3.1 The tangent bundle.

So far we have considered vectors and tensors at a point. We now wish to consider fields of vectors and tensors. The union of all tangent spaces is called the tangent bundle and denoted $TM$:

$$TM = \bigcup_{x \in M} T_x M.$$  \hfill (3.1.1)

The tangent bundle can be given a natural manifold structure derived from the manifold structure of $M$. Let $\pi : TM \to M$ be the natural projection that associates a vector $v \in T_x M$ to the point $x$ that it is attached to. Let $(U_A, \Phi_A)$ be an atlas on $M$. We construct an atlas on $TM$ as follows. The domain of a chart is $\pi^{-1}(U_A) = \bigcup_{x \in U_A} T_x M$, i.e. it consists of all vectors attached to points that belong to $U_A$. The local coordinates of a vector $v$ are $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ where $(x^1, \ldots, x^n)$ are the coordinates of $x$ and $v^1, \ldots, v^n$ are the components of the vector with respect to the coordinate basis (as in (2.2.7)). One can easily check that a smooth coordinate transformation on $M$ induces a smooth coordinate transformation on $TM$ (the transformation of the vector components is given by (2.3.10), so if $M$ is of class $C^r$, $TM$ is of class $C^{r-1}$).

In a similar way one defines the cotangent bundle

$$T^* M = \bigcup_{x \in M} T^*_x M,$$  \hfill (3.1.2)

denoted $T^*$.

The tensor bundles

$$TM^*_p = \bigcup_{x \in M} T^*_x M^*_p,$$  \hfill (3.1.3)

and the bundle of $p$-forms

$$\Lambda^p M = \bigcup_{x \in M} \Lambda^p_x M.$$  \hfill (3.1.4)

3.2 Vector and tensor fields.

A vectorfield on $M$ is a map $v : M \to TM$ such that for all $x$, $v(x) \in T_x M$. Such a map is called a section of $TM$ (this notion will be discussed in a more general context in 6.2). Similarly one defines tensor fields and fields of $p$-forms. In the domain of a chart $U_A$, the vectors $\{\partial_{\mu}\}$ and the one–forms $\{dx^\mu\}$ are smooth linearly independent local sections of $TM$ and $T^* M$ and can be used as local fields of bases for vectorfields and one–forms. Similarly, the tensor products $\{\partial_{\mu} \otimes \cdots \otimes \partial_{\nu}\}$ define fields of bases in the bundle of tensors of type $(r, s)$. A tensor field can thus be decomposed in a chart as

$$t(x) = t_{\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s}(x)dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_r} \otimes \partial_{\nu_1} \otimes \cdots \otimes \partial_{\nu_s},$$  \hfill (3.2.1)

where the components are smooth functions on $U_A$. The space of tensorfields of type $(r, s)$ is an infinite dimensional vectorspace denoted $T^r_s$. In particular, the space of vectorfields $T^0_1$ is also denoted $X(M)$ and the space of fields of $p$-forms is also denoted $\Omega^p(M)$.

If $v \in X(M)$ and $f \in C^\infty(M)$, we define a function $v(f) \in C^\infty(M)$ by $v(f)(x) = v(x)(f)$, the r.h.s. of this equation being defined as in (2.2.4). From the linearity and from (2.2.3) we conclude that vectorfields are derivations mapping the algebra of functions to itself.

One can define the commutator of two vectorfields

$$[v, w]f = v(w(f)) - w(v(f)).$$  \hfill (3.2.2)

In components we have

$$[v, w]f = v^\mu \partial_x^\mu w^\nu \partial_x^\nu - w^\nu \partial_x^\nu v^\mu \partial_x^\mu = \left(v^\mu \partial^\nu - w^\nu \partial^\mu\right)(v^\mu - w^\mu) \partial_x^\nu.$$  \hfill (3.2.3)

From this formula one can read off the components of the vectorfield $[v, w]$. The commutator of vectorfields is also called the Lie bracket. It satisfies the Jacobi identity

$$[v, [w, t]] + [w, [t, v]] + [t, [v, w]] = 0.$$  \hfill (3.2.4)

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Thus, $X(M)$ is an (infinite dimensional) Lie algebra.

### 3.3 Actions of maps on tensors.

Let $\phi : M \to N$ be a smooth map of manifolds, and $f \in C^\infty(N)$. We can define a function $\phi^* f \in C^\infty(M)$, called the pullback of $f$, by

$$\phi^* f = f \circ \phi.$$  \hfill (3.3.1)

Given a curve in $M$, i.e., a smooth map $c : R \to M$, we can construct a curve in $N$ by the composition $\phi \circ c$. If $v \in T_x M$ is the vector tangent to $c$ at $x$, the vector tangent to $\phi \circ c$ at $\phi(x)$ is called the push-forward of $v$ and will be denoted $T\phi(v)$ (or, as explained below, also $\phi_*(v)$). Its action on a real function $f \in C^\infty(N)$ is given by

$$T\phi(v)f = \frac{d}{dt}f(\phi(c(t)))\Big|_{t=0}.$$  \hfill (3.3.2)

It is easy to prove that $T\phi(\alpha v + bw) = aT\phi(v) + bT\phi(w)$, so given a map $\phi$, we have defined for each $x \in M$ a linear map $T\phi : T_x M \to T_{\phi(x)} N$ of the tangent spaces, called the differential of $\phi$. In the special case $N = R$ one can verify, using the natural identification of $T_{\phi(x)} R$ with $R$, that this agrees with the previously defined notion of differential of a real-valued function.

There is a relation between the pullback of functions and the push-forward of vectors:

$$v(\phi^* f) = \frac{d}{dt}\phi^* f((c(t)))\Big|_{t=0} = \frac{d}{dt}f(\phi(c(t)))\Big|_{t=0} = T\phi(v)f.$$  \hfill (3.3.3)

Given a linear map between vector spaces, one can immediately define a dual map between the dual spaces. In the present case, we can define a map $\phi^* : T^*_x N \to T^*_x M$ by requiring the contraction to be preserved. If $\alpha \in T^*_x N$, we define $\phi^* \alpha \in T^*_x (M)$ by

$$\phi^* \alpha(v) = \alpha(T\phi(v)).$$  \hfill (3.3.4)

It is called the pullback of $\alpha$ by $\phi$. Note that vectors and one-forms behave very differently under maps.

One can immediately generalize the definition of pullback to covariant tensors: if $t \in T^*_x M$, we define $\phi^* t \in T^*_x M$ by

$$\phi^* t(v_1, \ldots, v_r) = t(T\phi(v_1), \ldots, T\phi(v_r))$$  \hfill (3.3.5)

and one can also define a push-forward of contravariant tensors: if $t \in T_x M$ we define $\phi_*(t) \in T_{\phi(x)} N$ by

$$\phi_*(t)(\alpha_1, \ldots, \alpha_s) = t(\phi^*(\alpha_1), \ldots, \phi^*(\alpha_s)).$$  \hfill (3.3.6)

In the case $s = 1$ this coincides with $T\phi$: using the canonical identification of $T_x M$ with the dual of $T^*_x M$ we have, for any $\alpha$, $\phi_*\alpha = \alpha \circ T\phi$. If $x_1$ and $x_2 \in M$ are such that $\phi(x_1) = \phi(x_2)$, it is generally not the case that $T\phi(v(x_1)) = T\phi(v(x_2))$. It is therefore not possible to define unambiguously the image of $v$ at that point. No such issues arise with the pullback. In those cases when the image of a vector field $v$ is well-defined, one can verify that $T\phi$ is a homomorphism of Lie algebras:

$$[\phi_*(v_1), \phi_*(v_2)] = \phi_*([v_1, v_2]).$$  \hfill (3.3.7)

If $\phi$ is a diffeomorphism one can use the inverse $\phi^{-1}$ to pull-back vectors and push-forward one-forms. Furthermore, there cannot be any ambiguities of the sort described above, so one can define the transformation of a general tensor field under a diffeomorphism: if $t \in T_x M$ we define $\phi_*(t) \in T_{\phi(x)} N$ by

$$\phi_*(t(v_1, \ldots, v_r, \alpha_1, \ldots, \alpha_s)) = t(\phi_*^{-1}(v_1), \ldots, \phi_*^{-1}(v_n), \phi^*(\alpha_1), \ldots, \phi^*(\alpha_s)).$$  \hfill (3.3.8)
One can immediately check using the definition (2.3.4) that this map of tensors is a homomorphism of the tensor algebras:
\[ \phi_*(t_1 \otimes t_2) = \phi_* t_1 \otimes \phi_* t_2 \] (3.3.9)
and furthermore
\[ \phi_*(C^i_j) = C^i_j(\phi_* t) . \] (3.3.10)

By the definition of the differential, equation (2.3.1), and then using (3.3.3) we have
\[ (d(\phi^* f))(v) = v(\phi^* f) = (T \phi(v))f . \] Using again the definition of differential, and then using the definition of pullback (3.3.4), this is equal to
\[ df(T \phi(v)) = (\phi^* df)(v) . \] Since \( v \) is arbitrary, we have proven that
\[ d\phi^* f = \phi^* df , \] (3.3.11)
that is, the differential acting on functions commutes with the pull-back.

We now give local coordinate expressions for the action of maps on tensors. Let \( x^\mu \) and \( y^\nu \) be local coordinates in \( M \) and \( N \) respectively, as in section 2.1. Let \( (x^1(t), \ldots, x^m(t)) \) be the local coordinate representation of the curve \( c, (y^1(x^\mu), \ldots, y^n(x^\mu)) \) be the local representation of the map \( \phi \) and \( f(y^1, \ldots, y^n) \) be the local representation of the function \( f \). Then, applying the chain rule, we find
\[ T \phi(v)f = \frac{df(c(t))}{dt} \bigg|_{t=0} = \frac{\partial f}{\partial y^\nu} \frac{dx^\mu(t)}{dt} \bigg|_{t=0} \]
(note that the Jacobian matrix of the transformation is rectangular and does not have an inverse in general). Using equation (2.2.6) for the components of the tangent vector \( v \), one reads off the components \( v^\nu \) of \( T \phi(v) \):
\[ v^\nu(y) = \frac{\partial y^\nu}{\partial x^\mu} x^\mu(x) . \] (3.3.12)
Similarly, using (3.3.4) one finds for the pullback \( \alpha' = \phi^* \alpha \)
\[ \alpha'_\mu(x) = \frac{\partial y^\nu}{\partial x^\mu} \alpha_\nu(y) . \] (3.3.13)
and in the case when \( \phi \) is a diffeomorphism, using the definition (3.3.8) we find for the components of \( t' = \phi_* t \):
\[ t'_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s}(y) = \frac{\partial x^{\mu_1}}{\partial y^{\nu_1}} \cdot \frac{\partial x^{\mu_r}}{\partial y^{\nu_r}} \ldots \frac{\partial x^{\mu_1}}{\partial y^{\nu_1}} \ldots \frac{\partial x^{\mu_r}}{\partial y^{\nu_r}} \ldots \frac{\partial x^{\mu_1}}{\partial y^{\nu_r}} \ldots \frac{\partial x^{\nu_1}}{\partial y^{\nu_1}} \ldots \frac{\partial x^{\nu_s}}{\partial y^{\nu_s}} \ldots \frac{\partial x^{\nu_1}}{\partial y^{\nu_s}}(x) . \] (3.3.14)
which is identical to (2.3.10). The physical interpretation of the two formulae is as follows. In (2.3.10) we had a tensor \( t \) and we gave the formula for the change in the components of the tensor under a change of coordinates. Such a change of chart is called a passive transformation. In (3.3.14) the chart remains the same but we transform the tensor instead. This is called an active transformation. The two operations are physically indistinguishable. The mathematical expression of this fact is the identical form for the transformation of the local coordinate components of the tensor.

### 3.4 Maps of manifolds.

We can use the properties of the differential to classify maps between manifolds. Let \( \phi : M \to N \), where \( M \) is a \( m \)-dimensional manifold and \( N \) is a \( n \)-dimensional manifold. We define the rank of \( \phi \) at \( x \) to be the dimension of the image of \( T \phi \) or, in coordinates, the rank of the Jacobian matrix \( \frac{\partial x^\nu}{\partial y^\mu} \). A map \( \phi \) is said to be injective at \( x \) if \( r = m \). Clearly, this is only possible if \( m \leq n \). A map \( \phi \) is said to be surjective at \( x \) if \( r = n \). Clearly, this is only possible if \( m \geq n \). \( \phi \) is said to be an immersion (and \( M \) is then said to be an immersed submanifold of \( N \)) if it is injective at every point. \( \phi \) is said to be an embedding (and \( M \) is then said to be an embedded submanifold of \( N \)) if it is an immersion and furthermore it is a homeomorphism of \( M \) onto its image in the induced topology.

We have already defined the notion of diffeomorphism. By the inverse function theorem, if \( \phi \) is both injective and surjective at \( x \) \( (i.e. \) if the tangent map at \( x \) is an isomorphism) there exists a neighborhood \( U \) of \( x \) such that \( \phi|_U \) is a diffeomorphism.
3.5 Fields of bases

A field of bases (also called a field of frames) in $TM$ is a set of $n$ linearly independent vectorfields $e_a$. Fields of bases always exist locally. For example, within the domain of a chart, the vectorfields $\partial_a$ defined in Section 2.2 are smooth and linearly independent and therefore define a local field of bases. Any other field of bases is given by

$$e_a(x) = e_a^\mu(x)\partial_\mu,$$  \hspace{1cm} (3.5.1)

where $e_a^\mu(x)$ is a locally defined function with values in the group $GL(n)$.

In general on a manifold there do not exist smooth everywhere defined fields of bases. When such a field of bases exists, the manifold is said to be parallelizable. The reason for this term is that in the presence of a global field of bases $e_a$, we can define a notion of parallelism on $M$: if $v = v^ae_a(x) \in T_xM$, we define the parallel transport of $v$ to any other point $y$ to be the vector $v^ae_a(y)$, i.e. two vectors at different points are declared to be parallel if they have the same components in the global basis. The property of being parallelizable has a topological meaning, as we shall discuss in section 6.2.

Example 3.5.1. Let $G$ be a Lie group and let $\{e_a\}$ with $a = 1, \ldots, \text{dim } G$ be a basis in $T_eG$. Let $L_g$ be the diffeomorphism of $G$ defined by left multiplication: $L_g(g') = gg'$. We define a linear basis $\{L_a(g)\}$ by $L_a(g) = TL_g(e_a) \in T_gG$. This definition is smooth with respect to $g$, so $\{L_a(g)\}$ is a global field of frames. Thus, every Lie group is parallelizable.

Example 3.5.2. The two-dimensional sphere is not parallelizable. In fact, on the two-dimensional sphere there does not even exist a single nowhere-vanishing vectorfield.

If $e_a$ is a local field of frames, one can compute the Lie brackets of basis vectors and decompose them again on the same basis:

$$[e_a, e_b] = f_{ab}^c e_c,$$  \hspace{1cm} (3.5.2)

where

$$f_{ab}^c = (e_a^\mu \partial_\mu e_b^\nu - e_b^\mu \partial_\mu e_a^\nu)e_c^\nu$$  \hspace{1cm} (3.5.3)

are called the structure functions of the field of frames. A natural basis has vanishing structure functions.

3.6 Lie derivatives

In section 2.2 we have defined the derivative of a function in the direction of a vector at a given point. In section 3.2 this was generalized to the derivative of a function along a vectorfield. Notice that it is not possible to define the derivative of a general tensorfield in the direction of a vector at a point. The reason is that the values of a tensorfield at neighboring points are elements of different vectorspaces, and hence cannot be subtracted. The only type of tensors that do not have this problem are the tensors of type $(0,0)$, i.e. the functions. It is however possible to define the derivative of a generic tensorfield along a vectorfield. This is what we do next.

The first step is to define integral curves of the vectorfield, namely a family of curves having the given vectorfield as their tangent vector at each point. Since the problem is local, it will be enough to solve it in a particular coordinate system. Consider therefore the first order differential equation

$$\frac{dx^\mu(t)}{dt} = v^\mu(x(t)),$$  \hspace{1cm} (3.6.1)

where $x^\mu(t)$ is a local coordinate, parametric description of the curve and $v^\mu$ are the components of the vectorfield $v$.

This is an ordinary first order differential equation in $\mathbb{R}^n$, for which there exist well-known existence and uniqueness theorems. Translated into the language of manifolds, these theorems state that for every point $x \in M$ there exists a real number $\eta > 0$ and a unique smooth map $\phi : M \times [-\eta, \eta] \to M$, with $\phi(x, 0) = x$ and the semigroup property $\phi(\phi(x, t), s) = \phi(x, t+s)$ for all $t, s \in [-\eta, \eta]$ with $|s + t| < \eta$. In particular, there exists the inverse map $\phi(x, t)^{-1} = \phi(x, -t)$. There follows that for any fixed $x$, $\phi_x(t) = \phi(x, t)$ is the desired curve going through $x$ and having the vectorfield $v$ as tangent vectors.
On the other hand, for any fixed \( t < \eta \), the map \( \phi_t : M \to M \) defined by \( \phi_t(x) = \phi(x,t) \) is bijective, smooth and its inverse \( \phi_{-t} \) is smooth, so it is a diffeomorphism. The one-parameter family of diffeomorphisms \( \phi_t, -\eta < t < \eta \), is called the flow of \( \phi \).

We can use a flow to compare the values of tensor fields at different points and therefore construct a notion of derivative of a tensor field. We begin by vector fields. If \( v \) and \( w \) are vectorfields, using the flow generated by \( v \) we can compare the value of \( w \) at the point \( \phi_t(x) \) to its value at \( x \): \( T\phi_t^{-1}(w(\phi_t(x))) - w(x) \). By taking the limit \( t \to 0 \) of this difference one obtains the Lie derivative of \( w \) along \( v \):

\[
(\mathcal{L}_v w)(x) = \left. \frac{d}{dt} T\phi_t^{-1}(w(\phi_t(x))) \right|_{t=0}.
\]

(3.6.2)

In a similar way one defines the Lie derivative of a one-form \( \alpha \):

\[
(\mathcal{L}_v \alpha)(x) = \left. \frac{d}{dt} \phi_t^* (\alpha(\phi_t(x))) \right|_{t=0}.
\]

(3.6.3)

In general the Lie derivative of a tensor field \( t \) along the vectorfield \( v \) is

\[
(\mathcal{L}_v t)(x) = \left. \frac{d}{dt} T\phi_t^{-1}(t(\phi_t(x))) \right|_{t=0},
\]

(3.6.4)

where the transform of a tensor under a diffeomorphism is defined as in (3.3.8).

If we write (3.3.9) for the flow \( \phi_t \), take the derivative with respect to \( t \) and evaluate at \( t = 0 \) we get the Leibnitz rule

\[
\mathcal{L}_v (t_1 \otimes t_2) = \mathcal{L}_v (t_1) \otimes t_2 + t_1 \otimes \mathcal{L}_v (t_2).
\]

(3.6.5)

Acting similarly on equation (3.3.10) one obtains:

\[
\mathcal{L}_v (C^i_j t) = C^i_j (\mathcal{L}_v (t)).
\]

(3.6.6)

In order to write a formula for the Lie derivative in local coordinates, we begin by computing the Lie derivative of a natural basis. Let \( \Phi(x) = (x^1, \ldots, x^n) \) be the coordinates of \( x \) in a local chart. We denote \( \Phi(\phi_t(x)) = (x^1(t), \ldots, x^n(t)) \) the coordinates of \( \phi_t(x) \); then \( \Phi(\phi_t^{-1}(x)) = (x^1(-t), \ldots, x^n(-t)) \).

Using equation (3.3.12) for the coordinate representation of the push-forward of a vector we have

\[
\left. \frac{d}{dt} T\phi_t^{-1} \left( \frac{\partial}{\partial x^\mu} \right) \right|_{t=0} = \left. \frac{d}{dt} \frac{\partial x^\nu(-t)}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right|_{t=0}
\]

Commuting derivatives we can rewrite this as

\[
\left. \frac{\partial}{\partial x^\mu} \frac{dx^\nu(-t)}{dt} \frac{\partial}{\partial x^\nu} \right|_{t=0} = \left. \frac{\partial v^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right|_{t=0}
\]

Altogether, we have shown that

\[
\mathcal{L}_v \partial_\mu = -\partial_\mu v^\nu \partial_\nu.
\]

(3.6.7)

In a local coordinate system the vectorfield \( w \) can be decomposed as a linear combination of basis vectors with coefficients in \( C^\infty(U) \): \( w(x) = w^\mu(x) \partial_\mu(x) \). The product on the r.h.s. can be understood as the tensor product of a tensor of type \((0,0)\) and a tensor of type \((0,1)\). Applying the Leibnitz rule (3.6.5) we then get

\[
\mathcal{L}_v w = (v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \partial_\mu,
\]

(3.6.8)

which is the desired local coordinate expression of the Lie derivative. Comparing with (3.2.3) we see that

\[
\mathcal{L}_v w = [v, w].
\]

(3.6.9)
The local coordinate expression for the Lie derivative of a one-form can be derived by a similar calculation. Alternatively, one can use the properties (3.6.5) and (3.6.6) applied to $\alpha(w) = C_1^1(\alpha \otimes w)$:

\[
(\mathcal{L}_v \alpha)(w) = \mathcal{L}_v(\alpha(w)) - \alpha(\mathcal{L}_v w) \\
= v^\rho \partial_\rho (\alpha_\mu w^\mu) + \alpha_\rho (\mathcal{L}_v w)^\rho \\
= (v^\rho \partial_\rho \alpha_\mu + \alpha_\rho \partial_\mu v^\rho) w^\mu,
\]

whence

\[
\mathcal{L}_v \alpha = (v^\rho \partial_\rho \alpha_\mu + \alpha_\rho \partial_\mu v^\rho) dx^\mu .
\]

The general formula for the components of the Lie derivative of a tensor of type $(r, s)$ is

\[
(\mathcal{L}_v t)_{\nu_1 \nu_2 \cdots}^{\mu_1 \mu_2 \cdots} = v^\rho \partial_\rho t_{\nu_1 \nu_2 \cdots}^{\mu_1 \mu_2 \cdots} + t_{\rho \nu_2 \cdots}^{\mu_1 \mu_2 \cdots} \partial_{\nu_1} v^\rho + \cdots - t_{\nu_1 \nu_2 \cdots}^{\mu_2 \cdots} \partial_\rho v^{\mu_1} - \cdots ,
\]

with a term with a plus sign for every covariant index of $t$ and a term with a minus sign for every contravariant index.