Geometrical reduction for shape optimization problems

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Abstract

Recent work on shape optimization problems [3, 4] has shown that volume-based shape parameterizations, such as the free-form deformation, are a powerful tool in representing a large class of admissible domains, even at a low parametric dimension, and can be coupled to the reduced basis method to obtain rapid and reliable solutions of the state problem. However, the choice of the degrees of freedom of the parameterization is a critical issue: it can be based, in simple cases, on an a priori knowledge of some physically relevant properties of the problem; unfortunately, when the complexity of the model and/or of the domain increases, it is usually quite difficult to devise optimal choices. We present a sensitivity-driven algorithm for the automatic selection of the degrees of freedom of the free-form deformation, and some numerical results with applications to fluid dynamics problems. The efficiency of the method will be highlighted on a test case by a comparison with more classical shape deformation tools (e.g. local boundary variation) and the available analytical solution.

1. Introduction

Shape optimization problems have become extremely popular during the last decades, mainly because of several advanced applications in physics or engineering, in which the optimal design of a device greatly enhances the efficiency or the mechanical behavior of a system. In particular we consider the following shape optimization problem

$$\Omega_{\hat{\mu}} = \arg\min_{\Omega_{\mu} \in \mathcal{O}_{ad}} J(\Omega_{\mu}),$$

in the framework of computational fluid dynamics, dealing with three-dimensional steady Stokes flows and the minimization of the dissipated energy

$$j(\mu) = J(\Omega_{\mu}) = \frac{\nu}{2} \int_{\Omega_{\mu}} |\nabla \boldsymbol{u}(\Omega_{\mu})|^2 \, d\boldsymbol{x},$$

where $\boldsymbol{u}(\Omega_{\mu})$ is the velocity in the domain $\Omega_{\mu} \in \mathcal{O}_{ad}$, which is the image of a fixed reference domain under the free-form deformation (FFD) of parameters μ, under the additional constraint of constant volume.
Exploiting, as in [1], the formulation of the free-form deformation as a perturbation of identity $\boldsymbol{I} + \boldsymbol{\theta}_{\mu}$ and recalling, as in [5], that the optimality conditions of the shape optimization problem read

$$dJ(\Omega_{\mu}; \boldsymbol{\theta}_{\mu}) = -\frac{\nu}{2} \int_{\partial\Omega_{\mu}} \left| \frac{\partial}{\partial \boldsymbol{n}} \boldsymbol{u}(\Omega_{\mu}) \right|^2 \boldsymbol{\theta}_{\mu} \cdot \boldsymbol{n} = 0,$$

we can compute the sensitivity

$$-\frac{\partial j}{\partial \mu_i}(\mu)$$

of the cost functional to each displacement μ_i, $i = 1, \dots, P$, and design a sensitivity analysis averaging over a box $\mathbb{B} \subset \mathbb{R}^P$ by Monte Carlo integration methods with N uniformly distributed samples.

2. Body in a Stokes flow

Free-form deformation is a powerful tool for shape parameterization based on tensor product of *splines*, that allows global deformations of a reference domain (in this case, the complement of a spherical immersed body) by acting on a small set of control points, some of which will be fixed *a priori* (blue markers): we perform the sensitivity analysis for each displacement of the remaining control points (gray markers).

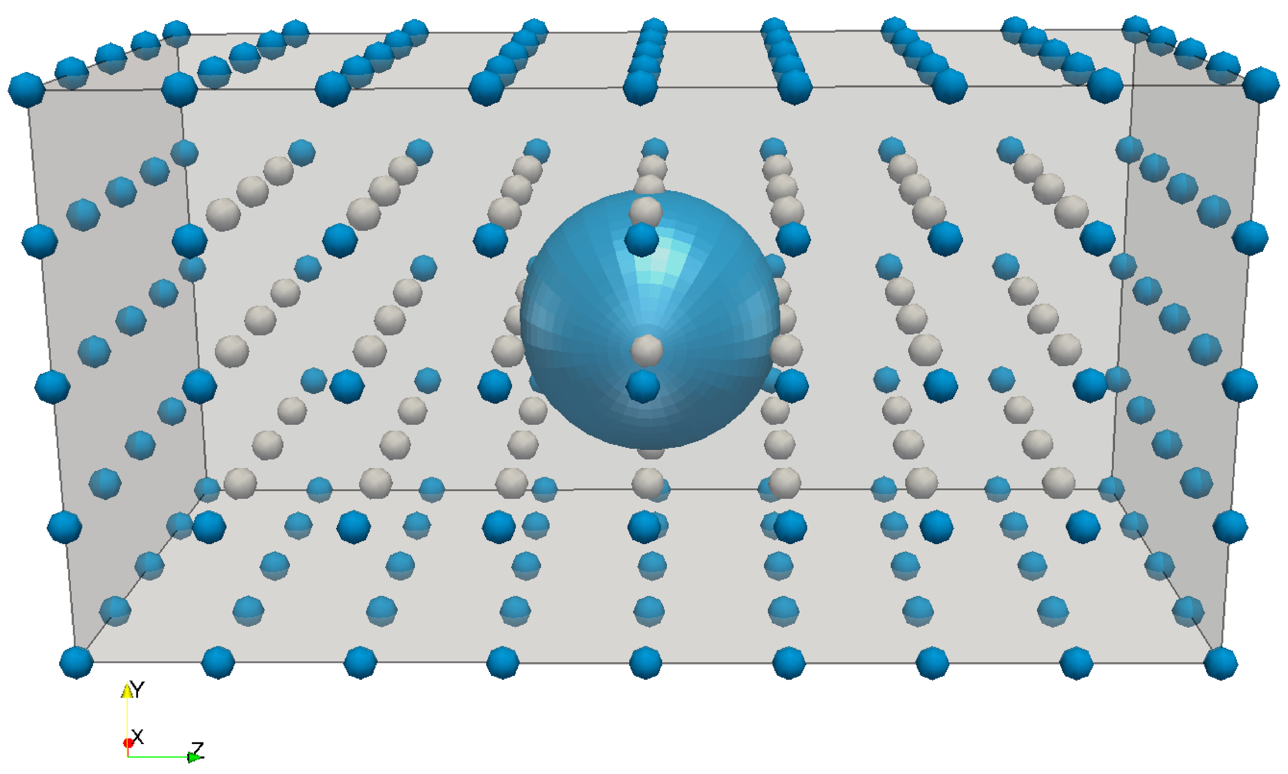

Figure 1: *FFD parameterization of an immersed body.*

The case of the body in a steady Stokes flow is interesting because, as for the sensitivity analysis, empirical considerations suggest that it is reasonable to enable the control points closest to the surface of the immersed body and, as for the shape optimization problem, we can compare the FFD optimal shape with the analytical one studied by Pironneau and Bourot in [2]. Moreover, FFD shape parameterization outperforms classical local boundary variation methods, because both reach convergence in a small number of iterations (less than 10) but FFD has a considerably smaller number of degrees of freedom and does not require any regularization of the shape between each iteration.

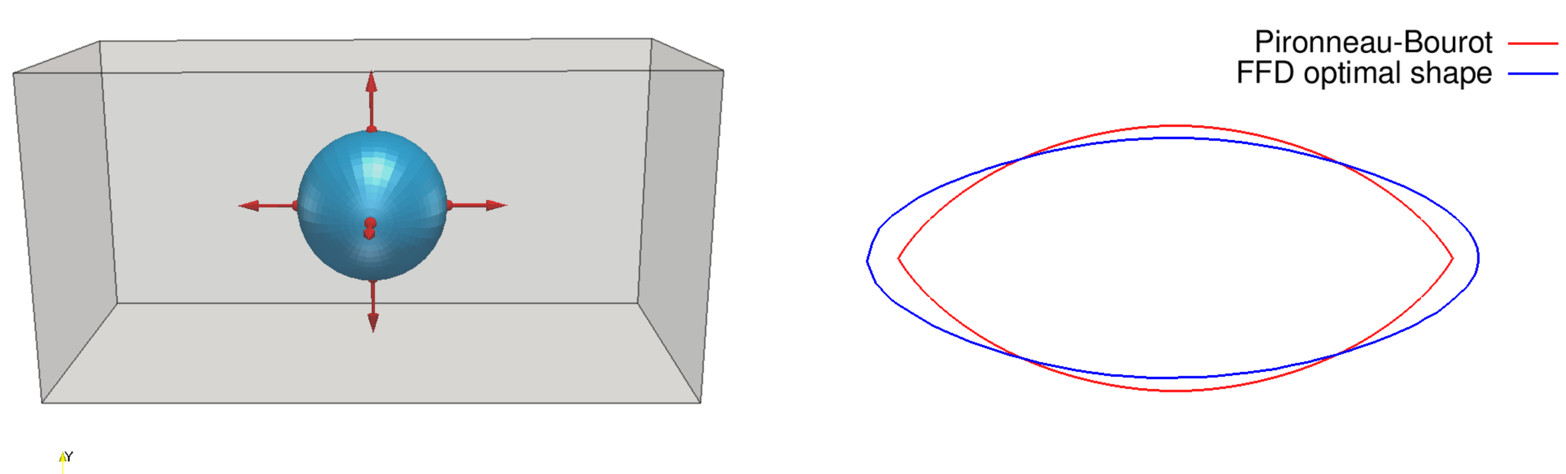

Figure 2: *Comparison between the FFD optimal shape and the Pironneau-Bourot profile, employing six displacements chosen according to empirical considerations.*

The simulations are carried out with the finite element library *LifeV*, considering a mesh of 60000 volume elements, $\mathbb{B} = [-1, 1]^P$ and $N = 200$.

The scatter plot of absolute value of the mean $E_i = \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} \frac{\partial j}{\partial \mu_i}(\mu) d\mu$ vs standard deviation $\sqrt{\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} \left(\frac{\partial j}{\partial \mu_i}(\mu) - E_i \right)^2 d\mu}$ of each gradient component $i = 1, \dots, P$ shows that the most significant displacements are in the direction of the flow (z direction).

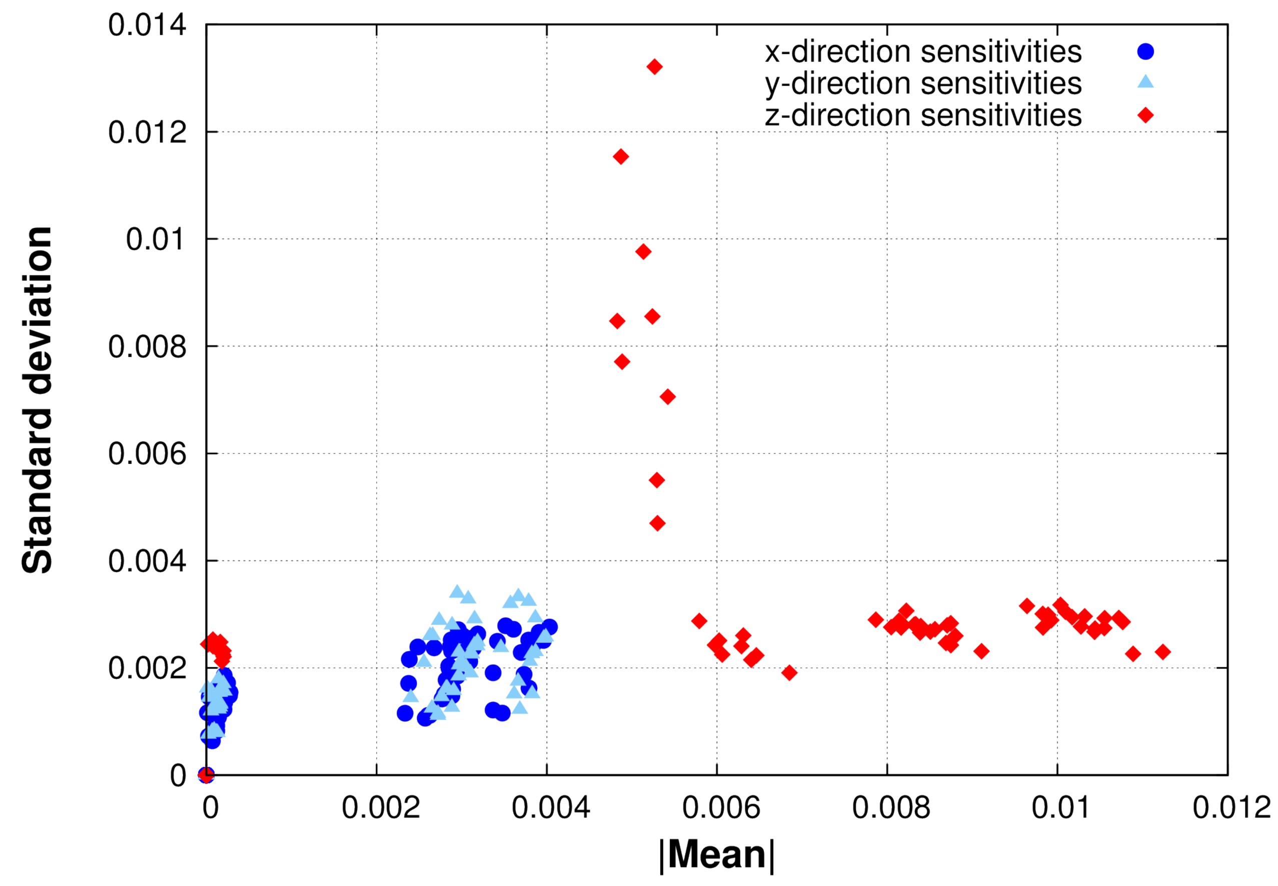

Figure 3: *Sensitivity analysis: absolute value of the mean vs standard deviation, for each component of the gradient of the (parametrized) cost functional $j(\mu)$.*

The analysis of the mean values shows that the sensitivities are symmetric, as expected since it can be shown that the optimal body is axisymmetric; on average, each steepest descent iteration will stretch the body in the direction of the flow. Moreover, considering each plane of (not fixed) control points which is orthogonal to the direction of the flow, the highest mean sensitivity values are obtained in the second and second-to-last planes. The diametral plane (the fourth one) is the only one with null mean sensitivity in the direction of the flow, and the mean sensitivity vector is tangent to the surface of the ball; however, even if the magnitude of the mean values in this plane is small, we are not willing to disable the four displacements suggested by our empirical considerations, because they grant us the validity of the volume constraint.

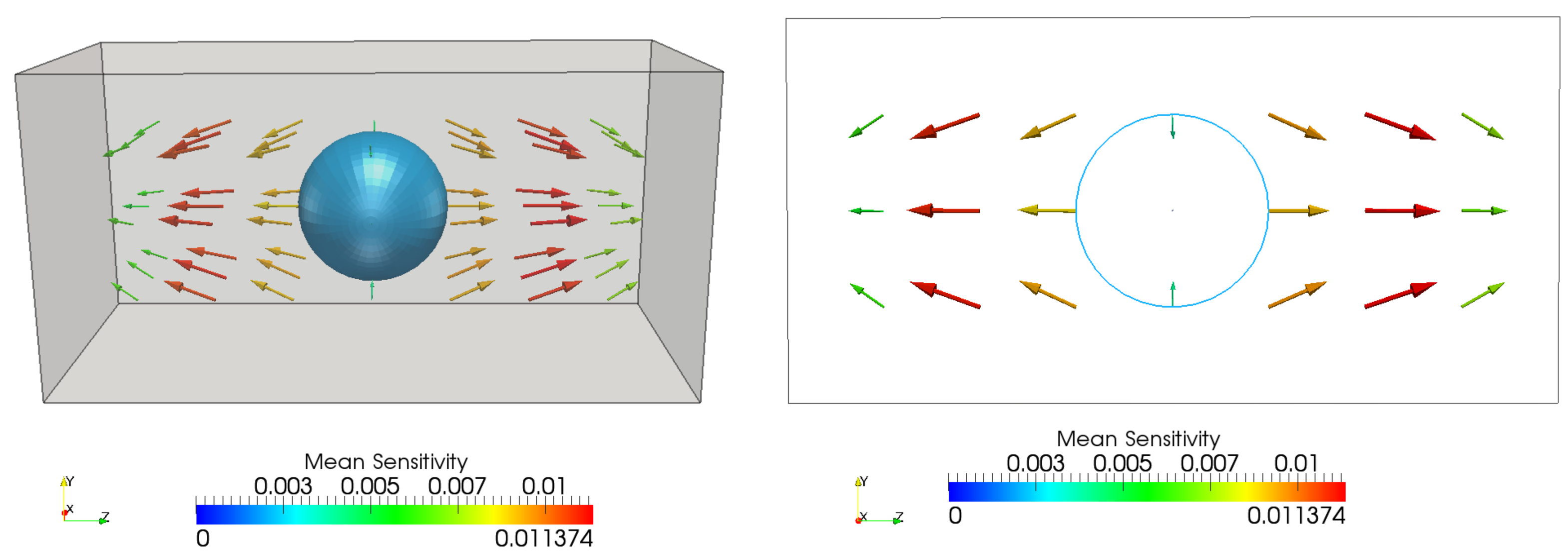

Figure 4: *Sensitivity analysis: mean values of each component of the gradient of $j(\mu)$.*

The results of the sensitivity analysis allow us to improve the FFD optimal shape: indeed, employing the same number of displacements, the Pironneau-Bourot profile is recovered with an higher accuracy.

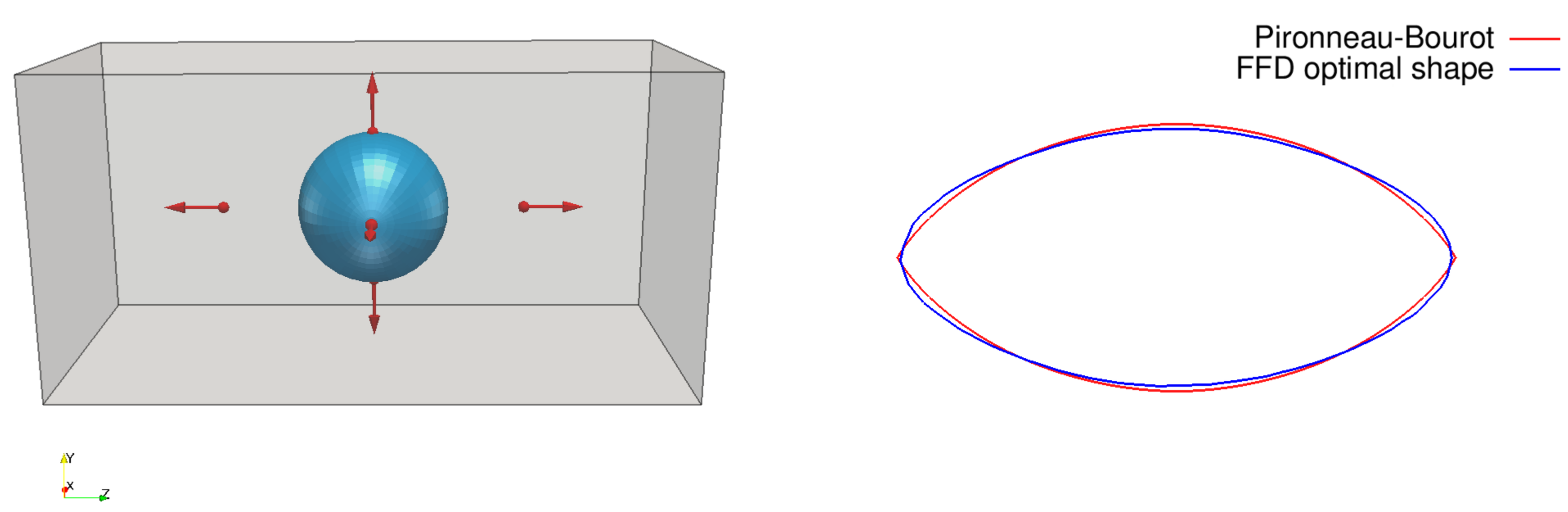

Figure 5: *Comparison between the FFD optimal shape and the Pironneau-Bourot profile, employing six displacements chosen according to the sensitivity analysis.*

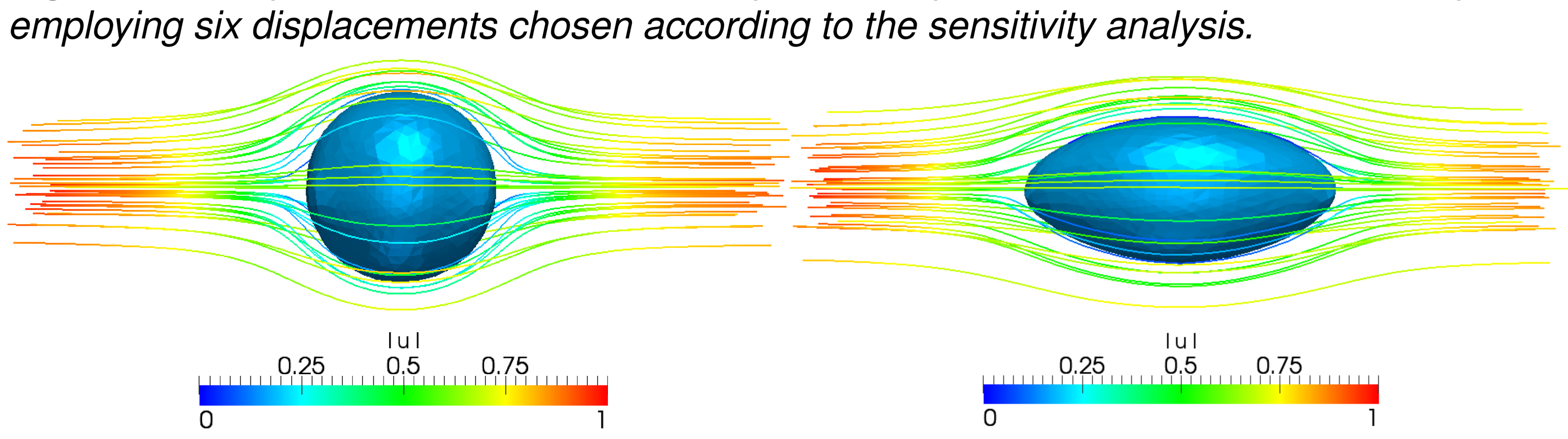

Figure 6: *Comparison of the flow in the reference and in the optimal shape.*

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