Isogeometric analysis based reduced order modelling for incompressible viscous flows in parametrized domains: applications to underwater shape design

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Introduction

Development of a new framework for the shape optimization in viscous flows, obtained by coupling advanced numerical techniques:

- Isogeometric viscous solver (IGA Stokes) [3]
- Shape deformation description (FFD) [2]
- Reduced order models (POD) [4]

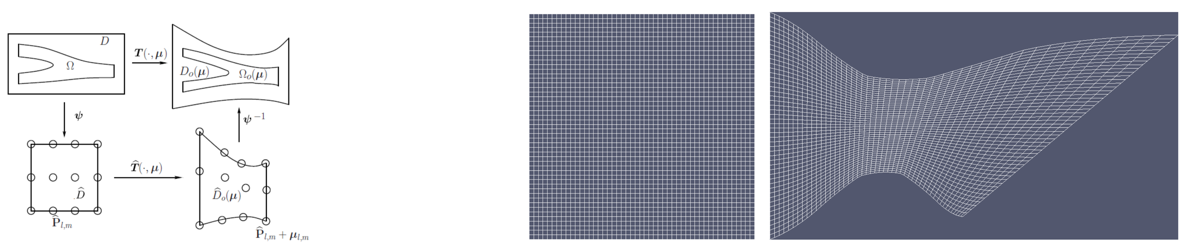
Methodology

IGA Isogeometric analysis is a very popular techniques in industrial field for CAD design. It allows to describe the geometry in an exact way (i.e. no mesh errors). The idea behind isogeometric analysis is to use the same basis functions $\phi_i(s)$ for the geometry description and the solution of the problem: $\boldsymbol{c}(s;\boldsymbol{\mu}) = \sum_{i=1}^{\mathcal{N}} \phi_i(s)\boldsymbol{P}_i(\boldsymbol{\mu})$, $\boldsymbol{P}_i$ control points. In such a way we provide a formulation of the problem on a reference domain (also necessary for the computational reduction).

FFD We have too many design parameters $\boldsymbol{P}_i$ to handle: for 2D $2\times\mathcal{N}$. We adopt a shape parametrization based on FFD for efficient geometrical reduction, defined as

$$\boldsymbol{P} = \boldsymbol{P}_0 + \mathcal{D}\sum_{l=0}^{L}\sum_{m=0}^{M} \mathsf{b}_{lm}(\boldsymbol{\psi}(\boldsymbol{P}_0))\boldsymbol{\mu}_{lm}$$

where b_{lm} are Bernstein polynomials and $\boldsymbol{\mu}_{lm}$ are the displacements of selected (few) FFD control points. Now the number of parameters is only $O(M\times L)$.



HF Stokes equations: $\begin{cases} -\nu\nabla^2\boldsymbol{u} + \nabla p = 0 \\ \nabla\cdot\boldsymbol{u} = 0 \end{cases}$ in Ω

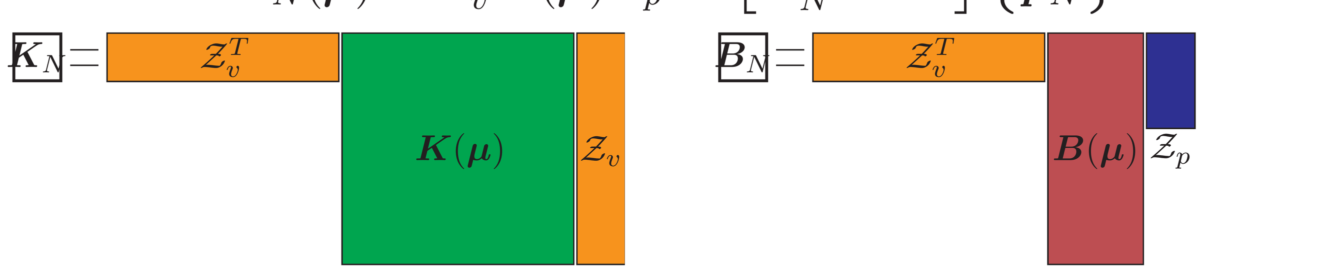
Variational formulation $\begin{cases} a(\boldsymbol{u},\boldsymbol{v};\boldsymbol{\mu}) + b(p,\boldsymbol{v};\boldsymbol{\mu}) = 0 & \forall v \in V_v \\ b(q,\boldsymbol{u};\boldsymbol{\mu}) = 0 & \forall q \in V_q \\ +\text{B.C.} & \end{cases}$

POD transforms the original variables into uncorrelated variables (POD modes). The modes are sorted by decreasing energy content. The steps necessary for the basis construction are:

1. Building the snapshots matrix $\mathcal{U} = [\boldsymbol{u}(\boldsymbol{\mu}_1), \cdots, \boldsymbol{u}(\boldsymbol{\mu}_n)]$,
2. Singular value decomposition of $\mathcal{U}$: $\mathcal{V}^T\mathcal{U}\mathcal{W} = \Sigma$,
3. From the columns of $\mathcal{W}$ we extract the basis matrix $\mathcal{Z}_v$, where each column is a reduced basis function (same procedure for $\mathcal{Z}_p$),
4. From Σ we extract the energy content $I(N) = \frac{\sum_{i=1}^{N}\sigma_i^2}{\sum_{i=1}^{\mathcal{N}}\sigma_i^2}$.

If we use the reduced basis functions $\mathcal{Z}_v = [\boldsymbol{\zeta}_1, \cdots, \boldsymbol{\zeta}_N] \in \mathbb{R}^{\mathcal{N}_v\times N}$; $\mathcal{Z}_p = [\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_N] \in \mathbb{R}^{\mathcal{N}_p\times N}$ in the variational formulation, we end up with the reduced system[a]:

$$\begin{matrix} \boldsymbol{K}_N(\boldsymbol{\mu}) = \mathcal{Z}_v^T\boldsymbol{K}(\boldsymbol{\mu})\mathcal{Z}_v \\ \boldsymbol{B}_N(\boldsymbol{\mu}) = \mathcal{Z}_v^T\boldsymbol{B}(\boldsymbol{\mu})\mathcal{Z}_p \end{matrix} ; \quad \begin{bmatrix} \boldsymbol{K}_N & \boldsymbol{B}_N \\ \boldsymbol{B}_N^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_N \\ \boldsymbol{p}_N \end{Bmatrix} = 0$$



[a] See section ROM-FFD-IGA-Stokes results

Computational details

IGA-Stokes space dimension $\mathcal{N}_v + \mathcal{N}_p$	1458 + 196
Number of FFD parameters	6
POD space dimension $N_v + N_p + (N_s)$ [1]	20 + 20 + (20)
POD tolerance $I(N)$	10^{-2}
IGA-Stokes evaluation time	0.7 s
POD construction time	65 s
POD evaluation time	0.07 s
Computational speedup POD	10

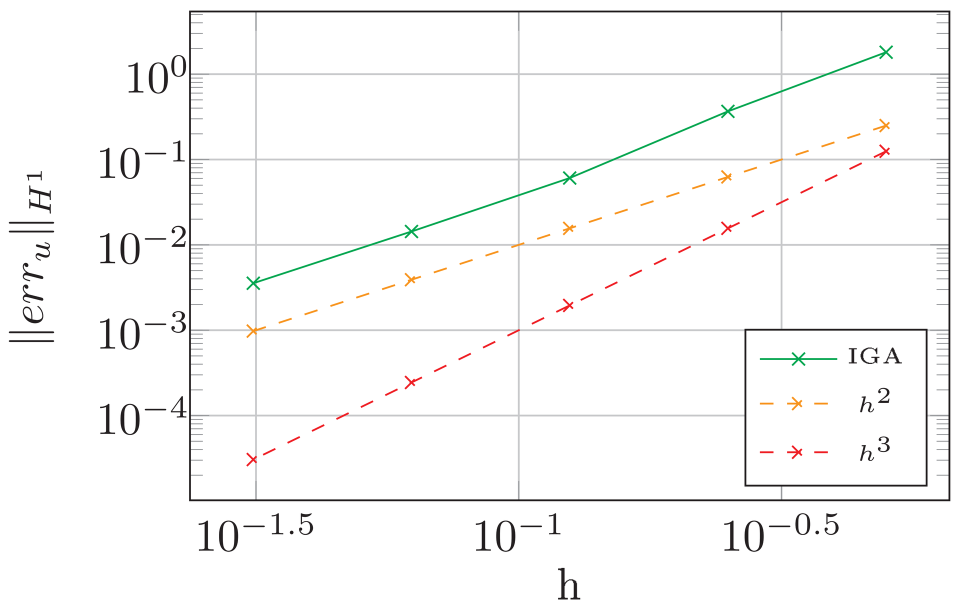
CPU: Intel Pentium G640 2.80 GHz, RAM: 4 GB

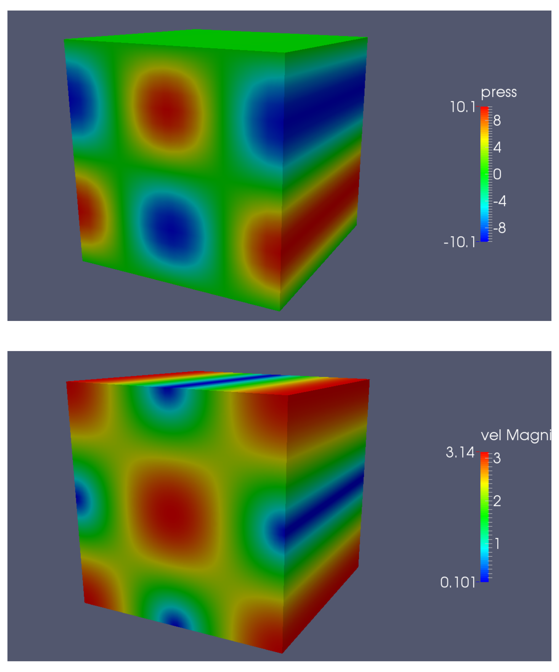
IGA-Stokes validation test

Benchmark: a divergence-free solution:

$$u_x = \pi\cos(\pi x)\cos(\pi y),$$
$$u_y = \pi\sin(\pi x)\sin(\pi y),$$
$$u_z = 0,$$
$$p = \pi^2\cos(2\pi x)\sin(2\pi y).$$

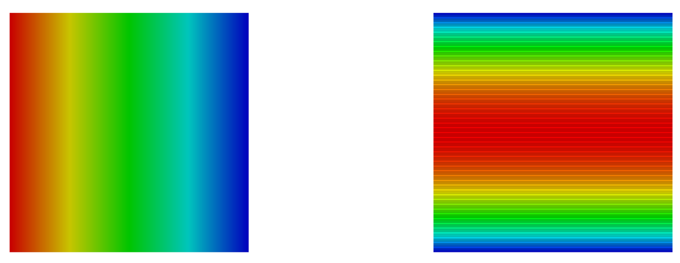
convergence test



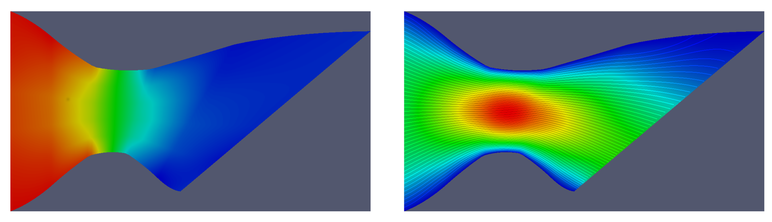


Pressure (top) and velocity magnitude (bottom) for the benchmark test: P2-P1 elements, dofs: 20577 for $\boldsymbol{u}$; 4913 for p

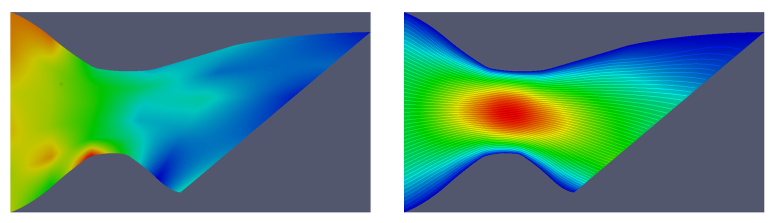
ROM-FFD-IGA-Stokes results



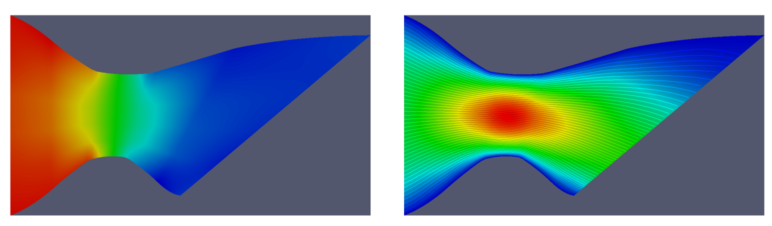
High-Fidelity solution on the reference domain; Pressure (left) and velocity (right).



High-Fidelity solution; Pressure (left) and velocity (right).

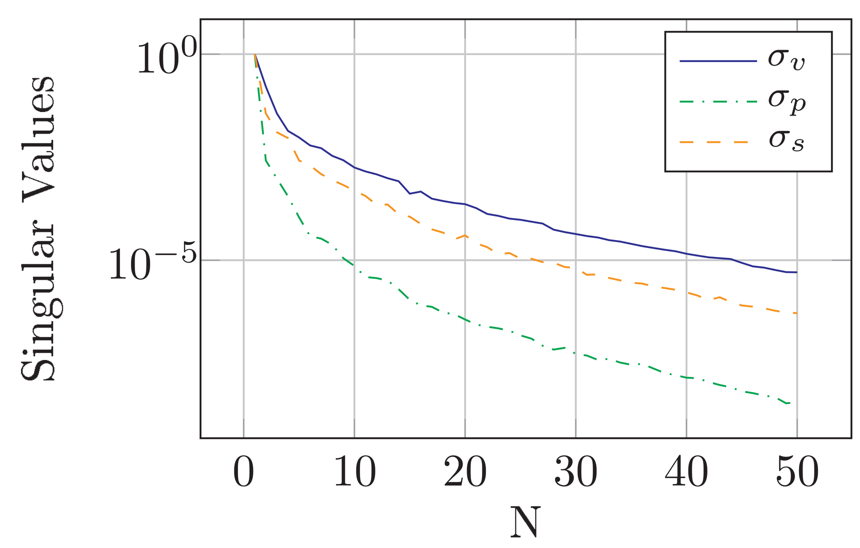


Reduced-Order solution, no supremizers; Pressure (left) and velocity (right).



Reduced-Order solution, supremizers; Pressure (left) and velocity (right).

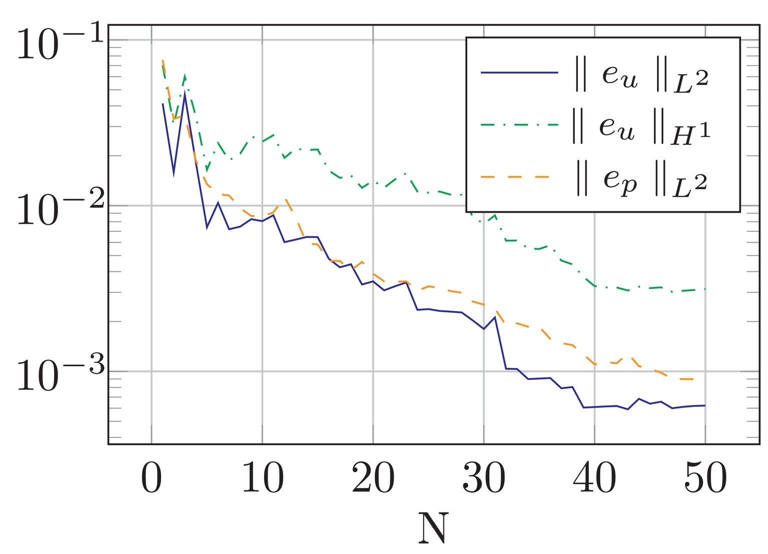
Singular Values of POD



Need to enrich the velocity space to fulfil an equivalent parametrized ROM Brezzi *inf-sup* stability condition to guarantee the approximation stability also at the reduced order level. [1]

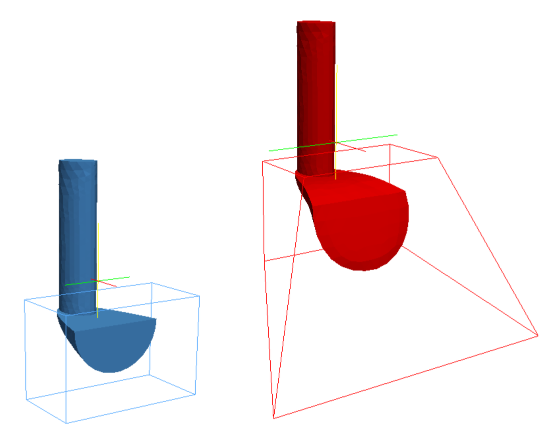
$$\begin{bmatrix} \mathcal{Z}_v^T\boldsymbol{A}\mathcal{Z}_v & \mathcal{Z}_v^T\boldsymbol{A}\mathcal{Z}_s & \mathcal{Z}_v^T\boldsymbol{B}\mathcal{Z}_p \\ \mathcal{Z}_s^T\boldsymbol{A}\mathcal{Z}_v & \mathcal{Z}_s^T\boldsymbol{A}\mathcal{Z}_s & \mathcal{Z}_s^T\boldsymbol{B}\mathcal{Z}_p \\ \mathcal{Z}_p^T\boldsymbol{B}^T\mathcal{Z}_v & \mathcal{Z}_p^T\boldsymbol{B}^T\mathcal{Z}_s & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{s} \\ \boldsymbol{p} \end{bmatrix} = 0$$

Error of the POD solution

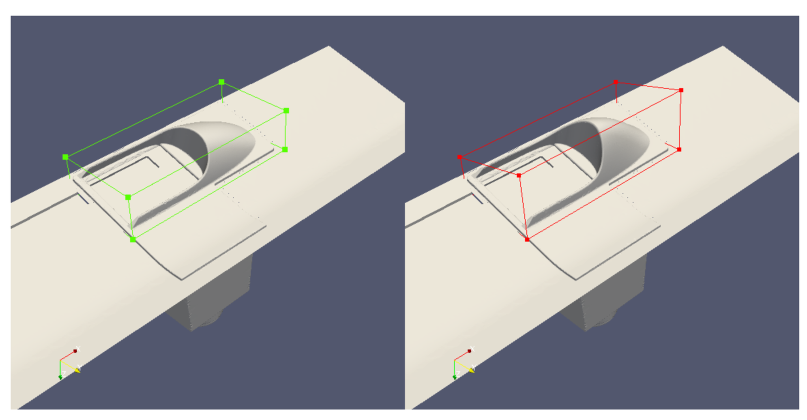


Forthcoming industrial applications

UBE (Underwater Blue Efficiency): shape optimization of immersed parts of motor yachts, including exhaust flow devices, for the reduction of emissions and vibrations and to increase on board comfort.



FFD on the exhaust gasses device: reference configuration (left) and deformed configuration (right)



FFD on a scoop: reference configuration (left) and deformed configuration (right)

Partners:

References

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[2] F. Ballarin, A. Manzoni, G. Rozza, and S. Salsa. Shape optimization by free-form deformation: existence results and numerical solution for Stokes flows. *Journal of Scientific Computing*, 60(3):537–563, 2014.

[3] J. A. Cottrell, T. J. Hughes, and Y. Bazilevs. *Isogeometric analysis: toward integration of CAD and FEA*. John Wiley & Sons, 2009.

[4] A. Quarteroni, G. Rozza, and A. Manzoni. Certified reduced basis approximation for parametrized partial differential equations and applications. *J. Math. Ind.*, 1(3), 2011.

Acknowledgements

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