Exercises are numbered as in the lecture notes of the course.

**Exercise 3.8.1.** For few instants of time $t \geq 0$ make a graph of the solution $u(x, t)$ to the wave equation with the initial data

$$
u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1, & x \in [x_0, x_1] \\ 0, & \text{otherwise} \end{cases}, \quad -\infty < x < \infty.$$ 

**Solution.** Let us consider the general D’Alembert formula for the solution of the vibrating string with $\rho = T = 1$: following the notation of the lecture notes, $\phi(x) = 0$ and $\psi(x) \neq 0$. Therefore

$$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds = \frac{\Psi(x + t) - \Psi(x - t)}{2}.$$ 

We now integrate this result and get the following cases

$$\Psi(y) = \begin{cases} 0, & y < x_0 \\ \int_{x_0}^{y} 1 dx, & y \in [x_0, x_1] \\ \int_{x_0}^{x_1} 1 dx, & y > x_1 \end{cases}$$

Thus we get

$$\Psi(y) = \begin{cases} 0, & y < x_0 \\ y - x_0, & y \in [x_0, x_1] \\ x_1 - x_0, & y > x_1 \end{cases}$$

We can now plot the solution $u(x, t) = \frac{\Psi(x+t) - \Psi(x-t)}{2}$ for different times (in the plot, $x_0 = -2$ and $x_1 = 2$) and the time increases from upper left to lower right.
Exercise 3.8.5. Prove that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad \text{for } 0 < x < 2\pi.$$ 

Compute the sum of the Fourier series for all other values of $x \in \mathbb{R}$.

Solution. Consider the Taylor expansion of the following real function:

$$\log(1 - t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$$

Now consider its analytic continuation by setting $t = e^{ix}$; we obtain the complex logarithm:

$$\Log(1 - e^{ix}) = -\sum_{n=1}^{\infty} \frac{e^{inx}}{n} = -\sum_{n=1}^{\infty} \frac{\cos nx}{n} - i \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$
Let us now take the imaginary part of both sides of the equation:

\[ \text{arg}(1 - e^{ix}) = -\sum_{n=1}^{\infty} \frac{\sin nx}{n}. \]

We expand the first term:

\[
\text{arg}(1 - e^{ix}) = \tan \left( -\frac{\sin x}{1 - \cos x} \right) = -\tan \left( \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) = -\tan \left( \cot \frac{x}{2} \right)
\]

\[ = -\tan(\tan(\frac{\pi}{2} - \frac{x}{2})) = -\left( \frac{\pi}{2} - \frac{x}{2} \right) \]

and this directly leads to the final result

\[ -\sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\left( \frac{\pi}{2} - \frac{x}{2} \right) \]

\[
\text{Exercise 3.8.6. Compute the sums of the following Fourier series:}
\]

\[ \sum_{n=1}^{\infty} \frac{\sin 2nx}{2n}, \quad 0 < x < \pi; \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \quad |x| < \pi. \]

\[ \text{Solution. We are going to exploit the previous exercise to easily compute the two series. We start with the first one. Notice that} \]

\[ \sum_{n=1}^{\infty} \frac{\sin 2nx}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin n2x}{n}, \quad 0 < x < \pi; \]

Set now \( y = 2x \) so that \( 0 < y < 2\pi \). Thus we get

\[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin ny}{n} = \frac{\pi - y}{4} = \frac{\pi - 2x}{4}. \]

We compute the second series in the interval \( 0 < x < \pi \) first. The summation can be split in two parts as follows:

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx = -\sum_{\text{odd}} \frac{\sin nx}{n} + \sum_{\text{even}} \frac{\sin nx}{n}. \]
Of course, in $0 < x < \pi$, we have that
\[
\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \sum_{\text{odd}} \frac{\sin nx}{n} + \sum_{\text{even}} \frac{\sin nx}{n}
\]
Notice also that the summation over $n$ even is exactly the first summation of the exercise. Hence, by the previous exercise, we get
\[
\sum_{\text{odd}} \frac{\sin nx}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} - \sum_{\text{even}} \frac{\sin nx}{n} = \frac{\pi - x}{2} - \frac{\pi - 2x}{4} = \frac{\pi}{4}
\]
In the interval $(0, \pi)$ we have obtained
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx = - \sum_{\text{odd}} \frac{\sin nx}{n} + \sum_{\text{even}} \frac{\sin nx}{n} = -\frac{\pi}{4} + \frac{\pi - 2x}{4} = -\frac{x}{2}, \quad 0 < x < \pi.
\]
Since the original series is odd in $(0, \pi)$, we can immediately extend our result to the full interval, since the result is odd. The exercise is concluded.

**Exercise 3.8.8.** Compute the sums of the following Fourier series:
\[
\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2},
\]
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.
\]

**Solution.** We proceed with the first summation.
\[
\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} = -\sum_{n=1}^{\infty} \int_{\pi/2}^{x} \frac{\sin(2n-1)y}{(2n-1)} \, dy = -\int_{\pi/2}^{x} \sum_{n=1}^{\infty} \frac{\sin(2n-1)y}{(2n-1)} \, dy
\]
By equation (1) of exercise 3.8.6, we know that the integrand is $\frac{\pi}{4}$ and thus we get
\[
\sum_{n=1}^{\infty} \int_{\pi/2}^{x} \frac{\sin(2n-1)y}{(2n-1)} \, dy = \int_{\pi/2}^{x} \frac{\pi}{4} \, dy = \frac{\pi}{4} x - \frac{\pi^2}{8}.
\]
The final result is:
\[
\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} = -\frac{\pi}{4} x + \frac{\pi^2}{8}, \quad \text{for } 0 < x < \pi.
\]
We are now going to use the results of exercise 3.8.5 to do the calculations of the second series:
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = -\sum_{n=1}^{\infty} \left( \int_{0}^{x} \frac{\sin ny}{n} \, dy - \frac{1}{n^2} \right)
\]
By the result of exercise 3.8.5, we get
\[ \sum_{n=1}^{\infty} \left( \int_{0}^{x} \frac{\sin ny}{n} \, dy - \frac{1}{n^2} \right) = \int_{0}^{x} \frac{\pi - y}{2} \, dy - \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{2} x - \frac{x^2}{4} - \frac{\pi^2}{6} \]
and our final result is:
\[ \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = -\frac{\pi}{2} x + \frac{x^2}{4} + \frac{\pi^2}{6}, \quad \text{for } 0 < x < 2\pi \]
\[ \diamond \]

**Exercise 3.8.10.** Prove the conservation of the quantity
\[ P(t) = \int_{0}^{l} \rho u_t(x,t)u_x(x,t) \, dx, \]
i.e. \( P(t) = P(0) \), for a vibrating string with FREE end points. This quantity can be interpreted as the total IMPULSE FLUX of the vibrating string.

**Solution.** We recall that the equation for a vibrating string \( u(x,t) \) of finite length – with \( 0 < x < l \) – and free end points is of the form
\[ Tu_{xx} = \rho u_{tt}, \quad u_x(0,t) = u_x(l,t), \quad u_t(0,t) = u_t(l,t). \]

We want to show that
\[ \frac{\partial P(t)}{\partial t} = 0 \]

Indeed we have that
\[
\frac{\partial P(t)}{\partial t} = \int_{0}^{l} \rho \left( u_{tt}(x,t)u_x(x,t) + u_t(x,t)u_{xt}(x,t) \right) \, dx \\
= \int_{0}^{l} \left( Tu_{xx}(x,t)u_x(x,t) + \rho u_t(x,t)u_{xt}(x,t) \right) \, dx \\
= \int_{0}^{l} \left( \frac{T}{2} \frac{\partial}{\partial x} u_x(x,t)^2 + \frac{\rho}{2} \frac{\partial}{\partial x} u_t(x,t)^2 \right) \, dx \\
= \frac{1}{2} \int_{0}^{l} \frac{\partial}{\partial x} \left( Tu_x(x,t)^2 + \rho u_t(x,t)^2 \right) \, dx \\
= T \frac{1}{2} u_x(x,t)^2|_0^l + \rho \frac{1}{2} u_t(x,t)^2|_0^l 
\]
(2)
The conditions \( u_x(0,t) = u_x(l,t) \) and \( u_t(0,t) = u_t(l,t) \) for all \( t \) imply that \( u_t(x,t)^2|_0^l = u_x(x,t)^2|_0^l = 0 \) and thus
\[ \frac{\partial P(t)}{\partial t} = T \frac{1}{2} u_x(x,t)^2|_0^l + \rho \frac{1}{2} u_t(x,t)^2|_0^l = 0. \]
\[ \diamond \]