Approximation of stability factors for μPDEs by two-level corrections and surrogate strategies

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The Idea

Stability factors of differential operators control the well-posedness of the equations and the magnitude of many a posteriori error estimates to certify the accuracy and reliability of the reduced basis (RB) method for example, under the form

$$||u_h(\mu) - u_h^N(\mu)|| \le \frac{||R(u_h^N(\mu))||_{X_h'}}{\beta_h(\mu)},$$

where the "truth" FE solution $u_h(\mu)$ is approximated by the RB solution $u_h^N(\mu)$, being $R(u_h^N(\mu))$ a discrete residual and $\beta_h(\mu)$ a stability factor to be bounded from below.

The efficient approximation of stability factors is required for both ensuring the reliability of the RB solution (Online) and the effectiveness of the greedy procedure used to select the snapshots (Offline). The Successive Constraint Method (SCM, [1, 2]) fits the Offline/Online decomposition, but its convergence is quite slow in the case of nonaffine, noncoercive and/or nonlinear operators, even for rather modest parametric complexity or few parameters.

We thus propose both (i) a two-level algorithm based on **two nested coarse/fine approximated affine decompositions** and suitable corrections for the stability factors and (ii) some surrogate procedures based e.g. on **Radial Basis Functions interpolations** to face the complexity arising in noncoercive and/or nonaffine problems.

1. Two-level affine decompositions

Consider two different affine approximations to the general parametric bilinear form $a(\cdot,\cdot;\mu)$:

$$a(v,w;\mu) = a^c(v,w;\mu) + e^c(v,w;\mu) \qquad \text{coarse-level}$$

$$a(v,w;\mu) = a^f(v,w;\mu) + e^f(v,w;\mu) \qquad \text{fine-level}$$

being

$$a^c(v,w;\mu) = \sum_{q=1}^{Q_c} \Theta_q^c(\mu) a_q^c(v,w), \tag{1}$$

$$a^f(v,w;\mu) = a^c(v,w;\mu) + \sum_{q=Q_c+1}^{Q_f} \Theta_q^f(\mu) a_q^f(v,w), \tag{2}$$

We assume that the error terms are bounded, for all $\mu \in \mathcal{P}$, by

$$|e^c(v,w;\mu)| \le \delta_c \|v\|\|w\|, \qquad |e^f(v,w;\mu)| \le \delta_f \|v\|\|w\|$$

and denote

$$\beta_c(\mu) = \inf_{v\in X_h} \sup_{w\in X_h} \frac{a^c(v,w;\mu)}{||v||_X||w||_X}, \quad \beta_f(\mu) = \inf_{v\in X_h} \sup_{w\in X_h} \frac{a^f(v,w;\mu)}{||v||_X||w||_X}.$$

By choosing δ_f very small, $\beta_f(\mu)$ also acts as a good estimate for the true discrete stability factor $\beta_h(\mu)$, but it may be hard to compute, since **SCM computational cost depends highly on Q_f**.

Idea: use the coarse-level as a surrogate for computing bounds for the stability factor bound, since

$$\beta_f(\mu) \ge \beta_c(\mu) + \varepsilon_{cf}(\mu)$$

being the correction term

$$\varepsilon_{cf}(\mu) := \inf_{v\in X_h} \frac{\sum_{q=Q_c+1}^{Q_f} \Theta_q^f(\mu) a_q^f(v, T_c^\mu v)}{||v||_X||T_c^\mu v||_X} \tag{3}$$

and $T_c^\mu : X_h \to X_h$ the coarse-level supremizer operator, such that $(T_c^\mu v, w)_X = a^c(v,w;\mu)\ \forall w \in X_h$. We propose [3] the following:

Two-level coarse-fine algorithm

1. Choose δ_f very small so that the fine-level affine problem is "indistinguishable" from the true nonaffine problem (Q_f very large).
2. Compute affine expansions (2) for any desired tolerance $\delta_f > 0$, e.g. by the Empirical Interpolation Method.
3. Choose $\delta_c \gg \delta_f$ such that the coarse-level affine problem has a manageable number of terms (Q_c small).
4. Perform the SCM on the coarse-level problem to obtain a **coarse lower bound** $\beta_c^{\mathrm{LB}}(\mu)$ and evaluate the **correction factor** $\varepsilon_{cf}(\mu)$ to obtain a **fine lower bound** $\beta_f^{\mathrm{LB}}(\mu)$ also for $\beta_f(\mu)$.

Goal: find an inexpensive approximation to $\tilde{\varepsilon}_{cf}(\mu)$, still giving reasonable lower bounds for the stability factor $\beta_f(\mu)$.

2. Coarse-level bounds for coercive PDEs

Let us consider the following *model* problem: find $u_h \in X_h$ s.t.

$$a(u_h, v_h; \nu(\mu,\mathbf{x})) = F(v_h;\mu) \quad \forall v_h \in X_h, \tag{4}$$

being $a(\cdot,\cdot;\nu)$ a μ-dependent, continuous, bilinear form:

$$a(u,v;\nu(\mu,\mathbf{x})) := \int_\Omega \nu(\mu;\mathbf{x})\nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})\, d\Omega;$$

s.t. $\nu(\mu;\mathbf{x}) \ge \nu_0 > 0$ for all $\mathbf{x}\in\Omega$, $\mu\in\mathcal{P}$ (uniform coercivity). In this case $T_c^\mu \equiv I$ (identity operator) and the stability factor simplifies to

$$\inf_{v_h\in X_h} \sup_{w_h\in X_h} \frac{a(v_h,w_h;\nu(\mu,\mathbf{x}))}{\|v_h\|_X\|w_h\|_X} = \inf_{v_h\in X} \frac{a(v_h,v_h;\nu(\mu,\mathbf{x}))}{\|v_h\|_X^2} =: \alpha(\nu(\mu)),$$

Coarse/fine-level approximations of $a(u,v;\nu(\mu,\mathbf{x}))$ are given by:

$$a^c(u,v;\mu) := a(u,v;\nu_c(\mu,\mathbf{x})), \quad a^f(u,v;\mu) := a(u,v;\nu_f(\mu,\mathbf{x})),$$

being $\nu_c = \nu_c(\mu,\mathbf{x})$ and $\nu_f = \nu_f(\mu,\mathbf{x})$ the coarse- and fine-level approximation of the parametrized tensor $\nu(\mu;\mathbf{x})$. In the same way:

$$\varepsilon_{cf}(\mu) = \inf_{v\in X_h} \frac{\sum_{q=Q_c+1}^{Q_f} \Theta_q^f(\mu) a_q^f(v,v)}{\|v\|_X^2}. \tag{5}$$

We can propose three different estimators for (5):

- **Constant correction (CC)**

$$\varepsilon_{cf}^{\mathrm{CC}}(\mu) := -\gamma C_a(\delta_c + \delta_f)$$

- **Global infimum (GI)**

$$\varepsilon_{cf}^{\mathrm{GI}}(\mu) := \gamma C_a \inf_{\mathbf{x}\in\Omega} \{\nu_f(\mu,\mathbf{x}) - \nu_c(\mu,\mathbf{x})\}$$

- **One-point correction (OP)**

$$\varepsilon_{cf}^{\mathrm{OP}}(\mu) := -\gamma C_a \left[\delta_f + \hat{\varepsilon}_{EIM}^c(\mu)\right]$$

where

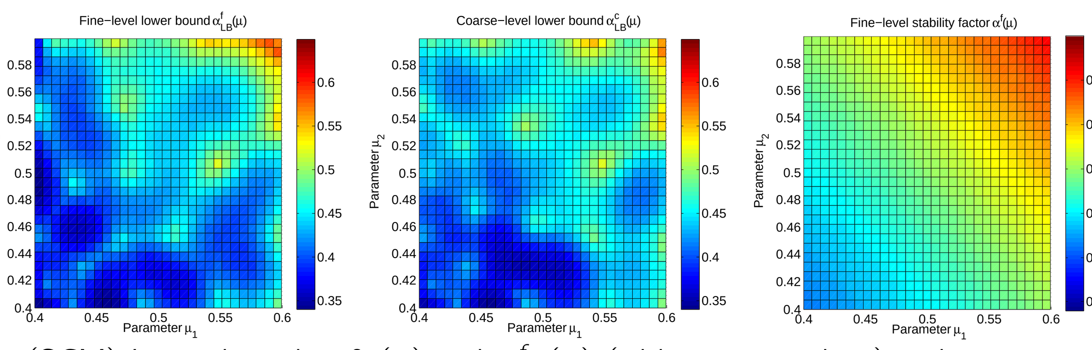
$$0 < C_a = \sup_{v\in X_h} \frac{a(v,v;1)}{\|v\|_X^2} < 1, \qquad \gamma \in [0,1]$$

and $\hat{\varepsilon}_{EIM}^c(\mu)$ is a measure of the error committed during the EIM.

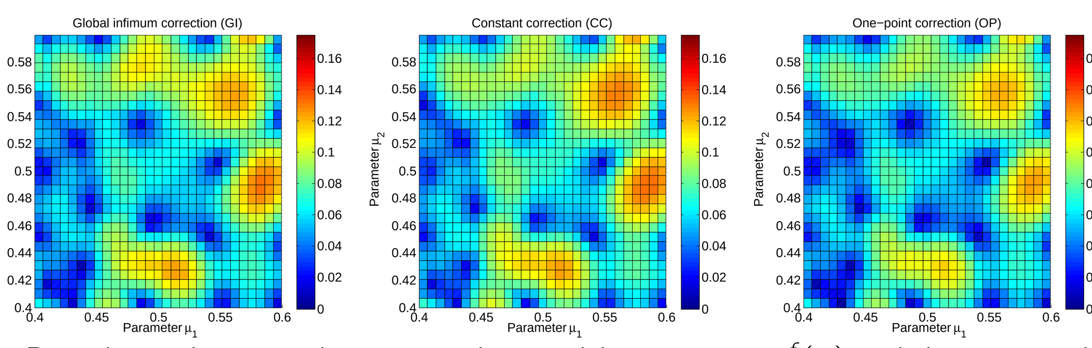
Example 1 : Poisson equation with $\mu \in \mathcal{P} := [0.4, 0.6]^2$, being

$$\nu(\mu,x) := \exp(\mu_1+\mu_2)\left[1 + \exp\left(-((x_1-\mu_1)^2 + (x_2-\mu_2)^2)/0.02\right)\right];$$

the number of terms in the EIM expansion grows rapidly w.r.t. the index q. We always obtain a positive lower bound for the coercivity constant, even using (OP), which is not a priori guaranteed to be rigorous. (CC) naturally gives always the largest bound gap, while (GI) gives the tightest bounds:



(SCM) Lower bounds $\alpha_{\mathrm{LB}}^c(\mu)$ and $\alpha_{\mathrm{LB}}^f(\mu)$ (without corrections) and true parametric coercivity constant $\alpha^f(\mu)$.



Bound gaps between the parametric coercivity factor $\alpha^f(\mu)$ and the corrected lower bounds for $\alpha_{\mathrm{LB}}^c(\mu)$ using the three methods (GI, CC, OP).

Computed efficiencies of the fine lower bound, the uncorrected and corrected coarse lower bounds over a range of 500 parameter points in $\mu \in \mathcal{P} := [0.4, 0.6]^2$, for $\delta_c = 10^{-2}$ and $Q_c = 20$, are as follows:

	effectivity $\alpha_f/\alpha_*^{\mathrm{LB}}$		
	min	avg	max
Coarse	1.0087	1.1587	1.3459
Fine	1.0088	1.1625	1.3350
Coarse + (CC) correction	1.0285	1.1862	1.3769
Coarse + (GI) correction	1.0184	1.1793	1.3818
Coarse + (OP) correction	0.9894	1.1588	1.3475

A speedup factor of 30% in the offline SCM step is observed while still obtaining effective stability factor lower bounds:

	tol δ	Q	SCM iter	μ^* points	CPU time (s)
Fine level	$5\cdot10^{-6}$	59	20	20	2 741
Coarse level	$1\cdot10^{-2}$	20	19	19	1 963

Offline computational complexity of the SCM in the coarse and fine level.

3. Coarse-level bounds for noncoercive PDEs

Let us consider the model problem (4) being now

$$a(u,v;\nu(\mu)) := \int_\Omega \left[\nu(\mathbf{x},\mu)\nabla u\cdot\nabla v - \omega^2 uv\right] d\Omega$$

a noncoercive, continuous bilinear form. We assume that the parametric dependence acts only on the elliptic part of the operator, which allows us to estimate (5) as

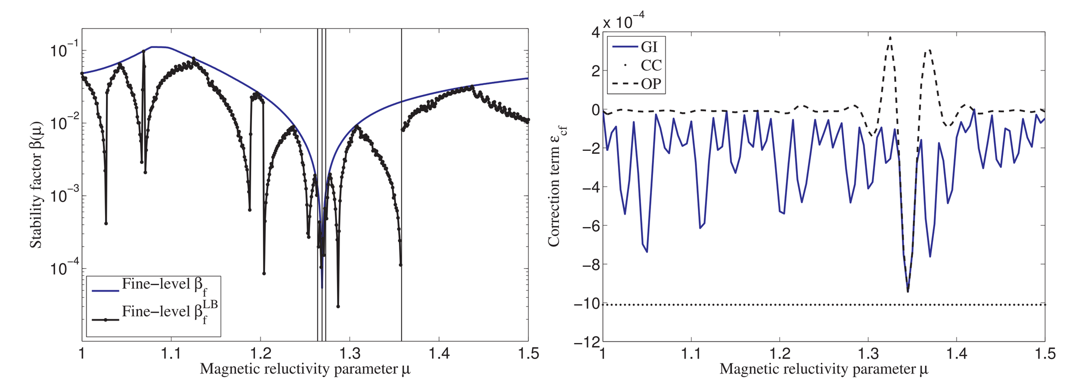
$$\varepsilon_{cf}(\mu) = \ge \gamma C_a \inf_{\mathbf{x}\in\Omega}\{\nu_f(\mu,\mathbf{x}) - \nu_c(\mu,\mathbf{x})\}$$

and thus obtain the same estimators for $\varepsilon_{cf}(\mu)$ as in the coercive case. However, since in practice $\beta_f(\mu) \ll 1$ and even negative for some finite number of parameter points μ, correction terms will be much more sensitive to the choice of δ_c.

Example 2: Parametrized Helmholtz equation modelling the reflection of time-harmonic waves on a stealth aircraft wing profile. The magnetic reluctivity ν is $\nu(\mu;z) = 0.9\cdot H_\varepsilon(|\kappa^{-1}(z)|;\mu) + 0.1$, whereas $\kappa : \mathbb{C}\to\mathbb{C}$ is a Kármán-Trefftz map and the smoothed radial step function H_ε is defined as

$$H_\varepsilon(r) := \begin{cases} 0, & r \le \mu - \varepsilon \\ \frac{1}{2}\left[1 + \frac{r-\mu}{\varepsilon} + \frac{1}{\pi}\sin\frac{\pi(r-\mu)}{\varepsilon}\right], & \mu-\varepsilon < r < \mu+\varepsilon \\ 1, & r \ge \mu+\varepsilon \end{cases}$$

with constant $\varepsilon = 0.1$; for $\mu \in [1, 1.5]$ it models an absorbing coat of paint on the surface of the airfoil of thickness $\mu - 1$. The location of the resonance frequencies depends on μ in quite a complicated way; for $\omega = 2.5$ we expect to find only one resonance point, near $\mu \approx 1.27$. By using (CC) a large number of sample points do not have a positive lower bound. The best correction is (OP), which adds no failed points – i.e. parameter points where a positive lower bound was not obtained – while remaining both reliable and more effective than (GI) with a cheaper Online evaluation cost:



Left: stability constant $\beta_f(\mu)$ and lower bound for the fine-level approximation with δ_f = 1e-5. Right: the three correction terms $\varepsilon_{cf}^{\mathrm{GI}}$, $\varepsilon_{cf}^{\mathrm{CC}}$, and $\varepsilon_{cf}^{\mathrm{OP}}$.

Computed efficiencies of the fine lower bound, the uncorrected and corrected coarse lower bounds over a range of 500 parameter points in $\mu \in [1, 1.5]$, for δ_c =1e-3 and Q_c = 27, are as follows:

	effectivity $\beta_f/\beta_*^{\mathrm{LB}}$			# failed
	min	avg	max	points
Coarse	1.0001	4.47	177.17	7
Fine	1.0000	5.54	271.01	4
Coarse + (CC)	1.0133	11.1	2698.1	39
Coarse + (GI)	1.0007	9.75	773.83	9
Coarse + (OP)	1.0003	9.34	972.17	7

Coarse-level procedure gives a CPU Offline time reduction of 60%:

	tol δ	Q	SCM iter.	μ^* points	CPU time (s)
Fine level	10^{-5}	119	297	16	14 567
Coarse level	10^{-3}	27	297	17	5 816

Offline computational complexity of the successive constraint method

4. A surrogate strategy for nonlinear PDEs

Even in the simpler affine case, estimating lower bounds of stability factors for nonlinear operators is a challenging computational task, since SCM convergence is even slower due to the additional eigenproblems arising from operator linearization. As in the linear case [1], the SCM interpolates $\beta_h(\mu)$ for each selected point $\mu^* \in \mathcal{S}$, i.e. $\beta_h(\mu^*) = \beta_h^{LB}(\mu^*)\ \forall \mu^* \in \mathcal{S}$, we set up the following procedure to compute a surrogate lower bound [4] through interpolation by using the Radial Basis Function (RBF) technique (e.g. *thin-plate splines*):

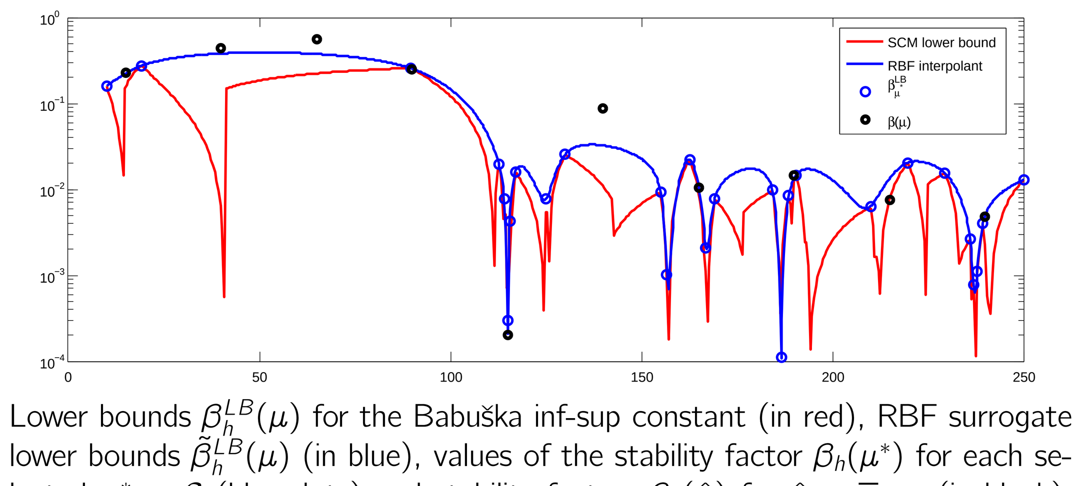
Surrogate lower bound algorithm

1.a Select a trial set $\Xi_{trial} \subset \mathcal{P}$ and compute the stability factors $\beta_h(\mu)$ (by solving an eigenproblem) for each $\mu \in \Xi_{trial}$.

1.b Alternatively, run n_{trial} initial steps of the SCM procedure.

2. Evaluate the RBF interpolant $\tilde{\beta}_h^{LB}(\mu)$.
3. In case, refine the interpolant, by taking a test set $\Xi_{test} \ne \Xi_{trial}$. If $\tilde{\beta}_h^{LB}(\mu)\beta_h(\mu) < 0$ for $\mu \in \Xi_{test}$, retain μ: $\Xi_{trial} \to \Xi_{trial} \cup \mu$.

Example 3: Navier-Stokes equations for a backward-facing step flow parametrized w.r.t. the Reynolds number μ = Re. We compute the surrogate by using the points $\mathcal{S} = \{\boldsymbol{\mu}^{j*}, j = 1,\dots,J\}$ selected through the SCM-greedy procedure.. Although the surrogate $\tilde{\beta}_h^{LB}(\mu)$ fails in at least one case, it is a good global surrogate of the SCM lower bound $\beta_h^{LB}(\mu)$, avoiding the lower peaks shown by SCM outputs and due to the imposition of the successive constraints.



Lower bounds $\beta_h^{LB}(\mu)$ for the Babuška inf-sup constant (in red), RBF surrogate lower bounds $\tilde{\beta}_h^{LB}(\mu)$ (in blue), values of the stability factor $\beta_h(\mu^*)$ for each selected $\mu^* \in \mathcal{S}$ (blue dots) and stability factors $\beta_h(\hat{\mu})$ for $\hat{\mu} \in \Xi_{test}$ (in black).

Nevertheless, the RBF surrogate (as well as constant lower bounds) might fail e.g. for separated flows also at moderate Reynolds number, because of possible undergoing bifurcation phenomena.

References

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