Reduced basis method for parametrized optimal control problems

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Abstract

We propose a reduced basis (RB) method for the rapid and reliable solution of parametrized optimal control problems governed by PDEs. In particular, we develop the methodology for parametrized quadratic optimization problems with either coercive elliptic equations or Stokes equations as constraints. Firstly, we recast the optimal control problem in the framework of saddle point problems in order to take advantage of the already developed RB theory for Stokes-type problems. Then the usual ingredients of the RB methodology are provided: a Galerkin projection onto a low-dimensional space of basis functions properly selected by an adaptive procedure; an affine parametric dependence enabling to perform competitive Offline-Online splitting in the computational procedure; an efficient and rigorous a posteriori error estimation on the state, control and adjoint variables as well as on the cost functional.

1. Problem definition

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with boundary Γ and $\mathcal{D} \subset \mathbb{R}^p$ be a p-dimensional parameter set, with $p \geq 1$. The state space Y is chosen such that $H^1_0(\Omega) \subset Y \subset H^1(\Omega)$, while $Q \equiv Y$ denote the space for the adjoint variable. The control space is given by $U = L^2(\omega)$, where $\omega \subset \Omega$ or $\omega \subset \Gamma$; finally, $\mathcal{Z}$ shall denote the observation space.
The state equation is given in the form

$$a(y, q; \boldsymbol{\mu}) = c(u, q; \boldsymbol{\mu}) + \langle G(\boldsymbol{\mu}), q\rangle \qquad \forall q \in Q, \tag{1}$$

where the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$ represents a linear elliptic operator, the bilinear form $c(\cdot,\cdot;\boldsymbol{\mu})$ expresses the action of the control while $G(\boldsymbol{\mu}) \in Q'$ acts as a forcing term. The quadratic cost functional to be minimized is given by

$$J(y, u; \boldsymbol{\mu}) = \frac{1}{2} m(y - y_d(\boldsymbol{\mu}), y - y_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(u, u; \boldsymbol{\mu}), \tag{2}$$

where $\alpha > 0$ is a given constant and $y_d(\boldsymbol{\mu}) \in \mathcal{Z}$ is a given parameter-dependent observation function.

The parametrized optimal control problem (OCP$_{\boldsymbol{\mu}}$) reads: for any given $\boldsymbol{\mu} \in \mathcal{D}$

$$\min J(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad (y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in Y \times U \text{ solves } (1).$$

- The main assumption in order to ensure the well-posedness of the problem is the (weakly) coercivity of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$.
- We shall make an additional assumptions, crucial to Offline-Online procedures, by assuming the bilinear and linear forms to be affine in the parameter $\boldsymbol{\mu}$.

2. Saddle-point formulation

We first define the product space $X = Y \times U$ and denote with $\underline{x} = (y, u) \in X$, $\underline{w} = (z, v) \in X$ its variables. We can reformulate the OCP$_{\boldsymbol{\mu}}$ as: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\begin{cases} \min \mathcal{J}(\underline{x};\boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(\underline{x},\underline{x};\boldsymbol{\mu}) - \langle \underline{F}(\boldsymbol{\mu}), \underline{x}\rangle, & \text{subject to} \\ \mathcal{B}(\underline{x}, q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in Q. \end{cases} \tag{3}$$

where $\underline{F}(\boldsymbol{\mu}) = m(y_d(\boldsymbol{\mu}), \cdot) \in X'$ and

$$\mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) = m(y, z; \boldsymbol{\mu}) + \alpha n(u, v; \boldsymbol{\mu}), \qquad \forall \underline{x}, \underline{w} \in X,$$

$$\mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = a(z, q; \boldsymbol{\mu}) - c(v, q; \boldsymbol{\mu}), \qquad \forall \underline{w} \in X, q \in Q.$$

The constrained optimization problem (3) falls into the framework of saddle-point problems. The assumptions of Brezzi theorem [1] can be easily verified [3] and therefore, for any $\boldsymbol{\mu} \in \mathcal{D}$, the optimal control problem has a unique solution $\underline{x}(\boldsymbol{\mu}) \in X$ that can be determined by solving the following saddle-point problem (i.e. the optimality system):

given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$ such that

$$\begin{cases} \mathcal{A}(\underline{x}(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w}\rangle & \forall \underline{w} \in X, \\ \mathcal{B}(\underline{x}(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in Q, \end{cases} \tag{4}$$

where $p(\boldsymbol{\mu})$ is the Lagrange multiplier associated to the constraint. Thanks to the affine parameter dependence assumption, an affine decomposition holds also for the bilinear and linear forms in (4).

3. Reduced basis approximation

The RB method [5] gives an efficient way to compute an approximation to the FE *truth* solution $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}))$ by considering only a small subspace of the FE space $X^{\mathcal{N}} \times Q^{\mathcal{N}}$. We thus take a suitably selected (by a greedy algorithm) set of parameter values $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N$ ($N \ll \mathcal{N}$) and the corresponding FE solutions $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^1), p^{\mathcal{N}}(\boldsymbol{\mu}^1)), \dots, (\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^N), p^{\mathcal{N}}(\boldsymbol{\mu}^N))$. The reduced basis control space is given by

$$U_N = \text{span}\{\lambda_n := u^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\},$$

while, in order to guarantee the stability of the RB approximation, we define the following *aggregated* reduced basis space for the state and adjoint variables

$$Y_N \equiv Q_N = \text{span}\{\zeta_n := y^{\mathcal{N}}(\boldsymbol{\mu}^n), \, \xi_n := p^{\mathcal{N}}(\boldsymbol{\mu}), \; n = 1, \dots, N\}.$$

Let $X_N = Y_N \times Q_N$, the reduced basis approximation reads:

given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{A}(\underline{x}_N(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w}\rangle & \forall \underline{w} \in X_N, \\ \mathcal{B}(\underline{x}_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in Q_N. \end{cases}$$

4. Offline-Online decomposition

Algebraic formulation:

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}}_{\mathbf{f}_N(\boldsymbol{\mu})}. \tag{5}$$

From the affine assumption, we can write

$$K_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_k} \Theta_k^q(\boldsymbol{\mu}) K_N^q, \qquad \mathbf{f}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) f_N^q,$$

where K_N^q and f_N^q are $\boldsymbol{\mu}$-independent. Offline-Online computational strategy:

- in the *Offline stage*, performed only once, we compute the basis function and form the $\boldsymbol{\mu}$-independent matrices K_N^q and the vectors f_N^q.
- in the *Online stage*, performed for each new value $\boldsymbol{\mu}$, we assemble the full matrix K_N and the vector $\mathbf{f}_N$ and then solve the reduced system of dimension $5N \times 5N$ to obtain $(\mathbf{x}_N, \mathbf{p}_N)$. The Online operation count depends on N and Q_* but is independent of $\mathcal{N}$.

5. A posteriori error estimation

Recasting the problem in the general Babuška framework [5, 4] we can provide an efficient and rigorous a posteriori error estimation on the solution variables:

$$\|(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu})) - (\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))\|_{X \times Q} \leq \frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|}{\hat{\beta}_{\mathsf{LB}}(\boldsymbol{\mu})} = \Delta_N(\boldsymbol{\mu})$$

- $0 < \hat{\beta}_{\mathsf{LB}}(\boldsymbol{\mu}) \leq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ is a *lower bound* of the inf-sup constant of the bilinear form $\mathsf{B}(\cdot,\cdot;\boldsymbol{\mu}) = \mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p; \boldsymbol{\mu}) + \mathcal{B}(\underline{x}, q; \boldsymbol{\mu})$ given by the Successive Constraint Method [4];
- $\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|$ is the dual norm of the residual of the optimality system.

With standard arguments we can also easily obtain the following a posteriori error estimation on the cost functional:

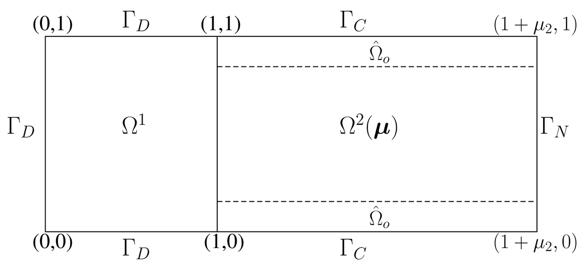
$$|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})| \leq \frac{1}{2}\frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|^2}{\hat{\beta}_{\mathsf{LB}}(\boldsymbol{\mu})} = \Delta_N^J(\boldsymbol{\mu}).$$

6. Numerical results: boundary control for a Graetz convection-diffusion flow

We consider the following optimal control problem

$$\min \; J(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\|y(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega})} + \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{L^2(\Gamma_C)}$$

$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_1}\Delta y(\boldsymbol{\mu}) + x_2(1 - x_2)\dfrac{\partial y(\boldsymbol{\mu})}{\partial x_1} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \\ \dfrac{1}{\mu_1}\nabla y(\boldsymbol{\mu}) \cdot \mathbf{n} = u(\boldsymbol{\mu}) & \text{on } \Gamma_C(\boldsymbol{\mu}) \\ + \text{BCs on } \Gamma_N \text{ and } \Gamma_D. \end{cases}$$



$$y_d(\boldsymbol{\mu}) = \mu_3 \chi_{\hat{\Omega}_o}$$

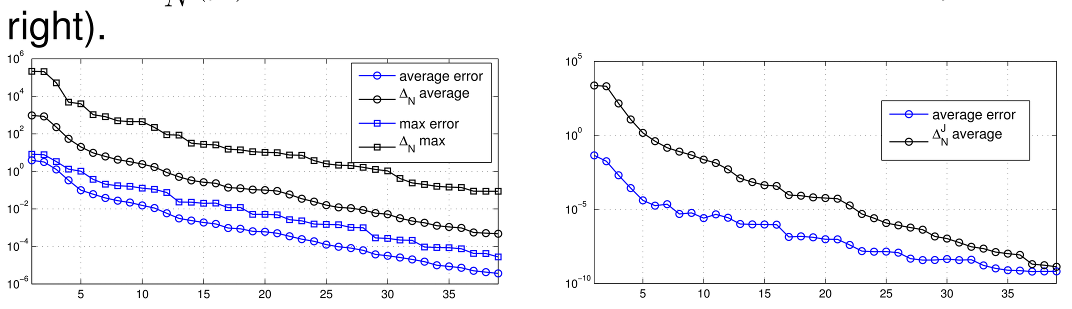
observation domain
$\hat{\Omega}_o(\boldsymbol{\mu}) \subset \Omega^2(\boldsymbol{\mu})$

$$\mathcal{D} = [6, 20] \times [1, 3] \times [0.5, 3]$$

By tracing the problem back to a reference domain (through affine geometrical mappings) we obtain the parametrized formulation (4).

No. of FE dof $\mathcal{N}$	8915		Linear system dim. reduction	$50:1$
No. of parameters P	3		FE evaluation t_{FE} (s)	14.5
No. of RB functions N	39		RB evaluation t_{RB}^{online} (s)	0.1

In the figure below we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the error (on the left) and the a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ with the error on the cost functional (on the right).



Implementation in the MATLAB environment: `MLife` + `rbMIT` libraries.

7. Stokes contraints: control of a Couette flow and an application in haemodynamics

The methodology has been extended to treat OCP$_{\boldsymbol{\mu}}$ with Stokes constraints:

$$\min \; J(\boldsymbol{v}, p, \boldsymbol{u}; \boldsymbol{\mu}) = \frac{1}{2} m(\boldsymbol{v} - \boldsymbol{v}_d(\boldsymbol{\mu}), \boldsymbol{v} - \boldsymbol{v}_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{\mu})$$

$$\text{s.t.} \begin{cases} a(\boldsymbol{v}, \boldsymbol{\xi}; \boldsymbol{\mu}) + b(\boldsymbol{\xi}, p; \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \boldsymbol{\xi}\rangle + c(\boldsymbol{u}, \boldsymbol{\xi}; \boldsymbol{\mu}) & \forall \boldsymbol{\xi} \in H^1 \\ b(\boldsymbol{v}, \tau; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), \tau\rangle & \forall \tau \in L^2. \end{cases}$$

The stability of the RB approximation can be fulfilled by introducing suitable supremizer operators [4] and by defining suitable aggregated spaces for the state and adjoint variables.

Numerical example: distributed control of a Couette flow

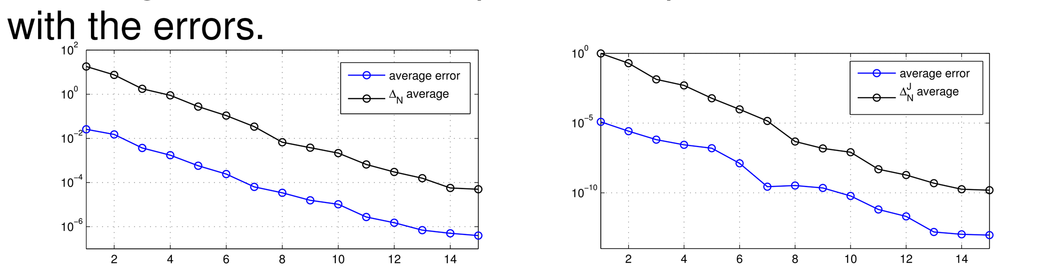
$$\min \; J(\boldsymbol{v}(\boldsymbol{\mu}), p(\boldsymbol{\mu}), \boldsymbol{u}(\boldsymbol{\mu})) = \frac{1}{2}\|v_1(\boldsymbol{\mu}) - x_2\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\boldsymbol{u}(\boldsymbol{\mu})\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \begin{cases} -\nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f}(\boldsymbol{\mu}) + \boldsymbol{u} & \text{in } \Omega(\boldsymbol{\mu}) \\ \text{div}\, \boldsymbol{v} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \\ + \text{BCs on } \Gamma_N \text{ and } \Gamma_D. \end{cases}$$

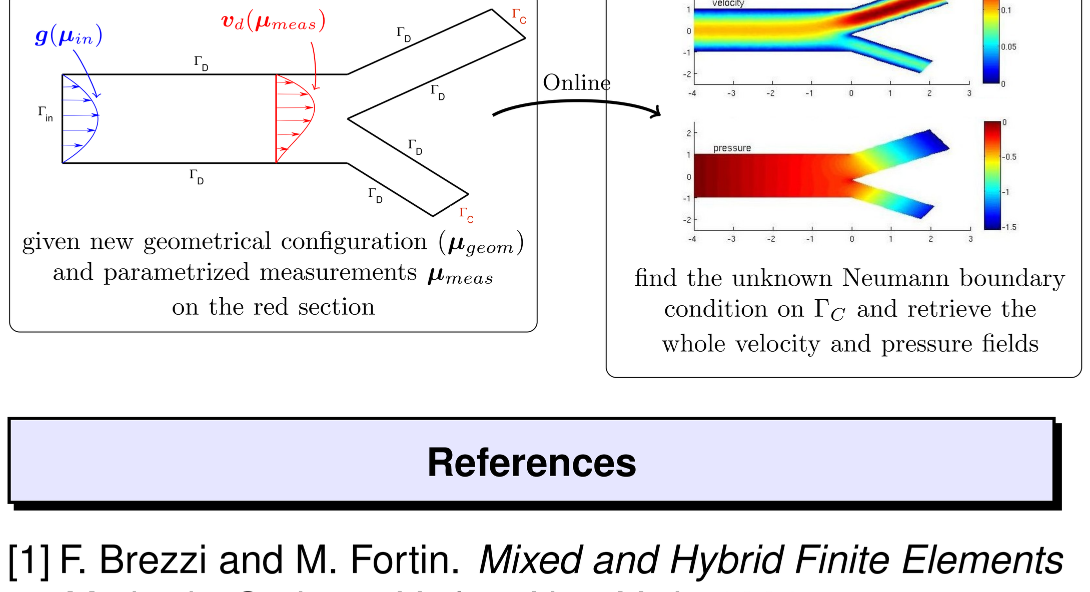
where $\boldsymbol{f}(\boldsymbol{\mu}) = (0, -\mu_2)$, $\Omega(\boldsymbol{\mu}) = [0, 1] \times [0, \mu_1]$ and $\mathcal{D} = [0.5, 2] \times [0.5, 1.5]$.

No. of FE dof $\mathcal{N}$	17439		Linear system dim. reduction	$80:1$
No. of parameters P	2		FE evaluation t_{FE} (s)	16.1
No. of RB functions N	15		RB evaluation t_{RB}^{online} (s)	0.1

In the figure below we compare the a posteriori error bounds with the errors.



As a possible application, we consider an inverse boundary problem in haemodynamics (inspired by the work in [2]) where the state equation models the blood flow (supposed to obey the Stokes equations) in a parametrized arterial bifurcation:



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