Reduced basis method for parametrized optimal control problems

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Abstract
We propose a reduced basis (RB) method for the rapid and reliable solution of parametrized optimal control problems governed by PDEs. In particular, we develop the methodology for parametrized quadratic optimization problems with either coercive elliptic equations or Stokes equations as constraints. Firstly, we recast the optimal control problem in the framework of saddle point problems in order to take advantage of the already developed RB theory for Stokes-type problems. Then the usual ingredients of the RB methodology are provided: a Galerkin projection onto a low-dimensional space of basis functions properly selected by an adaptive procedure; an affine parametric dependence enabling to perform competitive Offline-Onlin-splitting in the computational procedure; an efficient and rigorous a posteriori error estimation on the state, control and adjoint variables as well as on the cost functional.

1. Problem definition
Let \( \mathbb{R}^d \) be an open and bounded domain with boundary \( \Gamma \) and \( \mathbb{R}^d \) be a \( d \)-dimensional parameter set, with \( d \geq 1 \). The state equation is given in the form
\[
a \in \partial \mu \ (\nu, \psi):= a(\nu, \psi; \mu) + b(\nu, \psi; \mu) \quad \text{in} \quad \Omega, \quad \psi \in V \ni \int_{\Omega} \psi \ dV=1.\]
where \( \nu \) denotes the state space, \( \psi \) the test function, \( \mu \) the parameter set, \( a(\cdot, \cdot; \mu) \) is a bilinear form, \( b(\cdot, \cdot; \mu) \) a source term and \( V \) a (finite-dimensional) space.

2. Solution framework

2.1. Mathematical formulation
We first define the product space \( X := Y \times U \) and denote with \( x := (\nu, \psi) \in X \) its variables. We can re-consider the RB equations as:
\[
\min \ J(\nu, \psi, \mu) := \frac{1}{2} \int_{\Omega} (\nu - \nu^0)^2 + \int_{\Gamma} (\psi - \psi^0)^2 \ dS \quad \text{subject to} \quad (\nu, \psi) \in Q, \quad \mu \in \mathcal{M},
\]
where \( \mu \) denotes the RB parameter set, \( Q \) the RB space and \( \mathcal{M} \) the parameter set.

The constrained optimization problem (3) falls into the framework of saddle point problems. The assumptions of Brezzi theorem [1] can be easily verified [3] and therefore, for any \( \mu \in \mathcal{M} \), the optimal control problem has a unique solution \( (\nu^*, \psi^*) \) that can be determined by solving the following saddle-point problem (i.e. the optimality system):
\[
\begin{align*}
A(\nu^*, \psi^*) &= a(\nu^*, \psi^*; \mu) + b(\nu^*, \psi^*; \mu) = 0, \\
\nJ(\nu^*, \psi^*, \mu) &= \frac{1}{2} \int_{\Omega} (\nu^* - \nu^0)^2 + \int_{\Gamma} (\psi^* - \psi^0)^2 \ dS.
\end{align*}
\]

3. Reduced basis approximation
The RB method [5] gives an efficient way to compute an approximation to the FE truth solution \( (\nu^*, \psi^*) \) by considering only a small subspace of the FE space \( X \times U \). We thus take a suitably selected (by a greedy algorithm) set of parameter values \( \mu_1, \ldots, \mu_N \) (\( N \ll N \)) and the corresponding FE solutions \( (\nu_1, \psi_1), \ldots, (\nu_N, \psi_N) \). The reduced basis space is given by
\[
Y_R := \text{span}(\nu_1, \psi_1, \ldots, \nu_N, \psi_N), \quad n = 1, \ldots, N.
\]
while, in order to guarantee the stability of the RB approximation, we define the following aggregated reduced basis space for the state and adjoint variables
\[
Y_{R,\text{ag}} := \text{span}(\nu_1, \psi_1, \ldots, \nu_N, \psi_N, 0, \ldots, 0), \quad n = 1, \ldots, N.
\]
Let \( Y_R := X_R \times Y_R \) and \( Y_{R,\text{ag}} \) the reduced basis approximation reads:
\[
J(\nu_R, \psi_R, \mu) := \frac{1}{2} \int_{\Omega} (\nu_R - \nu^0)^2 + \int_{\Gamma} (\psi_R - \psi^0)^2 \ dS
\]
where \( \nu_1, \psi_1, \ldots, \nu_N, \psi_N \) are the reduced basis solutions.

4. Offline-Online decomposition

Algebraic formulation:
\[
\begin{align*}
A_N(\nu_R, \psi_R; \mu) :=& \sum_{n=1}^{N} \left[ a(\nu_1, \psi_1; \mu) \right]_{\nu_R}^{(n)} + \left[ b(\nu_1, \psi_1; \mu) \right]_{\psi_R}^{(n)} \ni V_R \ni \int_{\Omega} \psi_R \ dV=1, \quad n=1, \ldots, N. \end{align*}
\]

5. A posteriori error estimation
Revisiting the proof in the general Babuška framework [5, 4] we can provide an efficient and rigorous a posteriori error estimation on the solution variables:
\[
\|e(\nu_R)\|^2_{X,R} = J(\nu_R, \psi_R, \mu) - J(\nu, \psi, \mu) \leq \frac{C}{\gamma} \|\nu - \nu^0\|^2_{X,R} + \frac{C}{\gamma} \|\psi - \psi^0\|^2_{U,R}
\]
where \( C \) and \( \gamma \) are constants.

We consider the following optimal control problem
\[
\min \ J(\nu, \psi, \mu) := \frac{1}{2} \int_{\Omega} (\nu - \nu^0)^2 + \int_{\Gamma} (\psi - \psi^0)^2 \ dS \quad \text{subject to} \quad (\nu, \psi) \in Q, \quad \mu \in \mathcal{M},
\]

where \( \mu \) is the Lagrange multiplier associated to the constraint. Thanks to the affine parameter dependence assumption, an affine decomposition holds also for the bilinear and linear forms in (4).

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1. Suppose that the optimal control has a unique solution \( (\nu^*, \psi^*) \).

2. The reduced basis method is also referred to as the Reduced Basis Method (RBM) or the Reduced Basis (RB).

3. The RB approximation is also referred to as the Reduced Basis (RB) approximation or the Reduced Basis (RB) method.

4. The RB method is also referred to as the Reduced Basis (RB) method or the Reduced Basis (RB) method.

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6. The RB method is also referred to as the Reduced Basis (RB) method or the Reduced Basis (RB) method.

7. The RB method is also referred to as the Reduced Basis (RB) method or the Reduced Basis (RB) method.

References


