Comparison of reduced basis method and stochastic collocation method for parametric and stochastic elliptic problems

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1. Introduction

Reduced basis method and stochastic collocation method have been developed with many ideas in common to solve parametric and stochastic problems [1,2,3]. In this work, 1. we compare their convergence rate from the nonlinear approximation point of view and computational cost; 2. present similar computational reduction techniques such as sparse/adaptive/hierarchical collocation points and efficient greedy sampling strategies (e.g. "hp" type) in parameter space of reduced basis; 3. and analyze different strategies for dealing with high dimensional problems, e.g., dimension adaptation and analysis of sensitivity or variance (ANOVA).

2. Mathematical Formulation

Physical processes such as heat conduction with non-homogeneous conductivity or fluid flow through porous media with random porosity field can be characterized by the following parametric or stochastic elliptic problems:

$$\text{Find} \quad u(\mu) \in X \quad s.t. \quad a(u(\mu), v; \mu) = f(v), \quad \forall v \in X, \tag{1}$$

where a and f are the bilinear form and the linear functional; $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ can represent either parameters or random variables. We are interested in a functional $s(u; \mu)$ or its statistics, for instance, the expectation $E[s(u; \mu)]$.

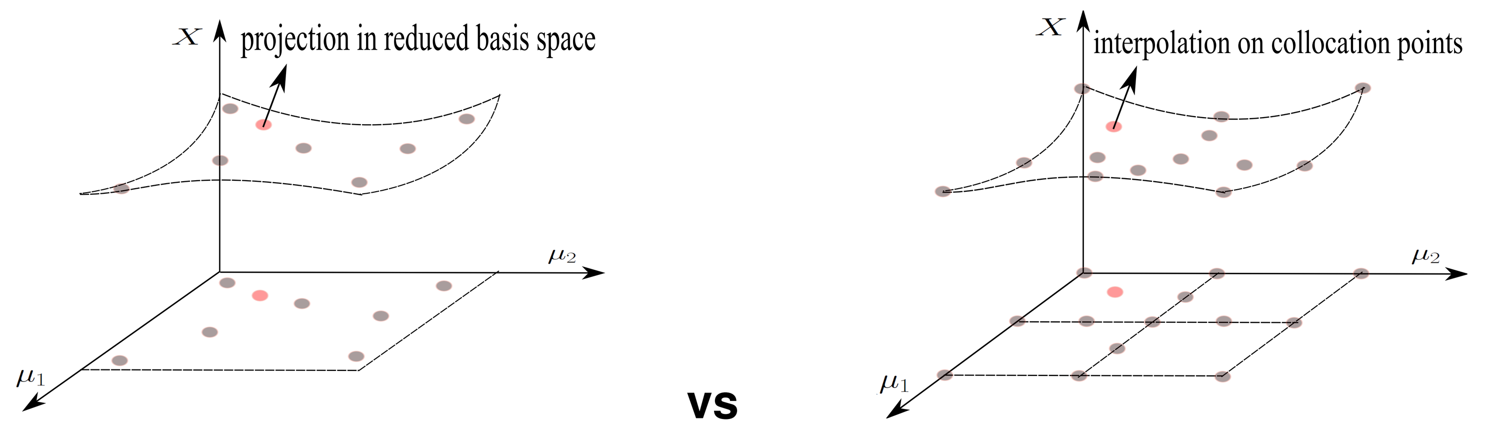

Figure 1: *Approximation by reduced basis (left) and stochastic collocation (right)*

Reduced basis method:[2] S1: Decompose the functionals by affine assumption S2: Compute and save the parameter free quantities. S3: Select samples and compute snapshots by a greedy algorithm based on a posteriori error estimate. S4: Evaluate $s(u; \mu)$ for new μ by Galerkin projection in the reduced basis space.

Stochastic collocation method:[1] S1: Choose appropriate collocation points, e.g. Clenshaw-Curtis, Gauss, Chebyshev, etc. S2: Compute and save solution at the collocation points. S3: Evaluate $s(u; \mu)$ for new μ by Lagrangian interpolation or compute statistics $E[s(u; \mu)]$ by numerical integration over the collocation points.

Methodological comparison: Galerkin projection on "snapshots" or the solution itself **vs** Lagrangian interpolation or projection on Lagrangian polynomials.

3. Convergence Rate and Computational Cost

Theorem 1 *Suppose that the solution $u(\mu)$ is analytic with respect to μ, we have[8]*

$$\left|E[s(u;\mu)] - E[s(u^N_{RB};\mu)]\right| \leq Ce^{-\alpha N^\beta} \quad \textbf{vs} \quad \left|E[s(u;\mu)] - E[s(u^N_{SC};\mu)]\right| \leq CN^{-r}, \tag{2}$$

provided that the Kolmogorov width of reduced basis approximation $d_N \leq Ce^{-\alpha N^\beta}$ holds[4], where u^N_{RB} and u^N_{SC} are reduced basis approximation and stochastic collocation approximation, respectively; C, α, β, r are constants independent of N.

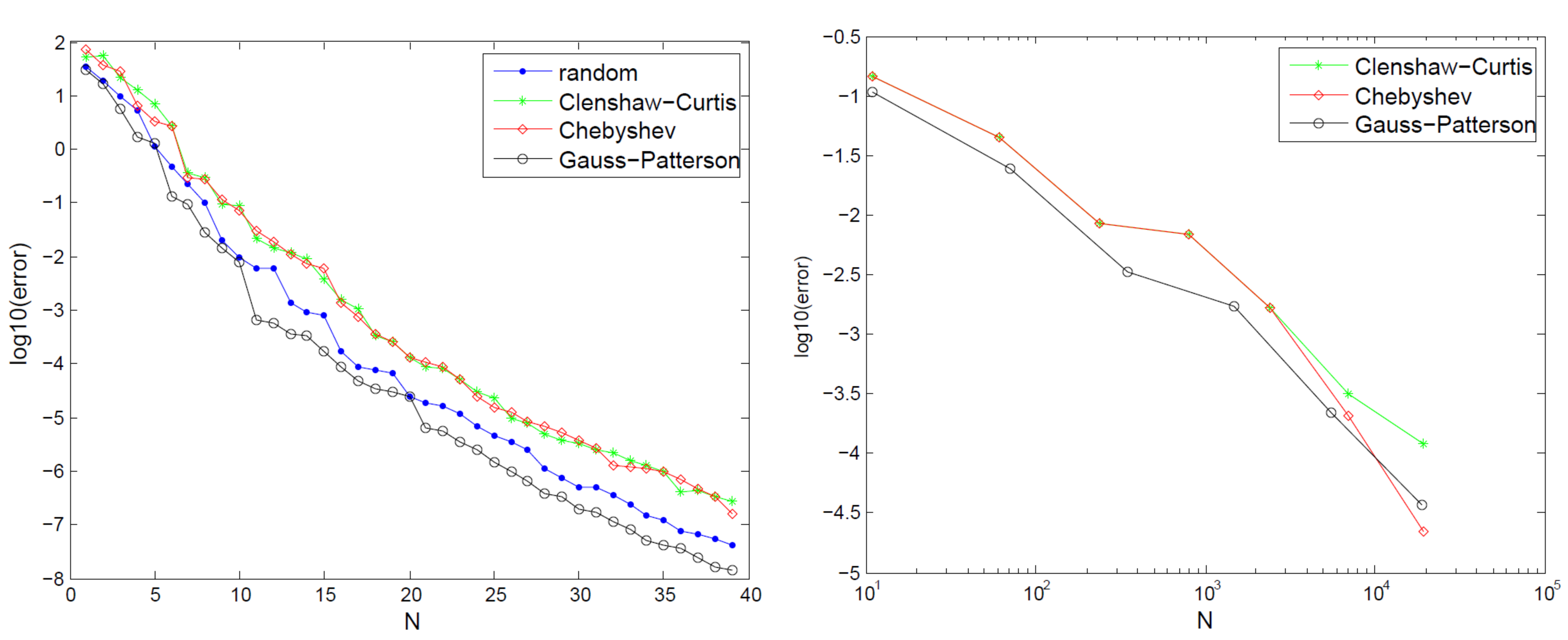

Figure 2: *Convergence rate for expectation $E[s(u; \mu)]$ with five affine terms by reduced basis method (left) and stochastic collocation method (right)*

Let N_{rb} := # reduced bases $\ll N_{sc}$:= # collocation points $\ll n_{train}$:= # training samples, Q_a := # affine terms, W_s := work for solving the elliptic problem once, W_p := work for evaluating a posteriori error estimate once $\ll W_s$, we have[8]

computational cost	construction	evaluation
reduced basis	$O(N_{rb} \times W_s) + O(N_{rb} \times n_{train} \times W_p)$	$O(Q_a N^2_{rb} + N^3_{rb})$
stochastic collocation	$O(N_{sc} \times W_s)$	$O(N_{sc})$

Table 1: *Approximate cost of reduced basis and stochastic collocation methods*

4. Computational Reduction Techniques

In order to diminish the computational cost for both the reduced basis method and the stochastic collocation method, we take advantage of the following techniques:

- tensor product $\rightarrow$ **hierarchical sparse structure**, reduction for construction;[1]
- isotropic $\rightarrow$ **anisotropic structure**, reduction for anisotropic problems;[1]
- global $\rightarrow$ **local adaptive p-refinement**, reduction for low regularity problems;[6]
- one $\rightarrow$ **domain decomposition h-refinement**, reduction for evaluation.[5]

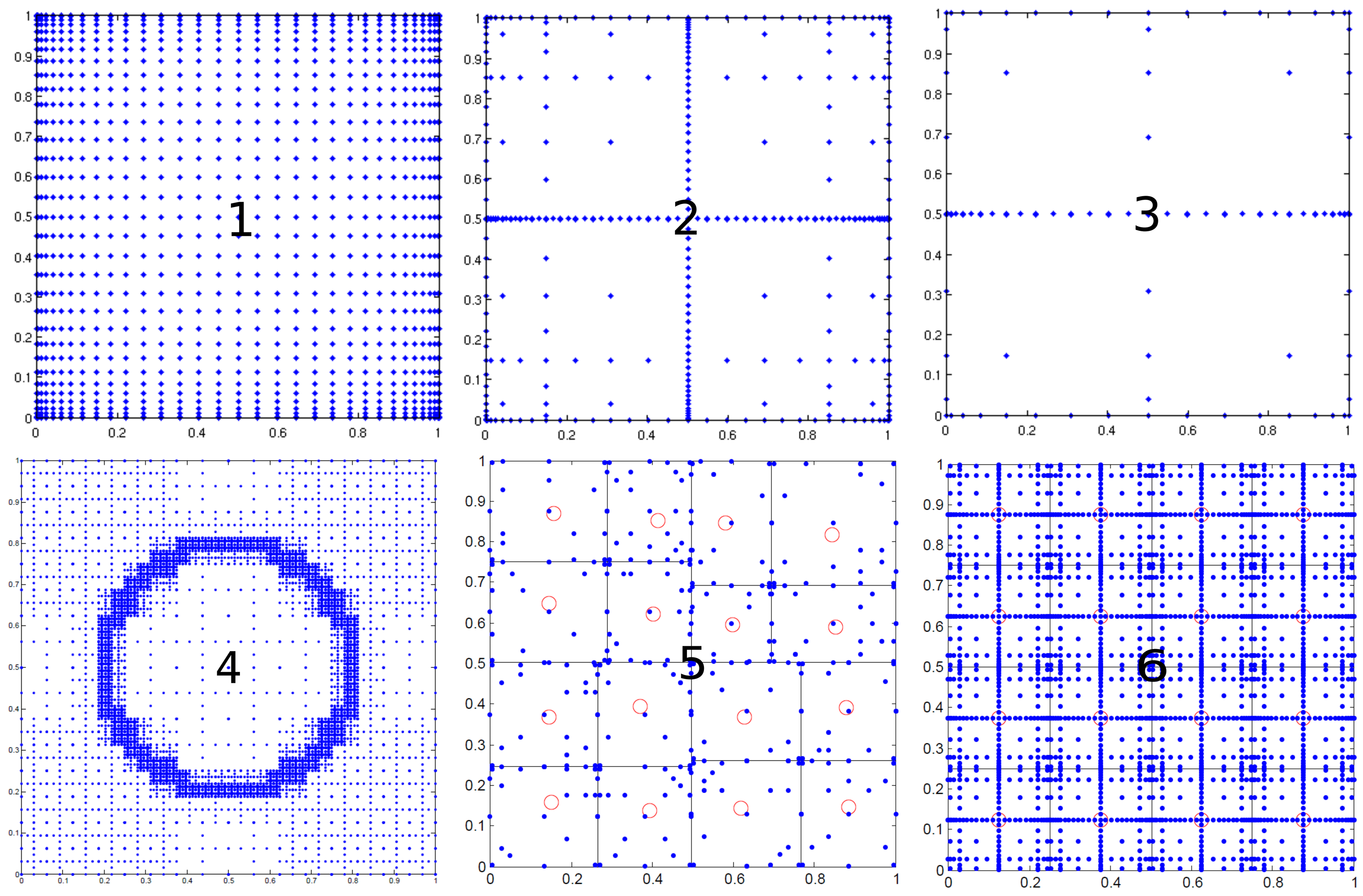

Figure 3: *1).full tensor product grid; 2). sparse grid; 3). anisotropic sparse grid; 4). local p-refinement; 5). h-refinement for RB; 6). h-refinement for SC.*

5. Toward High Dimensional Problems

Both methods suffer from "curse-of-dimensionality" for high dimensional problems. Recipe: **dimension adaptation**[6] and **sensitivity analysis or ANOVA**[7].

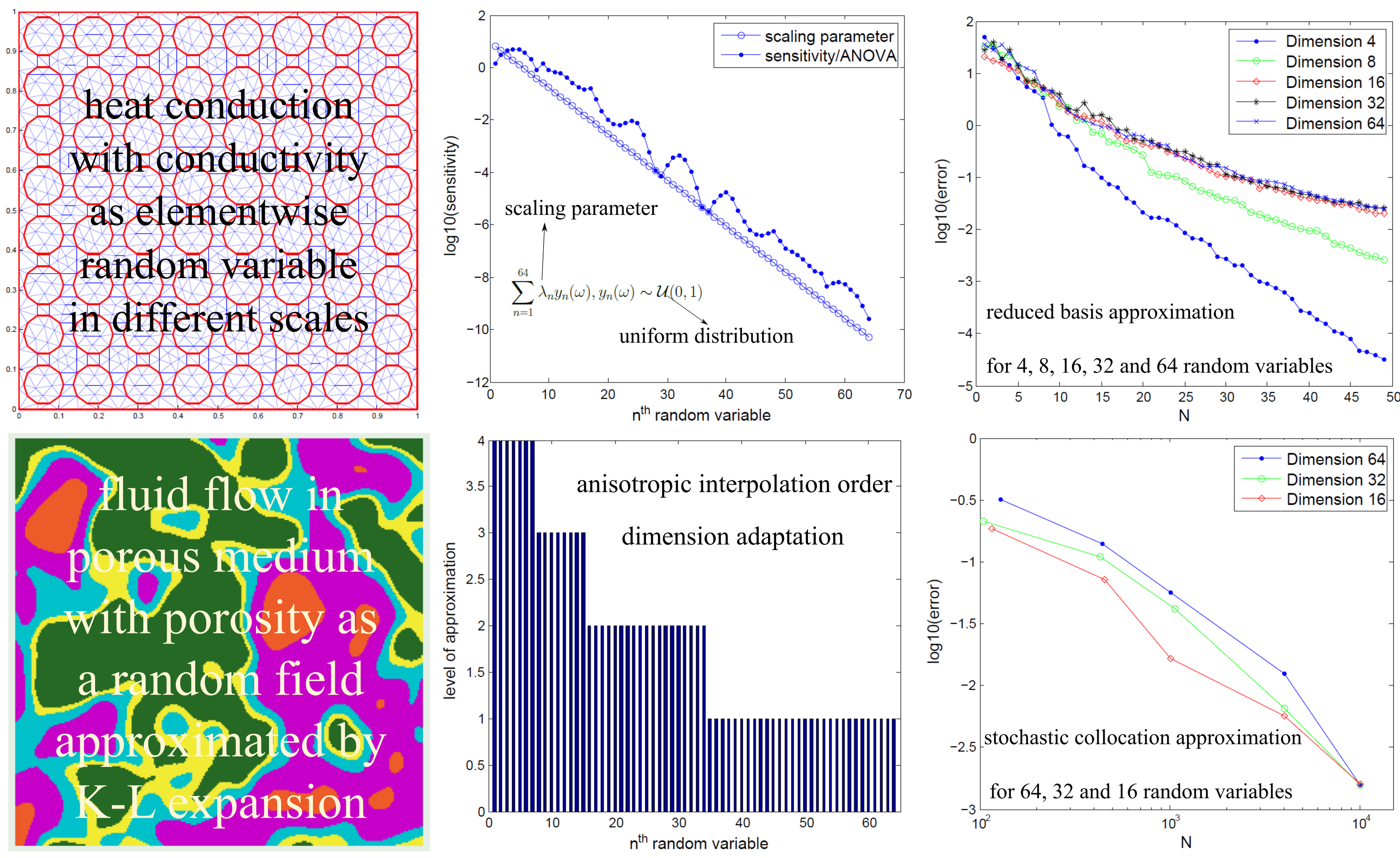

Figure 4: *1). heat conduction; 2). sensitivity or ANOVA; 3). RB approximation; 4). flow in porous medium; 5). dimension adaptation; 6). SC approximation.*

6. Concluding Remarks

In summary, reduced basis method achieves better accuracy than stochastic collocation method at comparable cost, while the latter one possesses hierarchical and anisotropic properties to deal with low regularity and high dimensional problems. A promising research is to combine them to achieve efficient sampling and computational reduction for solving parametric and stochastic problems.

7. References

[1] I. Babuska, F. Nobile, and R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, Siam J. Numer. Anal, 45 3 (2007)
[2] G. Rozza, D. B. P. Huynh and A. T. Patera, Reduced Basis Approximation and a Posteriori Error Estimation for Affinely Parametrized Elliptic Coercive Partial Differential Equations Arch Comput Meth Eng, 15 3 (2007)
[3] S. Boyaval, C. Le Bris, T. LeliÃÍvre, Y. Maday, N. C. Nguyen and A. T. Patera, Reduced Basis Techniques for Stochastic Problems. Arch Comput Meth Eng, 17 4 (2010)
[4] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk, Convergence Rates for Greedy Algorithms in Reduced Basis Methods SIAM J. Math. Anal. 43 3 (2011)
[5] J.L. Eftang, A.T. Patera, and E.M. Ronquist, An "hp" Certified Reduced Basis Method for Parametrized Elliptic Partial Differential Equations SIAM J. Sci. Comput, 32 6 (2010)
[6] X. Ma and N. Zabaras, An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations J Comp Phys, 228 8 (2009)
[7] J. S. Hesthaven and S. Zhang, On the use of ANOVA expansions in reduced basis methods for high-dimensional parametric partial differential equations Brown University (2011)
[8] P. Chen, A. Quarteroni and G. Rozza, Comparison of reduced basis method and stochastic collocation method based on parametric and stochastic elliptic problems. in preparation (2012)

CECAM workshop: Reduced Basis, POD and Reduced Order Methods for model and computational reduction: towards real-time computing and visualization?

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