Reduced order methods for optimal flow control problems

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Abstract

We propose a reduced basis (RB) framework for the numerical solution of optimal control problems for parametrized viscous incompressible flows. We mainly focus on control problems for the (Navier-) Stokes equations involving infinite-dimensional control functions, thus requiring a suitable reduction of the whole optimization problem, rather than of the sole state equation. The method is applied to the solution of a data assimilation problem arising in haemodynamics and a problem of vorticity minimization through suction/injection of fluid.

1. Problem definition

We consider the following parametrized optimal control problem:

given $\boldsymbol{\mu} \in \mathcal{D}$, find a triple $(\mathbf{v}, \pi, \mathbf{u})$ such that the cost functional

$$\mathcal{J}(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu}) + \mathcal{G}(\mathbf{u}; \boldsymbol{\mu}) \qquad (1)$$

is minimized subject to the steady (Navier-)Stokes equations:

$$\begin{aligned} -\nu\Delta\mathbf{v} + \delta(\mathbf{v}\cdot\nabla)\mathbf{v} + \pi &= \mathbf{0} && \text{in } \Omega_o(\boldsymbol{\mu}) \\ \operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega_o(\boldsymbol{\mu}) \\ \mathbf{v} &= \rho_1\mathbf{u}_1 && \text{on } \Gamma_D^o(\boldsymbol{\mu}) \\ -\pi\mathbf{n} + \nu\nabla\mathbf{v}\cdot\mathbf{n} &= \rho_2\mathbf{u}_2 && \text{on } \Gamma_N^o(\boldsymbol{\mu}). \end{aligned} \qquad (2)$$

Here $\Omega_o(\boldsymbol{\mu}) \subset \mathbb{R}^2$ is a parametrized domain with boundary $\partial\Omega_o = \Gamma_D^o \cup \Gamma_N^o$, being Γ_D^o the Dirichlet portion of the boundary and Γ_N^o the Neumann portion. The PDE constraint is given by the Navier-Stokes (Stokes) equations if $\delta = 1$ ($\delta = 0$); the state variables $\mathbf{v}$ and π denote the velocity and pressure fields, respectively. The boolean variables ρ_i satisfy $\rho_1 + \rho_2 = 1$, so that the control variable $\mathbf{u}$ may represent either a boundary velocity on Γ_D^o ($\mathbf{u}_1$) or a Neumann flux on Γ_N^o ($\mathbf{u}_2$). $\mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu})$ represents the objective to be minimized, while $\mathcal{G}(\mathbf{u}; \boldsymbol{\mu})$ is a regularization term ensuring the well-posedness of the problem. Possible choices for $\mathcal{F}$ are the viscous energy dissipation and vorticity type functionals or velocity tracking type functionals.

The optimal control problem (1)-(2) can be formulated in the following general form [2]: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{x \in X_o} \mathcal{J}_o(x; \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{E}_o(x) = 0 \text{ in } X_o', \qquad (3)$$

where X_o and Q_o are two Hilbert spaces, $x = (\mathbf{v}, \pi, \mathbf{u}) \in X_o$ is the optimization variable and the operator $\mathcal{E}_o\colon X_o \to Q_o'$ describes the state equation. We assume that the original domain $\Omega_o(\boldsymbol{\mu})$ can be obtained as the image of a reference (parameter independent) domain $\Omega = \Omega_o(\boldsymbol{\mu}_{\text{ref}})$ through a (affine or nonaffine) parametrized mapping $T(\cdot; \boldsymbol{\mu})\colon \mathbb{R}^2 \times \mathcal{D} \to \mathbb{R}^2$. The parametrized formulation of the optimal control problem can be derived by tracing (3) back on the reference domain Ω [6]:

$$\text{given } \boldsymbol{\mu} \in \mathcal{D}, \qquad \min_{x \in X} \mathcal{J}(x; \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{E}(x; \boldsymbol{\mu}) = 0, \qquad (4)$$

where the cost functional and the state operator are linked to the "original" ones through the mapping $T(\cdot; \boldsymbol{\mu})$.

By introducing the Lagrangian functional $\mathcal{L}(x, p; \boldsymbol{\mu}) = \mathcal{J}(x; \boldsymbol{\mu}) + \langle \mathcal{E}(x; \boldsymbol{\mu}), p \rangle$, we can derive the first order necessary optimality conditions:

$$\begin{cases} \mathcal{J}_x(x; \boldsymbol{\mu}) + \mathcal{E}_x(x; \boldsymbol{\mu})^* p = 0, & \text{in } X' \\ \mathcal{E}(x; \boldsymbol{\mu}) = 0, & \text{in } Q', \end{cases} \qquad (5)$$

being $p(\boldsymbol{\mu}) \in Q$ the Lagrange multiplier (adjoint variable) associated to the constraint.

2. Reduced basis approximation

The RB method gives an efficient way to compute an approximation to the FE *truth* solution $(x_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu}))$ by considering only a small subspace of the FE space $X_h \times Q_h$. We thus take a suitably selected (by a greedy algorithm) set of parameter values $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N$ ($N \ll N_h$) and the corresponding FE solutions $(x_h(\boldsymbol{\mu}^1), p_h(\boldsymbol{\mu}^1)), \dots, (x_h(\boldsymbol{\mu}^N), p_h(\boldsymbol{\mu}^N))$.

The stability of the RB approximation can be fulfilled by introducing suitable supremizer solutions [5] and by defining suitable *aggregated spaces* for the state and adjoint variables [6]. The reduced basis approximation reads:

given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{J}_x(x_N(\boldsymbol{\mu}); \boldsymbol{\mu}) + \mathcal{E}_x(x_N(\boldsymbol{\mu}); \boldsymbol{\mu})^* p_N(\boldsymbol{\mu}) = 0, & \text{in } X_N' \\ \mathcal{E}(x_N(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0, & \text{in } Q_N', \end{cases}$$

3. Stokes constraint: main ingredients

Since the cost functional is quadratic and the state equation is linear, the optimality system is linear and features a saddle-point structure. At the algebraic level we obtain the linear system

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}}_{\mathbf{f}_N(\boldsymbol{\mu})}. \qquad (6)$$

Thanks to the affine assumption, we can write

$$K_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_k} \Theta_k^q(\boldsymbol{\mu}) K_N^q, \qquad \mathbf{f}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) f_N^q,$$

where K_N^q and f_N^q are $\boldsymbol{\mu}$-independent, and we can therefore provide the usual Offline-Online computational decomposition.

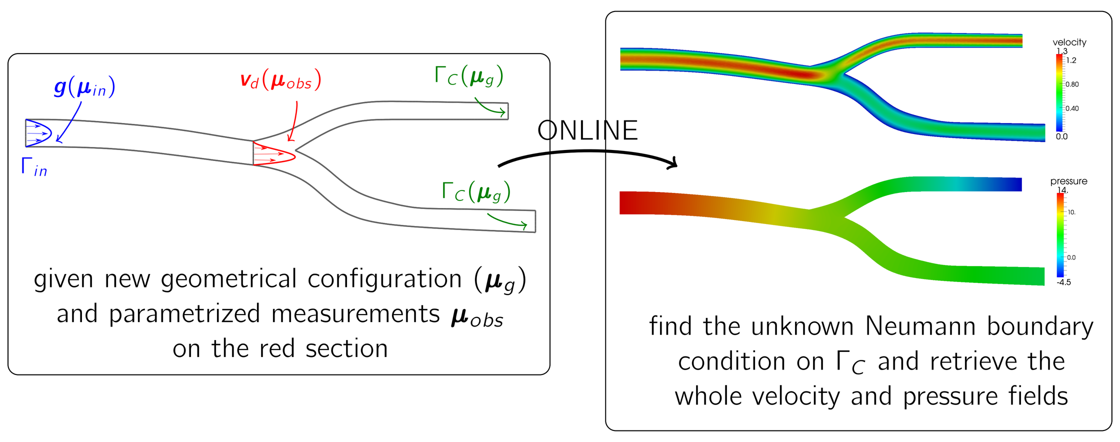
Recasting the problem in the general Babuška framework [5] we can provide an efficient and rigorous a posteriori error estimate on the solution variables (as well as on the cost functional [4]):

$$\|(x_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu})) - (x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))\|_{X \times Q} \leq \frac{\|r(\cdot; \boldsymbol{\mu})\|}{\beta_{LB}(\boldsymbol{\mu})} = \Delta_N(\boldsymbol{\mu})$$

where $0 < \beta_{LB}(\boldsymbol{\mu}) \leq \beta_h(\boldsymbol{\mu})$ is a *lower bound* of the inf-sup constant of the optimality system.

4. A boundary inverse problem

We consider an inverse boundary problem in haemodynamics (inspired by the work in [1]) where the state equation models the blood flow (supposed to obey the Stokes equations) in a parametrized arterial bifurcation and we suppose to have a measured velocity profile on the a transverse section, but not the Neumann flux on Γ_C that will be our control variable.

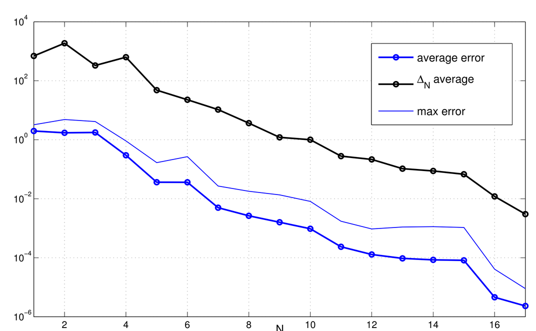


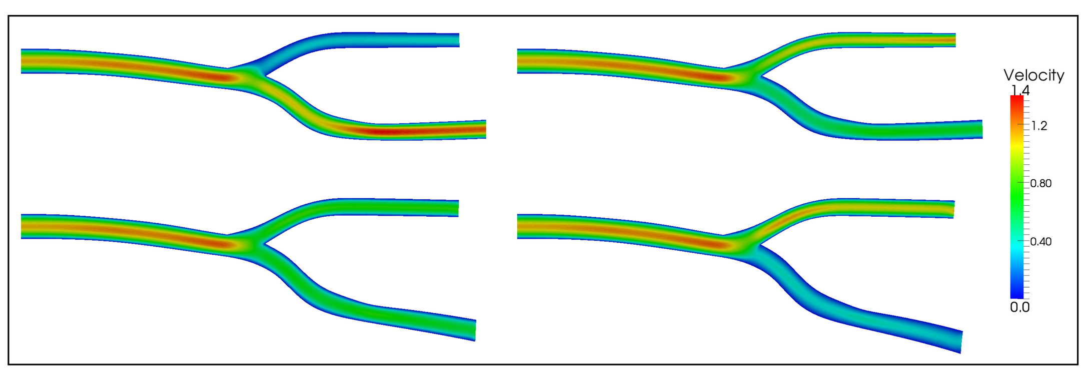
With respect to the formulation (1)-(2) we consider a Neumann control (thus $\rho_2 = 1$) and the following cost functional

$$\mathcal{J}(\mathbf{v}, \pi; \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2}\int_{\Gamma_{obs}} |\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}_{obs})|^2 \, d\Gamma + \frac{\alpha_1}{2}\int_{\Gamma_C} |\nabla\mathbf{u}|^2 d\Gamma + \frac{\alpha_2}{2}\int_{\Gamma_C} |\mathbf{u}|^2 d\Gamma.$$

- Geometrical parametrization: Free Form Deformation the geometrical parameter μ_g is related to the angle of rotation of the lower branch.
- Parametrized measured velocity profiles:

Number of FE dof N_h	$4 \cdot 10^4$
Number of parameters P	3
Number of RB functions N	17
Affine components Q_k	20
Linear system size red.	150:1





Examples of reconstructed velocity fields

5. Navier-Stokes constraint: main ingredients

In this case, the optimality system (5) forms a non-linear system to which we can apply Newton's method: for $k = 1, 2, \dots$, seek $(s_x^k, s_p^k) \in X \times Q$ such that

$$\begin{cases} \mathcal{L}_{xx}(x^k, p^k; \boldsymbol{\mu}) s_x^k + \mathcal{E}_x(x^k; \boldsymbol{\mu})^* s_p^k = -\mathcal{L}_x(x^k, p^k; \boldsymbol{\mu}), \\ \mathcal{E}(x^k; \boldsymbol{\mu}) s_x^k = -\mathcal{E}(x^k; \boldsymbol{\mu}), \end{cases}$$

and then set $(x^{k+1}, p^{k+1}) = (x^k, p^k) + (s_x^k, s_p^k)$.

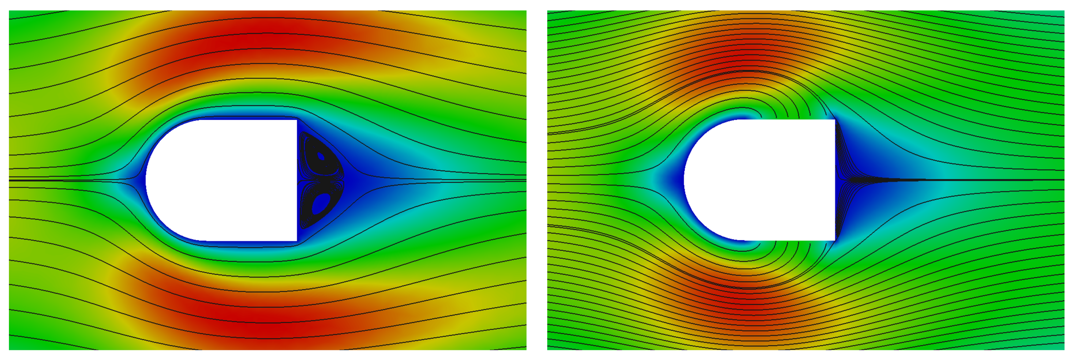
The main ingredients of our reduction strategy are the following:

- in the Online stage we solve the reduced optimization problem by first projecting the optimality system on the RB spaces and then applying Newton's method;
- the Offline/Online computational stratagem is preserved thanks to the quadratic nature of the nonlinear terms appearing in both first and second order derivatives;
- in order to derive an posteriori error estimate on the state, adjoint and control variables, we apply the Brezzi-Rappaz-Raviart theory on the optimality system.

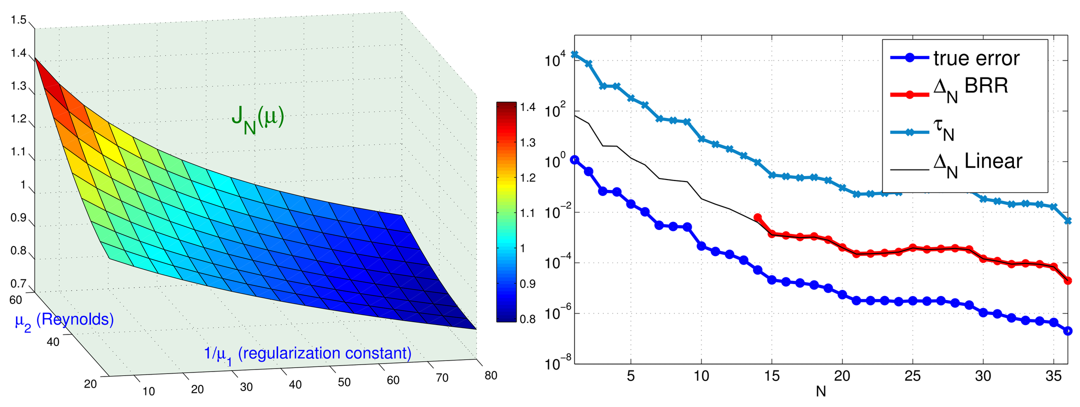
6. Vorticity minimization past a bluff body

As a preliminary benchmark test, we consider the problem of vorticity minimization through suction/injection of fluid in the downstream portion of a bluff body embedded in a 2-D flow. In this case we do not consider a geometrical parametrization, yet we take as parameters the Reynolds number and the regularization constant in the cost functional. In the figures below we show the convergence of the RB approximation, a comparison between an uncontrolled and a controlled flow, and the value of the cost functional with respect to the parameters.

Number of parameters P	2
FE evaluation t_{FE} (s)	≈ 60
RB evaluation t_{RB}^{online} (s)	0.9
Number of RB functions N	36



Uncontrolled (left) vs controlled (right) flow.



Cost functional $\mathcal{J}(\boldsymbol{\mu})$ (left) and convergence (right).

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