Non-Intrusive Polynomial Chaos Method Applied to Problems In Computational Fluid Dynamics and a Comparison with POD-Galerkin methods

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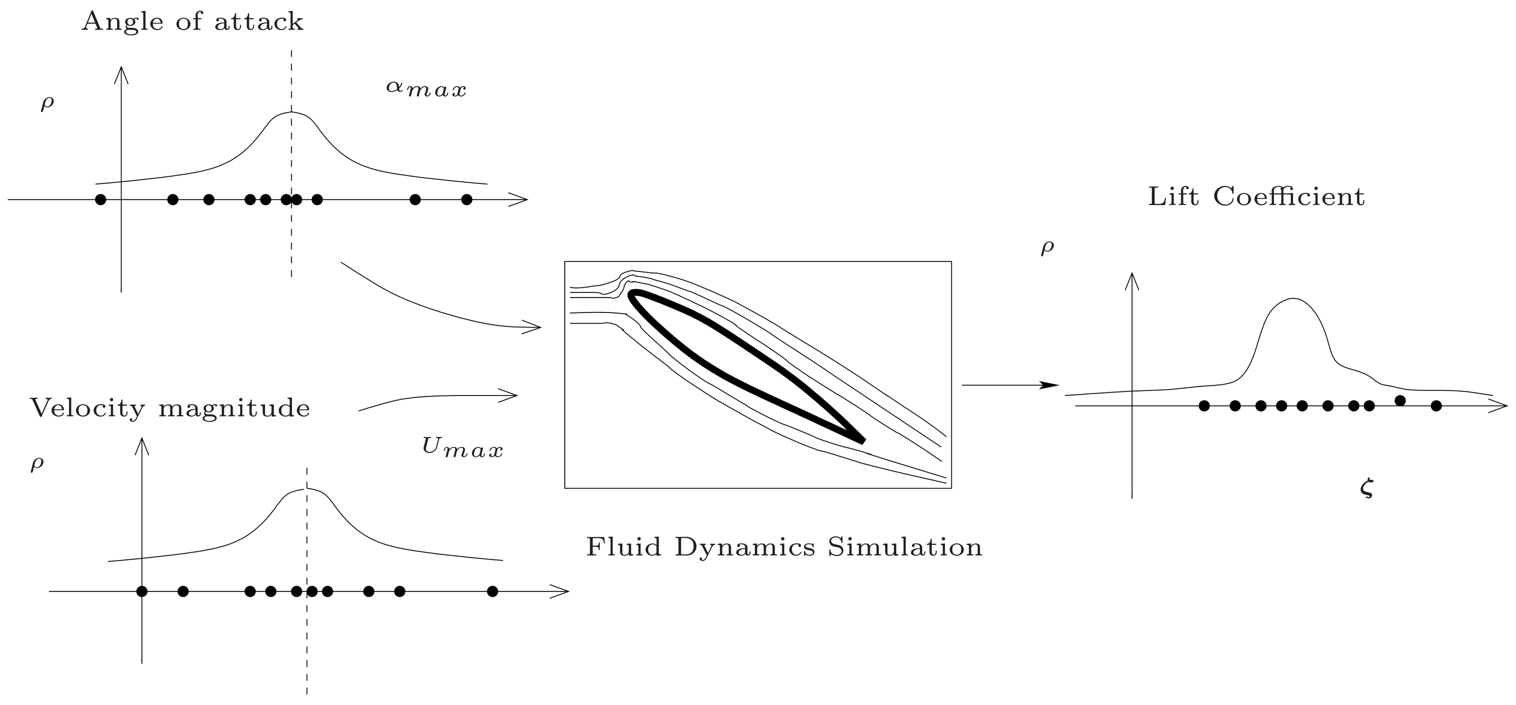
Introduction - 1

Motivations

- Uncertainty Quantification is a field of growing interest and it has an important rule in studying problems in Computational Fluid Dynamics CFD. Uncertainties in the input parameters may affect the results of the simulations.
- In aerospace engineering there is the problem of finding the angle of attack that maximizes the airfoil lift coefficient. Increasing the angle of attack starting from zero the lift coefficient increases up to a maximum value, after which stall occurs and lift is rapidly reducing.
- The angle of attack and the magnitude of the velocity in the above mentioned problem can be seen as input uncertainties that affect the output of interest that is the lift coefficient. In this work these uncertainties have been studied.

Methodology-Overview

- Proper Orthogonal Decomposition (POD) is which is a way for reconstructing reduced order spaces is used for building a reduced order model that will give lift coefficients as an output. **BOX 2**
- Non-Intrusive Polynomial Chaos Expansion (PCE) is a well known method for assessing how uncertainties in the input parameters propagate through the model and affect the output. A comparison with POD has been done. **BOX 3**



The POD-Galerkin reduced order model - 2

The physical problem is modelled using the steady incompressible Navier-Stokes equations which read as follows :

$$\begin{cases} (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\nabla^2\mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \boldsymbol{u}(\mathbf{x}) = (U_x, U_y) & \text{in } \Gamma_{dir} \end{cases}$$

The space discretization of the equations has been performed using a finite volume approach. The model reduction is performed using a POD-Galerkin approach. Within this method it is assumed that the reduced order solution for the velocity, pressure and fluxes fields is given by a linear combination of the bases functions $\varphi_i(\mathbf{x})$, $\psi_i(\boldsymbol{x})$ and $\chi_i(\boldsymbol{x})$ (which are respectively for velocity, flux and pressure and these bases depend only on space) multiplied by a coefficients $a_i(\zeta)$ for velocity and flux while $b_i(\zeta)$ for pressure:

$$\begin{pmatrix}\boldsymbol{u}(\boldsymbol{x},\zeta)\\ F(\boldsymbol{x},\zeta)\end{pmatrix} \approx \begin{pmatrix}\boldsymbol{u}_r(\boldsymbol{x},\zeta)\\ F_r(\boldsymbol{x},\zeta)\end{pmatrix} = \sum_{i=1}^{N_u} a_i(\zeta)\begin{pmatrix}\boldsymbol{\varphi}_i(\boldsymbol{x})\\ \psi_i(\boldsymbol{x})\end{pmatrix}, \tag{1}$$

$$p(\boldsymbol{x},\zeta) \approx p_r(\boldsymbol{x},\zeta) = \sum_{i=1}^{N_p} b_i(\zeta)\chi_i(\boldsymbol{x}), \tag{2}$$

The reduced basis space $\mathbb{V}_{rb} = \operatorname{span}(\varphi_i)$ is obtained by performing a POD that is on the snapshots matrices that is equivalent of solving the following minimization problem:

$$\mathbb{V}_{rb} = \arg\min \frac{1}{N_s}\sum_{n=1}^{N_s}\|\mathbf{u}_n(\mathbf{x}) - \sum_{i=1}^{N_r}(\mathbf{u}_n(\mathbf{x}),\varphi_i(\mathbf{x}))_{L^2}\varphi_i(\mathbf{x})\|^2_{L^2}$$

Where $\mathbf{u_n}(\mathbf{x},\zeta,u_{in})$ are field solutions (Snapshots) sampled at different inlet velocities magnitudes and angles of attack. Then the governing equations are projected onto the spatial bases and the original fields are replaced with the approximated fields. This operation generates the POD-Galerkin system:

$$\begin{cases} \boldsymbol{Ba} - \boldsymbol{a^T Ca} - \boldsymbol{Kb} = \boldsymbol{0} \\ \boldsymbol{Pa} = \boldsymbol{0} \end{cases}$$

Then the coefficients $a_i(\zeta)$ and $b_i(\zeta)$ can be computed and thus the reduced order solution for all the fields and lift coefficients will be obtained [3]. Stability is not preserved under Galerkin-projection and therefore to overcome this problem we used a supremizer pressure stabilization technique [4].

PCE technique - 3

PCE

As mentioned before, any stochastic process can be expanded into a series of a separable deterministic and stochastic terms in a way that is similar to Fourier expansion. Thus for a generic random variable $\alpha^\star$ can be expressed as follows :

$$\alpha^\star(\mathbf{x},t,\boldsymbol{\zeta}) = \sum_{i=0}^{\infty}\alpha_i(\mathbf{x},t)\psi_i(\boldsymbol{\zeta}) \tag{3}$$

where $\psi_i(\boldsymbol{\zeta})$ is the random basis function of the i^{th} mode which depends just on the random variables represented by the vector $\boldsymbol{\zeta}$. $\alpha_i(\mathbf{x},t)$ is the i^{th} fluctuation amplitude that is a function of the deterministic variables $\mathbf{x}$ and t. Of course, in practice this series is truncated and only its first P values are computed.
Hermite polynomials have been chosen in this study which form an orthogonal set of basis functions in terms of Gaussian distribution [1]. Here P is the number of Hermite polynomials used in the expansion and has to depend on the order of the polynomials chosen and the dimension of the random variable vector $\boldsymbol{\zeta}$.

Hermite polynomials in a space with random dimension n and with degree p are given by:

$$H_q(\zeta_1,\ldots,\zeta_n) = (-1)^p e^{\frac{1}{2}\boldsymbol{\zeta}^T\boldsymbol{\zeta}}\frac{\partial^p}{\partial(\zeta_1)^{c_1}\ldots\partial(\zeta_n)^{c_n}}e^{\frac{-1}{2}\boldsymbol{\zeta}^T\boldsymbol{\zeta}}$$

Where $\sum_{i=1}^{n} c_n = p$ and here P is given by $P+1 = \frac{(p+n)!}{p!n!}$ [1].

Coefficients Computation

The identification of the coefficients $\alpha_i(x,t)$ in 3 can be carried out with different ways. The one that is used here is based on sampling approach introduced by [2], which can be seen as a discretized version of equation 3:

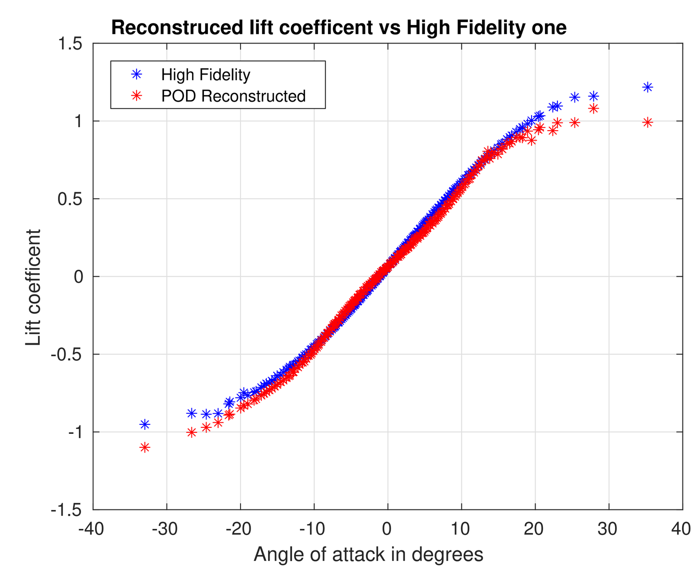
$$\begin{bmatrix}\alpha_0^*\\ \alpha_1^*\\ \vdots\\ \alpha_N^*\end{bmatrix} = \begin{bmatrix}\psi_1(\boldsymbol{\zeta}_0) & \psi_2(\boldsymbol{\zeta}_0) & \ldots & \psi_P(\boldsymbol{\zeta}_0)\\ \psi_1(\boldsymbol{\zeta}_1) & \psi_2(\boldsymbol{\zeta}_1) & \ldots & \psi_P(\boldsymbol{\zeta}_1)\\ \vdots & & \ddots & \\ \psi_1(\boldsymbol{\zeta}_N) & \psi_2(\boldsymbol{\zeta}_N) & \ldots & \psi_P(\boldsymbol{\zeta}_N)\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_P\end{bmatrix}$$

Where N is the number of the samples taken. The system above can be solved to determine the coefficients α_i from the known output coefficients $\alpha_i^\star$. In such case N has to coincide with the number of polynomials needed $P+1$. In practice more samples are considered and thus the system is solved in the least squares sense, namely:

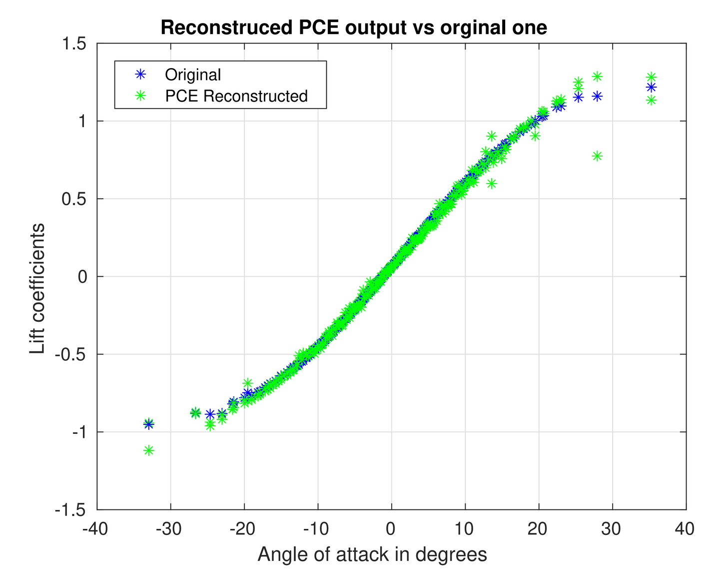
$$\alpha = (L^T L)^{-1} L^T \alpha^* \tag{4}$$

Where L, α and α^* denote the rectangular matrix in, the PCE coefficients vector and output vector respectively.

Results - 4



Comparison between lift coefficients obtained for different angles of attack by the full order solver with the ones computed using POD reduced order model with 6,4 and 4 modes used for velocity, pressure and supremizer respectively and relative error committed is 8.6%.



Comparison between lift coefficients obtained for different angles of attack by the full order solver with the ones computed using non intrusive PCE technique with third order degree polynomials and relative error committed is 7.6%.

Perspectives and Future Work - 5

Clearly, an interesting possibility which will be investigated in the next months is the application of PCE to the POD reconstructed output. Of course, this would dramatically speed up the offline phase of the PCE technique, but might lead to errors in the polynomials coefficients. An assessment of both the speed up and accuracy of such combined PCE and POD-Galerkin technique would be of great interest. An additional thing to be done is trying to apply another ways of computing the polynomials coefficients other than the least squares sampling approach.

References

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