9.1 The energy–momentum tensor

It will be useful to follow the analogy with electromagnetism (the same arguments can be repeated, with obvious modifications, also for nonabelian gauge theories). Recall that the coupling of charged matter fields to electromagnetism is of the form \( \int d^4x \, j^\mu A_\mu \), where the electromagnetic current can be defined as the current associated to invariance under global gauge transformations:

\[
j^\mu = \frac{\partial L}{\partial \partial_\mu \Phi} \delta \Phi .
\] (9.1.1)

Conservation of this current follows from Noether’s theorem. There is also another way of defining the current, namely as functional derivative of the matter action with respect to the gauge field:

\[
j^\mu = \frac{\delta S_{\text{matter}}}{\delta A_\mu}.
\] (9.1.2)

With this definition, current conservation follows from local gauge invariance. Let \( \epsilon(x) \) be an infinitesimal abelian gauge transformation (a function that tends to zero for \( x \to \infty \)) \(^{(1)}\). The variation of the gauge potential under such a transformation is \( \delta A_\mu = \partial_\mu \epsilon \). Invariance of the action implies

\[
0 = \delta_\epsilon S_{\text{matter}} = \int d^4x \left( \frac{\delta S_{\text{matter}}}{\delta A_\mu} \delta_\epsilon A_\mu + \frac{\delta S_{\text{matter}}}{\delta \Phi} \delta_\epsilon \Phi \right) .
\]

The second term vanishes upon using the equations of motion of the matter field, then the remaining term can be rewritten

\[
0 = \delta_\epsilon S_{\text{matter}} = \int d^4x \, j^\mu \partial_\mu \epsilon = - \int d^4x \, \epsilon \partial_\mu j^\mu ,
\] (9.1.3)

where we have used the asymptotic behaviour of \( \epsilon \) to drop the surface term in the partial integration. Since \( \epsilon \) is arbitrary, current conservation follows.

This procedure can be followed closely in the case of gravity. Given a certain action for matter fields coupled to the metric (in this context the electromagnetic field could itself be treated as a matter field) the energy–momentum tensor can be defined as the functional derivative

\[
T^\mu\nu(x) = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{matter}}}{\delta g^\mu\nu}(x) .
\] (9.1.4)

As in electromagnetism, the covariant conservation of this tensor follows from the gauge invariance of the gravitational theory, which in this formalism is just the invariance under general coordinate transformations (or, in the active point of view, diffeomorphisms). To see this we need an explicit formula for the infinitesimal variation of the metric. Let \( x' = x + \epsilon(x) \), with \( \epsilon \) a vectorfields that tends to zero at infinity, be an infinitesimal coordinate transformation. The variation of the metric is given by (3.5.12), one finds that the variation of the metric is

\[
\delta g^\mu\nu(x) = g'^\mu(x') - g^\mu\nu(x) .
\] (9.1.5)

The formula for the transformation of \( g \) is given by (8.4.2). To first order in \( \epsilon \), \( \frac{\partial x^\nu}{\partial x'^\sigma} = \delta^\nu_\sigma + \partial_\nu \epsilon^\sigma \) and furthermore, \( g'^\mu\nu(x') = g^\mu\nu(x') + \epsilon^\nu \partial_\lambda g^\mu\nu \). From here, using (3.5.12), one finds that the variation of the metric is given by the Lie derivative:

\[
\delta g^\mu\nu(x) = \epsilon^\lambda \partial_\lambda g^\mu\nu + \partial_\mu \epsilon^\nu g^\nu\rho + \partial_\nu \epsilon^\rho g^\rho\mu = \mathcal{L}_\epsilon g^\mu\nu .
\] (9.1.6)

This result, which has been derived for a covariant symmetric tensor, holds for any tensor: the variation under an infinitesimal diffeomorphism is given by the Lie derivative. In the second and third term in (9.1.6) one

\(^{(1)}\) A local gauge transformation, also called a gauge transformation of the second kind, is supposed to represent physically an unobservable operation and therefore has to tend to zero at infinity in order not to affect the values of the physically observable charges.
can take the metric tensor under the derivative; in this way one generates two more terms with derivatives of the metric that conspire with the first to reconstruct a Christoffel symbol. One can then rewrite the variation in the more useful form

\[ \mathcal{L}_e g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu . \]  

(9.1.7)

Here \( \nabla \) is the Levi–Civita covariant derivative. One can now write

\[ 0 = \delta_S \text{matter} = \int d^4x \left( \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S_{\text{matter}}}{\delta \Phi} \delta \Phi \right) . \]

The second term vanishes upon using the equation of motion of matter, while for the first term, using (9.1.4), (9.1.7), and the symmetry of \( T^\mu\nu \),

\[ 0 = \delta_S \text{matter} = \frac{1}{2} \int d^4x \sqrt{|g|} T^\mu\nu (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu) \]

\[ = \int d^4x \sqrt{|g|} T^\mu\nu \nabla_\mu \epsilon_\nu = - \int d^4x \sqrt{|g|} \epsilon_\nu \nabla_\mu T^\mu\nu . \]

(9.1.8)

Since \( \epsilon \) is arbitrary, there follows

\[ \nabla_\mu T^\mu\nu = 0 . \]

(9.1.9)

When (9.1.4) is applied to specific cases, one has to understand the behavior of functional derivatives under coordinate transformations. The basic rule of functional differentiation for fields on a \( n \) dimensional manifold is

\[ \frac{\delta \varphi(x)}{\delta \varphi(y)} = \tilde{\delta}^{(n)}(x - y) , \]

(9.1.10)

where \( \tilde{\delta}^{(n)}(x - y) = \prod_{\mu=1}^{n} \delta(x^\mu - y^\mu) \) and \( \delta(x^\mu - y^\mu) \) are the usual Dirac delta “functions”. Since \( \delta(x^\mu - y^\mu) \) is zero for \( x \neq y \), it is enough to consider the case when the points \( x \) and \( y \) are within the domain of a chart. The usual formal property \( \int d^n x \tilde{\delta}^{(n)}(x - y) = 1 \), means that \( \tilde{\delta}^{(n)}(x - y) \) can be interpreted as a scalar density of weight one. It is sometimes convenient to define a scalar delta function

\[ \delta^{(n)}(x, y) = \frac{1}{\sqrt{|g|}} \tilde{\delta}^{(n)}(x - y) , \]

(9.1.11)

such that \( \int d^n x \sqrt{|g|} \phi(x) \delta^{(n)}(x, y) = \phi(y) \).

With these definitions we can now tackle some specific cases. From the action (8.2.9) one obtains immediately

\[ T^\mu\nu(x) = m \int d\tau u^\mu(\tau)u^\nu(\tau)\delta^{(4)}(x, x(\tau)) , \]

(9.1.12)

where \( x^\mu(\tau) \) are the coordinates of the worldline of the particle.

By taking the continuum limit of a large number of particles, one finds that the energy-momentum tensor of a collisionless fluid (or “dust”) is \( T_{\mu\nu} = \rho u^\mu u^\nu \). For a perfect fluid with pressure \( p \) it is

\[ T^\mu\nu = (\rho + p)u^\mu u^\nu + p g_{\mu\nu} . \]

(9.1.13)

This can be shown by going to a reference system where the fluid is instantaneously at rest. The relation giving the pressure as a function of density is called the equation of state. For ordinary gases it has the form \( p = w \rho \), with \( 0 \leq w \leq 1/3 \), where the case \( w = 1/3 \) corresponds to the ultrarelativistic case.

For the applications to field theories one has to use the formula

\[ \delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} , \]

(9.1.14)

which can be easily proven by writing the determinant as the product of the eigenvalues of \( g \). Also, one has to use

\[ \delta g^{\mu\nu} = -g^{\mu\sigma} g^{\nu\rho} \delta g_{\rho\sigma} , \]

(9.1.15)
which follows from varying the equation \( g^{\mu \rho} g_{\rho \nu} = \delta^\mu_\nu \). With these formulae, one finds in the case of a scalar field with action (10.4.4):
\[
T^\rho_\sigma = \nabla^\rho \phi \nabla_\sigma \phi - \frac{1}{2} g^{\rho \sigma} (\nabla^\mu \phi \nabla_\mu \phi + m^2 \phi^2) .
\] (9.1.16)
This energy–momentum tensor is identical to the one obtained from Noether’s theorem. Varying the action (10.4.5) for the electromagnetic field with respect to the metric one obtains:
\[
T^\mu_\nu = F^{\mu \rho} F^\nu_\rho - \frac{1}{4} g^{\mu \nu} F^\rho_\sigma F^\rho_{\sigma} .
\] (9.1.17)
This energy momentum tensor agrees with the symmetric and gauge–invariant electromagnetic energy–momentum tensor obtained by “improving” the canonical one. Note that it is traceless: \( g^{\mu \nu} T^\mu_\nu = 0 \). Since a gas of photons is made up of electromagnetic field, its energy-momentum tensor must be traceless too, which implies that \( w = 1/3 \), as stated above.

9.2 The weak static field limit

In order to arrive at the generally covariant field equations for the gravitational field, we will exploit the fact that for a weak, static gravitational field they must reduce to the Poisson equation
\[
\Delta \Phi = 4\pi G \rho .
\] (9.2.1)
In the previous section we have discussed the tensor that contains \( \rho \), and thus must appear on the r.h.s. of the new equation. We now begin to discuss the tensor that must appear on the l.h.s.. The first priority is to determine which tensorial structure contains the gravitational potential \( \Phi \). To this end we consider the motion of a particle in a weak, static gravitational field. The metric will be of the form
\[
g^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu} ,
\]
with \( h^{\mu \nu} \ll 1 \) and \( \partial_t h^{\mu \nu} = 0 \). (We use here a cartesian coordinate system were the coordinates have dimension of length, and \( c = 1 \).) Since the field is weak, the velocity of the particles moving under the influence of the gravitational field will be small. Thus, we can approximate the proper time of the particle with the coordinate time:
\[
d\tau^2 = -g^{\mu \nu} dx^\mu dx^\nu = dt^2 \left( 1 - \left( \frac{dx^i}{dt} \right)^2 \right) - h^{\mu \nu} dx^\mu dx^\nu = dt^2 + O(v^2) + O(h)
\]
According to the discussion in section 10.2, a particle moving only under the influence of gravity follows the geodesic equation (10.2.7), which in the present case can be approximated as follows:
\[
\frac{d^2 x^i}{dt^2} \approx -\Gamma^i_0^\mu \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = -\Gamma^i_0^0 = 1 \frac{1}{2} \partial_i h^0_0 .
\]
Comparing to Newton’s equation \( \frac{d^2 x^i}{dt^2} = -\partial_i \Phi \) we get that \( h^0_0 = -2\Phi + K \), where \( K \) is a constant of integration. Since \( \Phi \to 0 \) far from a gravitating body, we must have \( K = 0 \). Then we obtain, for a weak static gravitational field
\[
g^0_0 = -1 - 2\Phi .
\] (9.2.2)
This agrees with equation (1.4.5) for the flow of proper time in a gravitational field. We also find
\[
\Gamma^i_0_0 = \partial_i \Phi .
\] (9.2.3)
Thus we see once more that the gravitational (as well as the inertial) forces are represented by the Christoffel symbols while the metric corresponds to the potential.

For later use, we also observe that under the assumptions made above, the 00 component of the Ricci tensor is
\[
R^i_0_0 = R^i_0_0^0 = \partial_i \Gamma^0_0 + O(h^2) \approx \Delta \Phi .
\] (9.2.4)
9.3 Einstein’s Equations

By analyzing the effect of gravity on matter, we have come to the conclusion that the gravitational field can be described by the metric $g_{\mu\nu}$. We now seek equations for the gravitational field of the form

$$K_{\mu\nu} = \kappa T_{\mu\nu} ,$$

(9.3.1)

where $K$ is a tensor containing at most two derivatives of the metric. The most natural candidate would be $\nabla_\rho \nabla^\rho g_{\mu\nu}$; however, for the sake of simplicity, we want to write the theory in terms of the metric alone, without treating the connection as an independent variable. Then, the connection must be the Levi–Civita connection, for which $\nabla_\rho g_{\mu\nu} = 0$.

Other tensors containing two derivatives of the metric can be obtained from the curvature of the Levi–Civita connection. For example, the Ricci tensor is a symmetric tensor with two indices formed from the metric and its first and second derivatives. This is still not the correct choice, however. To understand why consider again Maxwell’s equations

$$\partial_\mu F^{\mu\nu} = j^\nu .$$

The current on the r.h.s. is conserved when the field equations of the matter fields are satisfied. The l.h.s. is identically conserved due to symmetry reasons (2). The same must hold for gravity: the conservation of the l.h.s. of (9.3.1) must be an identity for the equations to be consistent.

Let us write the Bianchi identity (6.5.4) for the Levi Civita connection in the metric gauge:

$$\nabla_\lambda R_{\mu\nu}^{\rho\sigma} + \nabla_\mu R_{\nu\lambda}^{\rho\sigma} + \nabla_\nu R_{\lambda\mu}^{\rho\sigma} = 0$$

By contracting this identity twice, on the pairs of indices $(\mu, \rho)$ and $(\nu, \sigma)$, we find

$$\nabla_\mu \left( R^{\mu\nu} - \frac{1}{2} \delta^\mu_\nu R \right) = 0 .$$

(9.3.2)

The most general form of the tensor $K$ is therefore $K_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$, where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} \delta^\mu_\nu R$$

(9.3.3)

is called Einstein’s tensor and $\Lambda$ is a constant with dimensions of mass squared. The last point that need to be addressed is the value of the constant $\kappa$. To this end we set $\Lambda = 0$, trace the equation (9.3.1): $G^{\mu\mu} = -R = \kappa T^{\mu\mu}$ and use this back in (9.3.1) to obtain the equivalent equation

$$R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda \right) .$$

(9.3.4)

Consider the 00 component of this equation. In a static weak field $p \ll \rho$ and we have $T^\lambda_\lambda = T^0_0 = -\rho$, so

$$R_{00} = \frac{1}{2} \kappa T_{00} ,$$

which using (9.2.4) becomes

$$\Delta \Phi = \frac{1}{2} \kappa \rho .$$

This agrees with the Poisson equation (9.2.1) if we identify $\kappa = 8\pi G$. Einstein’s equations for the gravitational field, with $\Lambda = 0$, read therefore

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} .$$

(9.3.5)

(2) In nonabelian gauge theories $\nabla_\mu \nabla_\nu F^{\mu\nu} = \frac{1}{2} \left[ \nabla_\mu, \nabla_\nu \right] F^{\mu\nu} = \frac{1}{2} \left[ F^{\mu\nu} , F^{\mu\nu} \right] = 0$.  

4
As noted above, covariance permits an additional term on the l.h.s.:

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \, .$$  \hfill (9.3.6)

Note that the $\Lambda$–term could be moved to the r.h.s. of the equation where it could be interpreted as an “energy–momentum tensor of the vacuum” $T^{\text{vac}}_{\mu\nu} = -\Lambda g_{\mu\nu}$. This energy momentum tensor has the form of a perfect fluid with energy density and pressure

$$\rho_{\text{vac}} = \frac{\Lambda}{8\pi G} \, ; \quad p_{\text{vac}} = -\frac{\Lambda}{8\pi G} \, .$$  \hfill (9.3.7)

Note that this vacuum energy density is positive if $\Lambda$ is positive and the equation of state is $p = -\rho$. Removing a trace as before, equation (9.3.6) becomes

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda} \right) \, .$$

If the energy–momentum tensor of matter vanishes, and with $\Lambda = 0$, Einstein’s equations reduce to

$$R_{\mu\nu} = 0 \, ,$$  \hfill (9.3.8)

and admit the Minkowski metric as a solution. Note that this does not imply that spacetime is flat: the Weyl tensor (and hence the Riemann tensor) can still be nonzero. We shall see that this allows the existence of gravitational waves, in dimensions $n \geq 4$. A space satisfying (9.3.8) is said to be Ricci–flat. With $\Lambda \neq 0$ equations (9.3.6) without matter imply, in $n$ dimensions,

$$R_{\mu\nu} = \frac{2}{n-2} \Lambda g_{\mu\nu} \, .$$  \hfill (9.3.9)

A space whose Ricci tensor is proportional to the metric is called an Einstein space. Clearly, flat space cannot be a solution anymore. The typical curvature radius of spacetime in the absence of matter is $\Lambda^{-1/2}$. From the fact that the Newtonian potential works so well at planetary scales, one immediately obtains a bound $\Lambda \ll \ell^{-2}$, where $\ell$ is a typical radius of planetary orbits. The corresponding mass scale is therefore extremely small in atomic units. Given this bound, the new term can have an influence only on cosmological scales. The constant $\Lambda$ is called the cosmological constant.

### 9.4 The Hilbert action

The variational principle that gives rise to Einstein’s equations was found by Hilbert. Including also a cosmological term, it is given simply by

$$S_H(g) = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left( R - 2\Lambda \right) \, .$$  \hfill (9.4.1)

Under a variation of the metric $\delta g_{\mu\nu}$, the Christoffel symbols change by

$$\delta \Gamma^\mu_{\lambda \nu} = \frac{1}{2} g^{\mu\tau} \left( \nabla_\lambda \delta g_{\tau\nu} + \nabla_\nu \delta g_{\tau\lambda} - \nabla_\tau \delta g_{\lambda\nu} \right) \, .$$  \hfill (9.4.2)

The Riemann tensor then changes by

$$\delta R^{\rho \sigma}_{\mu \nu} = \nabla_\mu \delta \Gamma^\rho_{\nu \sigma} - \nabla_{\nu} \delta \Gamma^\rho_{\mu \sigma} \, .$$  \hfill (9.4.3)

The trace of the variation of the Ricci tensor is a total derivative:

$$g^{\mu\sigma} \delta R_{\mu\sigma} = \nabla_\mu \delta \Gamma_{\nu \mu} - \nabla_{\nu} \delta \Gamma_{\mu \nu} = (g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\rho\sigma}) \nabla_\mu \nabla_\nu \delta g_{\rho\sigma} \, .$$  \hfill (9.4.4)
Now consider the total action of gravity and matter $S_H + S_{\text{matter}}$. Using (9.1.4), (9.1.10), (9.1.11) and neglecting the total derivative (9.4.4) we obtain

$$\delta(S_H + S_{\text{matter}}) = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \delta g_{\mu\nu} \left[ \frac{1}{2} g^{\mu\nu} (R - 2\Lambda) - R^\mu_\nu \right] + \frac{1}{2} \int d^4x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu}.$$ 

Requiring stationarity of the total action implies equation (9.3.6).

### 9.5 Structure of Einstein’s equations

In this section we restrict ourselves to four dimensional spacetimes with Lorentzian signature. Einstein’s equations (11.3.4) or (11.3.5) are a system of ten second order, nonpolynomial semilinear partial differential equations for the ten unknowns $g_{\mu\nu}$. They are nonpolynomial because they contain, for example, the inverse of the metric. They are semilinear in the sense that the second derivatives appear at most linearly. This can be seen directly inserting formula (7.5.5) for the Christoffel symbols in formula (7.6.1) for the Riemann tensor. The part of the Riemann tensor that contains second derivatives is
can be seen directly inserting formula (7.5.5) for the Christoffel symbols in formula (7.6.1) for the Riemann tensor. They are semilinear in the sense that the second derivatives appear at most linearly. This can be seen directly inserting formula (7.5.5) for the Christoffel symbols in formula (7.6.1) for the Riemann tensor. The part of the Riemann tensor that contains second derivatives is

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\mu\sigma} \right) + \text{first derivatives},$$

so contracting two indices to obtain the Ricci tensor and then subtracting a trace part to obtain the Einstein tensor we find

$$G_{\mu\nu} = \frac{1}{2} g^{\mu\sigma} \left( \partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\mu\sigma} \right) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \left( \partial_\alpha \partial_\beta g_{\gamma\delta} - \partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} + \partial_\beta \partial_\gamma g_{\alpha\delta} \right) + \text{first derivatives}$$

which is indeed linear in second derivatives. Since there are as many equations as unknowns one might expect that given initial conditions, the solution is uniquely determined. However, not all ten equations are functionally independent, because they are related by the four differential identities

$$\nabla_\mu (G^{\mu\nu} - 8\pi G T^{\mu\nu}) = 0.$$ 

These are just a consequence of the gauge invariance of the action, as shown in (9.1.8) (by gauge invariance we mean here the invariance under coordinate transformations or equivalently under diffeomorphisms). The equations fail to determine the solution uniquely because if $g_{\mu\nu}$ is a solution, also any metric obtained from $g_{\mu\nu}$ by a gauge transformation must be a solution. In order to select one solution among all these equivalent solutions one has to impose four gauge conditions. For example one can use the harmonic coordinate condition:

$$\Delta x^\mu = 0,$$

where $\Delta$ is the Laplacian on scalars. Writing the Laplacian in the form $\nabla^\nu \nabla_\mu$, the gauge condition can also be written

$$0 = g^{\lambda\nu} \Gamma_\lambda^\mu = -\frac{1}{\sqrt{|g|}} \partial_\nu (\sqrt{|g|} g^{\mu\nu}),$$

where we have used (8.4.2). Alternatively one can arrive at the same formula writing the Laplacian in the form (7.2.14) and specializing to the case $f = x^\mu$.

These issues become particularly clear when one tries to set up the initial value problem, for example in view of solving the equations numerically with a computer. Since the equations are of second order, one would have to give the metric and its first time derivative at time $t = 0$ and then the equations should determine the metric and its first time derivative at any subsequent time. For this to happen, the equations should contain second time derivatives of all ten components of the metric. However, four out of ten components of Einstein’s tensor contain only first time derivatives. To see this, contract two indices in (9.5.1) and split all contracted indices into their time and space values $(0, i)$. Then one sees that

$$R_{00} = -\frac{1}{2} g^{ij} \partial_0^2 g_{ij} + \dot{R}_{00}, \quad R_{0k} = \frac{1}{2} g^{ij} \partial_0^2 g_{ik} + \dot{R}_{0k}, \quad R_{ij} = -\frac{1}{2} g^{00} \partial_0^2 g_{ij} + \dot{R}_{ij},$$

where
and
\[ R = (g^{0i}g^{0j} - g^{00}g^{ij})\partial^2_{ij} + \text{first derivatives} \] (9.5.7)
where \( \dot{R}_{ij} \) (not a tensor!) are the remaining terms, depending at most on first time derivatives of the metric. From here one finds that
\[ G_{0}^{0} = \frac{1}{2}(g^{00}\ddot{R}_{00} - g^{ij}\dddot{R}_{ij}) \quad ; \quad G_{i}^{0} = g^{00}\ddot{R}_{i0} + g^{ij}\dddot{R}_{ij} \] (9.5.8)
i.e. these components of \( G_{\mu}^{\nu} \) do not contain second time derivatives of the metric. (This property does not hold for the covariant components \( G_{\mu0} \). However, when indices of a tensorial system of equations are raised and lowered, one is merely taking linear combinations of the equations. It can be shown that the components \( G_{\mu}^{0} \) are proportional to the linear combinations \( n^{\nu}G_{\mu\nu} \), where \( n \) is the unit normal to the surfaces of constant time. See exercise 9.5.1 below.) So we see that the four equations
\[ G_{\mu}^{0} = 8\pi GT_{\mu}^{0} \] (9.5.9)
are not dynamical equations but rather constraints on the initial data.

On the other hand, \( G_{ij} \) contains second time derivatives of \( g_{ij} \), so the six equations
\[ G_{ij} = 8\pi GT_{ij} \] (9.5.10)
can be used to determine \( \partial^2_{ij} \). The second time derivatives of the metric components \( g_{ij} \) must be determined by imposing four gauge conditions. For example taking the time derivative of the harmonic coordinate gauge condition one has
\[ \partial^2_{\mu}(\sqrt{|g|g^{\mu\nu}}) = -\partial_\mu\partial_\nu(\sqrt{|g|g^{\mu\nu}}) \]
and from this, together with the space components of Einstein’s equations, one can determine all second time derivatives of the metric.

When the four constraint equations are imposed on the initial data (i.e. at time \( t = 0 \)), they are automatically satisfied at all times. We prove this in vacuum (\( T_{\mu\nu} = 0 \)). The Bianchi identity implies that
\[ \partial_\mu G_{\mu}^{0} = -\Gamma_{0}^{0} g_{\mu}^{0} - \Gamma_{0}^{0 \, k} G_{\mu}^{k} + \Gamma_{0}^{\lambda} \mu G_{\lambda}^{0} - \nabla_{k} G_{\mu}^{k} \] (9.5.12)
The first and third term on the r.h.s. are already proportional to the constraint equations. When the dynamical equation \( \dot{R}_{ij} = 0 \) are satisfied,
\[ G_{0}^{0} = \frac{1}{2}g^{00}R_{00} \quad ; \quad G_{i}^{0} = g^{0i}R_{00} \quad ; \quad G_{0}^{i} = g^{00}R_{i0} + g^{ij}R_{0j} \quad ; \quad G_{i}^{j} = g^{ij}R_{00} - \frac{1}{2}g^{ij}(g^{00}R_{00} + 2g^{0k}R_{0k}) \] .
From here one can express the components \( G_{\mu}^{k} \) as linear combinations of the components \( G_{\mu}^{0} \) with coefficients involving only the metric. Therefore the r.h.s. of (9.5.12) is a linear combination of \( G_{\mu}^{0} \) and \( \partial_\mu G_{\mu}^{0} \).

Exercise 9.5.1. Assume that \( M \) can be foliated by timelike surfaces \( \Sigma_t \) labelled by a time coordinate \( t \), with coordinates \( x^i \) in the surface \( \Sigma \). We denote \( (\partial_t, \partial_i) \) a natural basis for vectors. Let \( h(t) = h_{ij}dx^i dx^j \) be the induced metric in \( \Sigma_t \) (this is called the first fundamental form of the submanifold \( \Sigma_t \) and let \( u^i \) be the unit normal to the surface \( t=\)constant. Decompose the basis vector \( \partial_i \) into its normal and tangent components as follows: \( \partial_i = N^i + N^i \partial_i \). The function \( N \) is called the “lapse” and the three-vector \( N^i \) is called the “shift”. This can be inverted to give \( n = \frac{1}{N}(\partial_i - N^i \partial_i) \). From this, and the relations \( g(n, n) = -1 \) and \( g(n, \partial_i) = 0 \), one can read off the components of the metric:
\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + N^iN_i & N_i h_{ij} \\ N_i h_{ij} & h_{ij} \end{pmatrix} \quad ; \quad g^{\mu\nu} = \begin{pmatrix} -\frac{N^i}{N} & \frac{N^j}{N} \\ \frac{N^i}{N} & h^{ij} - \frac{N^iN^j}{N} \end{pmatrix} \]
where space indices are raised and lowered with \( h \). Using these relations show that the components \( G_{\mu}^{0} \) are proportional to \( n^{\nu}G_{\mu\nu} \), as stated in the text.

(1) This discussion can be formulated also in a Hamiltonian formalism, see R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. 117, 1595 - 1602 (1960).
9.6 The linearized Einstein equations

Consider fluctuations of the metric around a background field $\bar{g}_{\mu\nu}$: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where we write $\delta g_{\mu\nu} = h_{\mu\nu}$. (1) The first variation of the Christoffel symbols is

$$\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\tau} \left( \nabla_\mu h_{\tau\nu} + \nabla_\nu h_{\tau\mu} - \nabla_\tau h_{\mu\nu} \right).$$

(9.6.1)

Here $\nabla$ denotes the covariant derivative of $\bar{g}$, and $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. Indexes are raised and lowered with the background metric. From here one gets the first variations of the Ricci tensor and of the Ricci scalar:

$$\delta R_{\mu\nu} = \delta \left[ \nabla_\alpha \Gamma^\alpha_{\mu\nu} - \nabla_\mu \Gamma^\alpha_{\alpha\nu} - \nabla_\nu \Gamma^\alpha_{\mu\alpha} \right].$$

(9.6.2)

$$= \frac{1}{2} \left[ - \nabla^2 h_{\mu\nu} + \nabla^\rho \nabla_\rho h_{\mu\nu} + \nabla_\mu \nabla_\rho h_{\rho\nu} - \nabla_\nu \nabla_\rho h_{\mu\rho} \right].$$

In the rest of this section we restrict ourselves to the case of a flat background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\Lambda = 0$. The linearized Einstein equations read then:

$$\partial^2 h_{\mu\nu} - \partial_\rho \partial^\rho h_{\mu\nu} - \partial_\nu \partial^\mu h_{\rho\mu} + \eta_{\mu\nu} \partial_\rho \partial^\rho h + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial^2 h = -16\pi G T_{\mu\nu}.$$

(9.6.3)

It is convenient to define a bar operation on symmetric tensors: in four dimensions,

$$\bar{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} t_{\rho\sigma}.$$

It has the property that $\bar{t} = t$. Then (9.6.3) can be written more compactly as

$$\partial^2 \bar{h}_{\mu\nu} - \partial_\rho \partial^\rho \bar{h}_{\mu\nu} - \partial_\nu \partial^\mu \bar{h}_{\rho\mu} + \eta_{\mu\nu} \partial_\rho \partial^\rho \bar{h} = -16\pi G T_{\mu\nu}.$$

(9.6.4)

The propagator for the linearized perturbations is obtained by inverting the operator that appears in this equation. However, this operator is not invertible: it has a large kernel consisting of fluctuations of the form

$$h_{\mu\nu} = \partial_\rho \epsilon^\rho + \partial_\sigma \epsilon_\mu,$$

(9.6.5)

or equivalently

$$\bar{h}_{\mu\nu} = \partial_\rho \epsilon^\rho + \partial_\sigma \epsilon_\mu - \eta_{\mu\nu} \partial_\chi \epsilon^\lambda.$$

(9.6.6)

This is seen by simply inserting (9.6.6) in the l.h.s. of (9.6.4) and observing that all the terms cancel. The fluctuations of the form (9.6.6) are simply gauge transformations of the background metric and the existence of the kernel is a consequence of the gauge invariance of the action.

As usual, this difficulty is circumvented by imposing a gauge condition. The operator is then invertible on the subspace of fluctuations that satisfy the gauge condition. We have introduced the harmonic coordinate gauge in (9.5.4-5). The Minkowski metric, written in Cartesian coordinates, certainly satisfies this condition. Varying (9.5.5) with respect to $g$ we see that the fluctuating metric will satisfy the gauge condition provided

$$\partial_\mu h^{\mu\nu} = 0.$$

(9.6.7)

This is also called the de Donder gauge condition. Given a fluctuation $h_{\mu\nu}$ which does not satisfy this condition, one looks for an infinitesimal gauge transformation $\epsilon_\mu$ such that $h_{\mu\nu} + \partial_\rho \epsilon^\rho + \partial_\nu \epsilon_\mu$ satisfies it. For this, $\epsilon$ must satisfy the equation

$$\partial^2 \epsilon_\nu = - \partial_\rho \bar{h}^{\rho\nu}.$$

(9.6.8)

(1) One has to pay attention to the fact that the variation of the inverse metric is $\delta g^{\mu\nu} = - h^{\mu\nu} + O(h^2)$, where the indices on $h$ are raised with $\bar{g}$.

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This equation always admits a solution. In fact, the solution is determined only up to a solution of the homogeneous equation
\[ \partial^2 \epsilon_\nu = 0 , \] (9.6.9)
indicating that the gauge condition (9.6.7) leaves some residual gauge freedom that has to be fixed separately.

To summarize then, the linearized fluctuations of the metric around flat space, in the harmonic coordinate gauge, satisfy the simple equation
\[ \partial^2 \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu} . \] (9.6.10)
We will now discuss the solutions of this equation in vacuo.

### 9.7 Plane waves

Taking the bar of equation (9.6.10), we find in vacuo
\[ \partial^2 h^{\mu\nu} = 0 , \] (9.7.1)
whose solution can be Fourier expanded
\[ h^{\mu\nu}(x) = \int \frac{dk}{(2\pi)^2} \left( \Pi^{\mu\nu}_\epsilon e^{ik_\nu x^\nu} + \Pi^{\mu\nu}_* e^{-ik_\nu x^\nu} \right) . \] (9.7.2)
The polarization tensors \( \Pi^{\mu\nu}_\epsilon \) are complex constants. The wave equation requires that \( k^2 = 0 \) and the gauge condition requires that \( k^\mu \Pi^{\mu\nu}_\epsilon = 0 \). The solutions of (9.6.9) are the residual gauge freedom. They can be expanded in the form
\[ \epsilon_\mu(x) = \int \frac{dk}{(2\pi)^2} \left( \epsilon_\mu e^{ik_\nu x^\nu} + \epsilon^*_\mu e^{-ik_\nu x^\nu} \right) . \] (9.7.3)
and we can exploit this additional freedom to impose four additional conditions on \( \Pi \). Under a transformation (9.7.3), \( \Pi \) changes into
\[ \Pi'_\mu = \Pi_\mu + i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu) \] (9.7.4)
We can choose the constant vector \( \epsilon_\mu \) such that \( U^\mu \Pi'_\mu = 0 \), for some constant vector \( U \). These are only three independent equations since \( k^\mu \Pi^{\mu\nu}_\epsilon U^{\nu} \) is invariant under the residual gauge transformations. We can impose one last condition, and we require that \( \Pi \) be traceless. This can be achieved since the trace of \( \Pi \) changes by \(-2ik \cdot \epsilon \). Since, \( \Pi \) is traceless, \( \bar{\Pi} = \Pi \), so we can summarize the gauge conditions as follows
\[ \Pi^\lambda_\lambda = 0 ; \quad k^\mu \Pi^{\mu\nu}_\epsilon = 0 ; \quad U^\mu \Pi_{\mu\nu} = 0 . \] (9.7.5)
Let us consider for example a plane wave propagating in the \( z \) direction. The wave vector has components \( k^\mu = (k,0,0,k) \); we choose \( U^\mu = (1,0,0,0) \). Then the conditions (9.7.5) imply that the polarization tensor has only two degrees of freedom which we call \( A \) and \( B \):
\[ \Pi^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \] (9.7.6)
The two remaining free parameters \( A \) and \( B \) correspond to the two physical degrees of freedom of the gravitational field, after all gauge freedom has been fixed. They are the two polarization states of the gravitational wave. A rotation by an angle \( \theta \) in the \((x,y)\) plane, given by the Lorentz transformation
\[ \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \]
transforms (9.7.6) into a matrix with the same form, but

\begin{align*}
A' &= A \cos 2\theta - B \sin 2\theta \\
B' &= A \sin 2\theta + B \cos 2\theta
\end{align*}

so that, defining \( e_\pm = A \pm iB \), \( e_\pm \mapsto e'_\pm = e_\pm e^{2i\theta} \). The coefficient in the exponent means that the gravitational waves have helicity 2. The line element of the plane wave propagating in the z direction, with \( A \)-polarization and frequency \( k \) is

\[ ds^2 = -dt^2 + (1 + 2A \sin(k(z - t)))dx^2 + (1 - 2A \sin(k(z - t)))dy^2 + dz^2, \]

thus the distance from the origin of points on the \( x \) axis and the distance from the origin of points on the \( y \) axis oscillate in time with a phase shift of \( \pi \). Using (9.7.7) we see that the polarization states \( A \) and \( B \) are equivalent up to a rotation by \( \pi/4 \). A ring of freely falling matter is deformed under the effect of such waves as shown in the following figure:

These elliptic deformations are reminiscent of the effect of tidal forces near the surface of the Earth, as discussed in section 1.3 and example 10.3.1. In fact they are the hallmark of the curvature tensor being of Weyl type, as befits a vacuum solution. It is worth emphasizing that local propagating gravitational degrees of freedom exist only in dimension \( n \geq 4 \). For \( n = 2, 3 \) the Riemann tensor is entirely determined by the Ricci tensor and therefore vanishes on shell, in vacuo.

We conclude with the observation that the Lagrangian for such a transverse plane wave is given just by the first term in (1.6.2):

\[ -\frac{1}{4} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu}. \]

This agrees with the result of the linearization of the Hilbert action (9.4.1), up to a factor \( 16\pi G \). The sign of this Lagrangian is such that the corresponding Hamiltonian is a sum of squares and thus positive. This criterion determines the overall sign of the Hilbert action.