A reduced basis hybrid method for viscous flows in parametrized complex networks

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Abstract

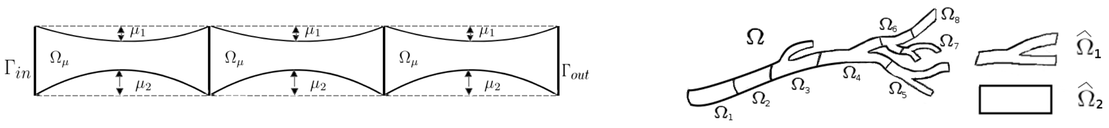
Reduced order techniques speed up the computational time with rapid and efficient models which allow to approximate numerically fluid flows in complex and realistic configurations where geometrical features are recurrent and similar. The proposed reduced basis hybrid method (RBHM) is a reduced order strategy developed upon techniques combining reduced basis method and domain decomposition methodologies. It takes advantage of repetitive geometries of the computational domain when solving incompressible fluid flows problems modeled by steady Stokes equations. In particular it allows to improve the regularity of the global solution at the interfaces of the subdomains by minimizing the jump of the velocity and of the pressure solutions, and by combining reduced basis solutions on a fine grid with a coarse finite element solution.

1. Reduced Basis Hybrid Method

Carrying out simulations almost in real time and in a many query context with a reasonable level of accuracy may really increase the importance of cardiovascular simulations in daily diagnosis or risk evaluation procedure. The RBHM takes advantage of both the reduced basis (RB) [5] and the domain decomposition (DD) concepts by adding some new extension to the reduced basis element method (RBEM) [4]. We consider the following steady Stokes problem in a domain $\Omega \subset \mathbb{R}^2$ with mixed boundary conditions on $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_w$:

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_w \\ \nu\frac{\partial u}{\partial \hat{n}} - p\hat{n} = \bar{\sigma} & \text{on } \Gamma \setminus \Gamma_w \end{cases} \quad \text{with } \bar{\sigma} = \begin{cases} \bar{\sigma}^{in} \text{ on } \Gamma_{in} \\ \bar{\sigma}^{out} \text{ on } \Gamma_{out} \end{cases} \tag{1}$$

where Ω is a domain occupied by a fluid of constant density, u is the fluid velocity, p the pressure, f a force field, ν the kinematic viscosity and $\hat{n}$ the normal unit vector to the domain boundary. The domain Ω is given by a non-overlapping union of R subdomains Ω_r, and each sub–domain is a deformation of a reference domain $\hat{\Omega}_k$ through regular enough, and non-affine, maps $T^k_{\mu_r} : \hat{\Omega}_k \to \Omega_r$, so that $\Omega = \bigcup_{r=1}^R \Omega_r = \bigcup_{r=1}^R T^k_{\mu_r}(\hat{\Omega}_k)$.



The parameter $\mu \in \mathcal{D} \subset \mathbb{R}^P$ (with $P \geq 1$) represents the geometric deformation associated to each sub–domain. The parametric maps, their Jacobians $\mathcal{J}_k$ and the corresponding determinants J_k allow to define the parametric weak formulations of problem (1) for each sub-domain Ω_r on $\hat{\Omega}_k$ after the transformation. Find $(u, p) \in Y^k \times M^k$ such that

$$\begin{cases} a^k(u, v, \mu_r) + b^k(v, p, \mu_r) = f^k(v, \mu_r, \sigma_r^{in}, \sigma_r^{out}) & \forall v \in Y^k = H^1_{\Gamma_w}(\hat{\Omega}_k) \\ b^k(u, q, \mu_r) = 0 & \forall q \in M^k = L^2(\hat{\Omega}_k) \end{cases} \tag{2}$$

where $a^k(v, w, \mu) = \nu \int_{\hat{\Omega}_k} \mathcal{J}_k^{-T} \nabla v \cdot \mathcal{J}_k^{-T} \nabla w |J_k| \; d\hat{\Omega}$, $b^k(v, q, \mu) = -\int_{\hat{\Omega}_k} q \nabla \cdot (\mathcal{J}_k^{-1} v) |J_k| d\hat{\Omega}$ and $f^k(w, \mu, \sigma^{in}, \sigma^{out}) = \int_{\hat{\Omega}_k} f \cdot v |J_k| d\hat{\Omega} + \int_{\hat{\Gamma}_{in}} \sigma^{in} v \cdot n |J_k| d\hat{\Gamma} + \int_{\hat{\Gamma}_{out}} \sigma^{out} v \cdot n |J_k| d\hat{\Gamma}$.

If Ω_r is a sub-domain associated with the inflow boundary, we use $\sigma_r^{in} = \bar{\sigma}^{in}$ and $\sigma_r^{out} = 0$; if Ω_r is a central sub-domain, we impose $\sigma_r^{in} = 0$ and $\sigma_r^{out} = 0$; while if Ω_r is a sub-domain associated with the outflow boundary, we impose $\sigma_r^{in} = 0$ and $\sigma_r^{out} = \bar{\sigma}^{out}$, so that the local solutions have **zero stress** on the internal interfaces. The RB method allows to find the solution of parametrized problem (2) as projection of previously precomputed solutions $(v_h(\mu^i), p_h(\mu^i))$ for certain instances of the parameters $\mu_i, i = 1, \cdots, N$ (properly selected by an exploration in the parametric space, carried out through a greedy algorithm based on residuals, [6]) using $\mathbb{P}_2$- $\mathbb{P}_1$ Taylor-Hood finite elements (FE) on an accurate **fine mesh**. We define the reduced basis spaces in each reference domain: $Z_N^k = span\left\{u_h^k(\mu^i), i = 1, ..., N_k\right\}, M_N^k = span\left\{p_h^k(\mu^i), i = 1, ..., N_k\right\}$.
In order to fulfill the inf-sup condition the velocity basis has been enriched, [6]:
$Y_N^k = Z_N^k \oplus X_N^k, \quad X_N^k = span\left\{v_h^k(\mu^i), i = 1, ..., N_k\right\}, \quad v_h^k(\mu^i) = \arg\sup_{w \in Z} \frac{b(w, p_h^k(\mu^i), \mu^i)}{||w||_Z}$.
Each basis function in each reference subdomain is computed considering zero-stress conditions at the internal interfaces, the **continuity of the stresses** (non-zero) at the interfaces is recovered by a coarse finite element solution on the global domain Ω, while the **continuity of velocities** is guaranteed by Lagrange multipliers. A Galerkin FE solution (u_H, p_H) of the problem (1) is obtained in a very quick way by using a **coarse mesh** for the whole domain Ω. Together with the supremizer solution v_H, we restrict these functions to each subdomain, we map them back to the corresponding reference domain and we include them in the reduced basis spaces:
$Y_N^{kr} = span\left\{u_H|_{\Omega_r}, v_H|_{\Omega_r}, u_h^k(\mu_k^i), v_h^k(\mu_k^i), i = 1, ..., N_k\right\}, M_N^{kr} = span\left\{p_H|_{\Omega_r}, p_h^k(\mu_k^i), i = 1, ..., N_k\right\}$.
The problem (1) in the reduced basis hybrid approximation is: find $u_N(\boldsymbol{\mu}) = \bigcup_{r=1}^R u_N(\mu_r)$ and $p_N(\boldsymbol{\mu}) = \bigcup_{r=1}^R p_N(\mu_r)$ such that, for $r = 1, \cdots, R$, $(u_N(\mu_r), p_N(\mu_r)) \in (Y_N^{kr} \times M_N^{kr})$

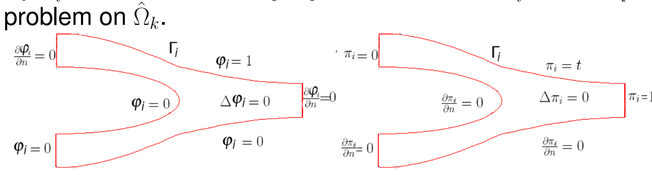
$$\begin{cases} a^k(u_N(\mu), w, \mu_r) + b^k(w, p_N(\mu), \mu_r) = f^k(w, \mu_r, \sigma_r^{in}, \sigma_r^{out}) & \forall w \in Y_N^{kr} \\ b^k(u_N(\mu), q, \mu_r) = 0 & \forall q \in M_N^{kr} \end{cases} \tag{3}$$

and $\int_{\Gamma_{kl}} (u_N(\boldsymbol{\mu})|_{\Omega_k} - u_N(\boldsymbol{\mu})|_{\Omega_l}) \psi ds = 0, \quad \forall \psi \in W_{k,l}, \quad \forall k, l \in \{1, \cdots, R\}$ where Γ_{kl} is the interface between two adjacent sub-domains denoted with the indexes k and l respectively and $W_{k,l}$ is a low order polynomial space defined on this interface.

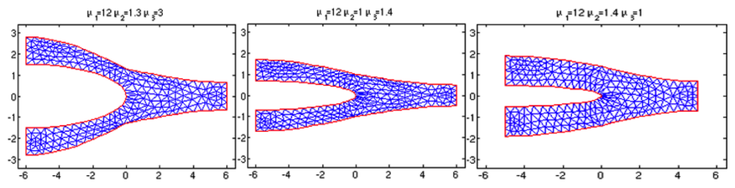
2. Transfinite map

The geometrical parametrization of the domain is obtained through the transfinite map which induces non-affine parameter dependence: an empirical interpolation technique (EIM), [1], is used to recover an affine parameter dependence and a subsequent offline/online decoupling of the reduced basis procedure in order to reduce considerably the problem complexity and the computational times. The blocks deformations are obtained, through the transfinite map $T_{\mu_r}^k : \hat{\Omega}_k \to \Omega_r$, by a linear combination of parametrized functions describing the boundaries of Ω_r [3].

- Each boundary $\Gamma_i^r \subset \partial\Omega_r = \cup_{i=1}^{n_r} \Gamma_i^r$ is described by a parametrized function $\psi_i^r(\cdot, \mu) : [0, 1] \to \Gamma_i^r$;
- For each boundary $\hat{\Gamma}_i^k \subset \partial\hat{\Omega}_k$ we define a **weight function** $\varphi_i^k : \hat{\Omega}_k \to [0, 1]$ solving a proper Laplace problem on $\hat{\Omega}_k$;
- For each boundary $\hat{\Gamma}_i^k \subset \partial\hat{\Omega}_k$ we build a **projection function** $\pi_i^k : \hat{\Omega}_k \to [0, 1]$ as the solution of a second Laplace problem on $\hat{\Omega}_k$.

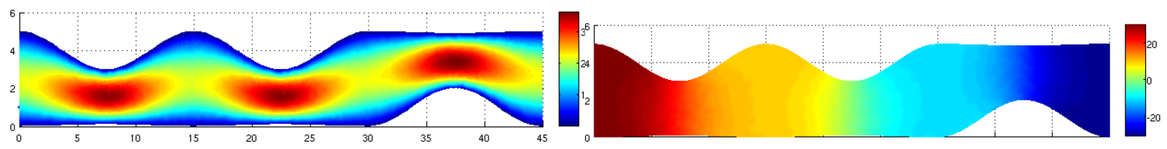


Denoting with n_r the number of boundaries of Ω_r, $\mathbf{x} \in \Omega_r$ is defined as $\mathbf{x} = T_{\mu_r}^k(\hat{\mathbf{x}}) = \sum_{i=1}^n [\varphi_i^k(\hat{\mathbf{x}}) \psi_i^r(\pi_i^k(\hat{\mathbf{x}}), \mu_r) - \varphi_i^k(\hat{\mathbf{x}}) \varphi_{i+1}^k(\hat{\mathbf{x}}) \psi_i^r(1, \mu_r)]$, where $\hat{\mathbf{x}} \in \hat{\Omega}_k$, so that $T_{\mu_r}^k : \hat{\Omega}_k \to \Omega_r$. In the figures below we show an example of deformed bifurcation obtained with the transfinite map using three parameters: μ_1 = length, μ_2 = thickness, μ_3 = bifurcation span.

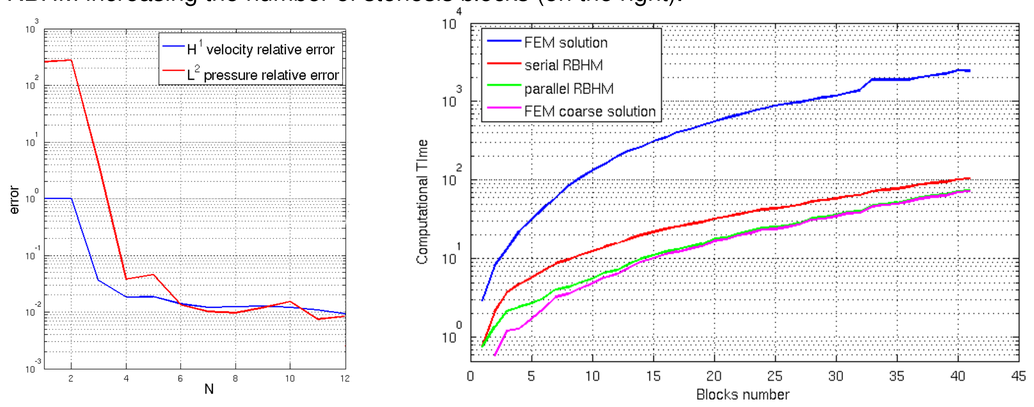


3. Numerical results

We solve a Stokes problem to model the blood flow through a stenotic artery, a multi-block domain has been considered by combining pipes with a curved and parametrized wall. In an off-line stage, we compute all the expensive ingredients involving high resolution: the EIM for the affine decomposition, the FE matrices and the basis functions, the greedy algorithm for choosing the parameters $\mu_1, \cdots, \mu_N$. In the online stage we find the coarse solution for the Stokes problem in the whole domain and we use the stored matrices to assemble and solve the system (3), [2]. The deformation of the stenotic blocks depends on the amplitudes, $\mu_1 \in [0, 2]$ and $\mu_2 \in [0, 2]$, on the upper and lower walls.



In the figures below we report the reductions of the H^1 and L^2 relative errors on velocity and pressure for an increasing number of basis (on the left) and the computational times of FE solutions and the online RBHM increasing the number of stenosis blocks (on the right).



In order to show the advantages of the reduced model proposed, we report in the Table the values of the CPU times for different number K of sub-domains. The computational time for the matrix assembling is 0.76s for each of the sub-domains considered.

K	Fine FEM solution	Coarse FEM solution	RB Linear System	RBHM	%
5	31.13	1.73	0.06	2.72	8.76
10	132.18	4.86	0.14	5.68	4.30
15	311.44	10.18	0.23	11.08	3.56
20	557.57	16.77	0.28	17.81	3.19
25	880.54	23.86	0.60	25.22	2.86
30	1183.5	34.81	0.78	36.35	3.07
35	1895.7	49.74	1.02	51.52	2.71
40	2484.6	70.44	1.56	72.76	2.92

The error tolerance for the EIM is $\epsilon_{tol}^{EIM} = 10^{-6}$. The greedy tolerance is $\epsilon_{tol} = 10^{-5}$. The number of the mesh nodes is 200 for the coarse grid and 1583 for the fine grid. We can see that, in terms of computational cost, the fine solution computed with RBHM is comparable to the one obtained on the coarser grid with standard FEM method. The CPU time spent for FEM simulation with 5 blocks is on par with a RBHM simulation with 27 blocks, still retaining the continuity of velocities and pressure at interfaces.

4. References

[1] M. Barrault, Y. Maday, N.C. Nguyen, and A.T. Patera. An "empirical interpolation" method: application to efficient reduced-basis discretization of partial differential equations. *C.R. Acad. Sci. Paris Anal. Numerique 339*, 667-672, 2004
[2] L. Iapichino, A. Quarteroni and G. Rozza. A reduced basis hybrid method for the coupling of parametrized domains represented by fluidic networks. In preparation, 2010.
[3] A.E. Lovgren, Y. Maday and E.M. Ronquist. A reduced basis element method for complex flow systems. *ECCOMAS CFD, P. Wesseling, E. Onate, J. Periaux (Eds.) TU Delft, The Netherlands*, 2006.
[4] A.E. Lovgren, Y. Maday, and E.M. Ronquist. A reduced basis element method for the steady Stokes problem. *Mathematical Modelling and Numerical Analysis*, 40(3):529-552, 2006.
[5] G. Rozza. Reduced basis methods for Stokes equations in domains with non-affine parameter dependence. *Comput. Vis. Sci.*, 12(1):23-35, 2009.
[6] G. Rozza and K. Veroy. On the stability of the reduced basis method for Stokes equations in parametrized domains. *Comput. Meth. Appl. Mech. Engr.*, 196(7):1244-1260, 2007.

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