mathcal{N}}$, $k = 1, \dots, N$. Then, the evaluation of the output will not depend on $\mathcal{N}$ but only on $N$.
There exists numerous way to choose the snapshots to construct the reduced basis. Here, we consider two of them. The Greedy-RB and the POD. The Greedy-RB is an algorithm which chooses at each step the parameter $\boldsymbol{\mu_K}$ which maximizes the error $u^{\mathcal{N}}(\boldsymbol{\mu}) - u_K^{\mathcal{N}}(\boldsymbol{\mu})$, where $u^{\mathcal{N}}$ is the finite element solution and $u_K^{\mathcal{N}} \in X_K$ is the reduced basis solution and it minimizes the error in $L^\infty$-norm. For the POD, we have to solve an eigenvalue problem and it minimizes the error in $L^2$-norm.

## Reference

[1] G. ROZZA, D.B.P HUYNH and A.T. PATERA. *Reduced Basis Approximation and a Posteriori Error Estimation for Affinely Parametrized Elliptic Coercive Partial Differential Equations : Application to Transport and Continuum Mechanics.* Archives of Computational Methods in Engineering, 15(3):229-275, 2008.

[2] A. QUARTERONI, G. ROZZA. *Numerical solution of parametrized Navier-Stokes equations by reduced basis methods.* Numerical Methods for PDEs, 23(4): 923-948, 2007.