A reduced computational framework for optimal control problems

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Abstract

Solving optimal control problems for many different scenarios obtained by varying a set of parameters in the state system is a computationally extensive task. We present a reduced framework for the numerical solution of parametrized PDE-constrained optimization problems. The proposed framework is based on the greedy algorithm and the reduced basis method for optimal control problems and exploits the reduced basis method, leading to a relevant computational reduction with respect to traditional discretization techniques such as the finite element method. The setting is applied to the solution of two problems arising from haemodynamics, dealing with both data reconstruction and data assimilation over domains of variable shape (for which a further geometrical reduction is pursued), which can be recast in a common PDE-constrained optimization formulation.

1. Problem definition and saddle point formulation

Let $\Omega \subset \mathbb{R}^d$ be a spatial domain and $\mathcal{D} \subset \mathbb{R}^d$ a $d$-dimensional parameter set. Let $\mathcal{Y}, \mathcal{D}$ be two Hilbert spaces for the state and control variables $y$ and $u$ respectively, and $\mathcal{Z}$ shall denote the observation space. We consider the case of a quadratic cost functional to be minimized

$$J(y, u; \mu) = \frac{1}{2}(y - y_0)(y - y_0) + \frac{1}{2}(u - u_0)(u - u_0), \quad (1)$$

where $a > 0$ is a given constant and $y_0(\mu), u_0(\mu) \in \mathcal{Z}$ are given parameter-dependent observation functions. Let $\mathcal{D} \ni \mu$ denote the test space, we define the linear constraint equation

$$B(y, u; \mu) = (G(y), \varphi) \in \mathcal{Q} \subset \mathbb{R}, \quad (2)$$

where the bilinear form $B(\cdot, \cdot)$: $\mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}$ is given by the sum of two contributions

$$B(y, u; \mu) = (y, -\mu) + (u, \varphi); \mu \in \mathcal{D} \quad (3)$$

the (weakly) coercive bilinear form $a(\cdot, \cdot)$ represents a linear elliptic operator while the bilinear form $b(\cdot, \cdot)$ expresses the action of the control.

The parametrized optimal control problem (OCP) reads for any $\mu \in \mathcal{D}$

$$\min_{y, u \in \mathcal{Y} \times \mathcal{D}} J(y, u; \mu) \quad s.t. \quad B(y, u; \mu) = (G(y), \varphi) \in \mathcal{Q} \subset \mathbb{R}. \quad (4)$$

In order to formulate the optimal control problem as a saddle-point problem, we first define the product space $\mathcal{X} = \mathcal{Y} \times \mathcal{D}$ and denote with $x = (y, u) \in \mathcal{X}$, $w = (z, v) \in \mathcal{X}$ its variables. We can reformulate the OCP as given $\mu \in \mathcal{D}$

$$\min_{x \in \mathcal{X}} J(x; \mu) = \frac{1}{2}A(x, x; \mu) + B(x; \mu), \quad (x, \mu) \in \mathcal{X} \times \mathcal{D} \quad (5)$$

where $F(\cdot, \cdot)$ is the linear constraint equation $B(\cdot, \mu) = (G(\cdot), \varphi)$ in $Q \subset \mathbb{R}$, $\mathcal{X} = \mathcal{Y} \times \mathcal{D}$.

The constrained optimization problem (3) falls into the framework of saddle-point problems. The assumptions of Brezzi theorem can be easily verified [4] and therefore, for any $\mu \in \mathcal{D}$, the optimal control problem has a unique solution $x(\mu) \in \mathcal{X}$. This can be solved by following the saddle-point problem (i.e. the optimality system):

$$\begin{cases}
\min_{y, u \in \mathcal{X}} J(y, u; \mu) = \frac{1}{2}A(y, y; \mu) + B(y; \mu), \quad s.t. \quad B(y, u; \mu) = (G(y), \varphi) \in \mathcal{Q}, \quad (y, u) \in \mathcal{X},
\end{cases}$$

where $A(\cdot, \cdot)$ is a saddle point bilinear form $\mathcal{X} \times \mathcal{D} \rightarrow \mathbb{R}$ and $B(\cdot, \cdot)$ is the action of the control $\mathcal{D} \rightarrow \mathbb{R}$.

The RB methodology has been extended to treat OCPs with Stokes constraints. The stability of the RB approximation can be fulfilled by introducing suitable supenmer operators [5] and by defining suitable aggregated spaces for state and adjoint variables [6]. As a possible application, we consider an inverse boundary problem in haemodynamics (inspired by the work in [6]) where the state equation models the blood flow (supposed to obey the Stokes equations) in a paramterized arterial bifurcation and we suppose to have a measured velocity profile on a transverse section, but not the Neumann flux on $\Gamma_C$ that will be our control variable.

2. Reduced basis approximation

The RB method gives an efficient way to compute an approximation to the FE true solution $(y(\mu), u(\mu))$ by considering only a small subspace of the FE space $X_0 \times Q_0$. We thus take a suitably selected (by a greedy algorithm) set of parameter values $\mu_1, \ldots, \mu_N$ ($N \ll M$) and the corresponding FE solutions $(y(\mu_1), u(\mu_1)), \ldots, (y(\mu_N), u(\mu_N))$. The reduced basis control space is defined by

$$U_0 = \text{span}(u(\mu_1), \ldots, u(\mu_N)) \quad n = 1, \ldots, N.$$