11.1 Static spherically symmetric spacetimes

A metric is said to be stationary if it admits a timelike Killing vector $K$. It is said to be static if it is stationary and there exists a surface $\Sigma$ of codimension one which is everywhere orthogonal to $K$. In a neighborhood of $\Sigma$ every point $x$ belongs to a unique flow line of $K$. This line passes through a point $\bar{x} \in \Sigma$, and $x = \phi(\bar{x}, t)$, where $\phi$ is the flow of $K$ (section 3.6). We can use $t$ and the coordinates of $\bar{x}$ as coordinates for $x$. In these coordinates $K = \frac{\partial}{\partial t}$ and the metric is

$$
d s^2 = -V^2(x^1, x^2, x^3)dt^2 + h_{ij}(x^1, x^2, x^3)dx^i dx^j, \tag{11.1.1}
$$

where $V^2 = -g(K, K)$ and the condition of being static implies the absence of mixed terms $dt \, dx^i$.

A metric has spherical symmetry if it admits three spacelike Killing vectors satisfying the algebra of $SO(3)$, whose orbits are two dimensional spheres. The induced metric in the orbits is the standard metric in $S^2$, up to rescalings. The three Killing vectors are given explicitly in (5.4.2.1). Let $A$ be the area of an orbit computed with the induced metric. Neighboring orbits will generally have different areas. We can define a coordinate $r$ labelling different orbits by $r = \sqrt{A}$. It is called “area radius” because by definition, the area of an orbit is equal to $4\pi r^2$. One should not assume however that $r$ is a “radius” in the usual sense, nor that the topology of space is the topology of $R^3$.

If a spacetime is both static and spherically symmetric, the timelike Killing vector $K$ must commute with the generators of rotations. In $S^2$ there does not exist any vectorfield commuting with all generators of rotations, so $K$ must have zero components in the orbits of $SO(3)$. This implies that the orbits lie in the surfaces $t = const$. Then the metric must have the form

$$
d s^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{11.1.2}
$$

This is as far as one can get in general. We have reduced the ten independent components of the metric, which in general are functions of all the coordinates, to just two functions of a single coordinate. In order to determine these functions we have to insert the ansatz (11.1.2) into Einstein’s equations.

11.2 The Schwarzschild solution

We are interested in the gravitational field of a point particle. We tentatively assume that space is $R^3$ and that the particle lies at the origin of the coordinates, $r = 0$. Clearly for $r \neq 0$ there is no matter, so we have to solve Einstein’s equations in vacuum

$$
R_{\mu \nu} = 0. \tag{11.2.1}
$$

We thus need the components of the Ricci tensor of the metric (11.1.2). One way to compute them is to work out all the Christoffel symbols (9.2.5) and then all the components of the Riemann tensor (9.3.1). A contraction then yields the Ricci tensor (9.3.5). This is generally a rather tedious, albeit fully straightforward, procedure. It is sometimes convenient to use instead the vierbein gauge and to obtain the connection by solving the equations (9.2.1). In the vierbein gauge, from (9.1.3) and (9.1.12) we see that the components of the soldering form can be regarded as the components of an orthonormal frame for $g_{\mu \nu}$. In this gauge it is natural to decompose all tensor components in the orthonormal basis, but one can also choose to refer some components to an orthonormal basis and others to a natural basis. Recall that in section 6.5 we found it convenient to hide the Lie algebra indices and treat $A_{\mu}$ as a matrix–valued form. We are now going to do the opposite: we will hide the form indices and keep the algebra indices in sight. Thus, the Levi–Civita connection is a one form $\Gamma^a_{\ b} = \Gamma^a_{\ \mu b} dx^\mu$ and its curvature is $R^a_{\ \ b} = \frac{1}{2}R_{\mu \nu a b} dx^\mu \wedge dx^\nu$. The soldering form is $\theta^a = \theta^a_{\ \mu} dx^\mu$. Because of the metricity condition (9.2.1b), $\Gamma_{ab} = -\Gamma_{ba}$ (and $R_{ab} = -R_{ba}$). Then the condition of vanishing torsion (9.2.1a) can be written

$$
d \theta^a + \Gamma^a_{\ b} \wedge \theta^b = 0 \tag{11.2.2}
$$

and the definition of the curvature, equation (6.5.4), becomes

$$
R^a_{\ \ b} = d\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b}. \tag{11.2.3}
$$
With some practice, using the properties of differential forms can be much more efficient than working out the Christoffel symbols. We illustrate this by computing the Levi–Civita connection. The vierbein for the metric (11.1.2) is
\[ \theta^t = \sqrt{f} \, dt \; ; \; \theta^r = \sqrt{h} \, dr \; ; \; \theta^\theta = r \, d\theta \; ; \; \theta^\varphi = r \sin \theta \, d\varphi . \]

Then (11.2.2) becomes
\[ \frac{f'}{2\sqrt{f}} \, dr \wedge dt + \Gamma^t_r \wedge \sqrt{f} \, dr + \Gamma^t_\theta \wedge r \, d\theta + \Gamma^t_\varphi \wedge r \sin \theta \, d\varphi = 0 ; \quad (11.2.4a) \]
\[ \Gamma^r_t \wedge \sqrt{f} \, dt + \Gamma^r_\theta \wedge r \, d\theta + \Gamma^r_\varphi \wedge r \sin \theta \, d\varphi = 0 ; \quad (11.2.4b) \]
\[ dr \wedge d\theta + \Gamma^\theta_t \wedge \sqrt{f} \, dt + \Gamma^\theta_r \wedge \sqrt{h} \, dr + \Gamma^\theta_\varphi \wedge r \sin \theta \, d\varphi = 0 ; \quad (11.2.4c) \]
\[ dr \wedge \sin \theta \, d\varphi + r \cos \theta \, d\theta \wedge d\varphi + \Gamma^\varphi_\theta \wedge \sqrt{f} \, dt + \Gamma^\varphi_r \wedge \sqrt{h} \, dr + \Gamma^\varphi_\varphi \wedge r \, d\theta = 0 ; \quad (11.2.4d) \]

Since we know that the solution will be unique, we are free to make guesses about the coefficients \( \Gamma^{\alpha}_\beta \) and check them a posteriori. One reasonable guess is that the spherical coordinates do not mix with time, in the sense that \( \Gamma^t_\theta = 0 \) and \( \Gamma^t_\varphi = 0 \). Inserting in (11.2.4a) we find that
\[ \Gamma^t_r = -\Gamma^r_t = -A dr - \frac{f'}{2\sqrt{h}f} \, dt \]
where \( A \) is a function of \( r \). Using this in (11.2.4b), the latter becomes
\[ \sqrt{f} A \, dr \wedge dt + \Gamma^r_\theta \wedge r \, d\theta + \Gamma^r_\varphi \wedge r \sin \theta \, d\varphi = 0 \]
(11.2.5)

Since the first term does not contain \( d\theta \) or \( d\varphi \), we must have \( A = 0 \). Next we assume that
\[ \Gamma^r_\theta = F d\theta + G d\varphi ; \quad \Gamma^r_\varphi = H d\theta + L d\varphi . \]

which inserted in (11.2.5) yields \( G = H \sin \theta \). Inserting in (11.2.4c) gives
\[ dr \wedge d\theta - (F d\theta + G d\varphi) \wedge \sqrt{h} \, dr + \Gamma^\varphi_\theta \wedge r \sin \theta \, d\varphi = 0 ; \]

The coefficient of \( dr \wedge d\theta \) must vanish, and this implies that \( F = -\frac{h}{r \sin \theta} \). The remaining terms tell us that
\[ \Gamma^\theta_\varphi = -\frac{G \sqrt{h}}{r \sin \theta} \, dr + M d\varphi \]
for some function \( M \). Inserting all the above in (11.2.4d) gives
\[ \sin \theta \, dr \wedge d\varphi + r \cos \theta \, d\theta \wedge d\varphi - \left( \frac{G}{\sin \theta} \, d\theta + L d\varphi \right) \wedge \sqrt{h} \, dr + \left( \frac{G \sqrt{h}}{r \sin \theta} \, dr - M \, d\varphi \right) \wedge r \, d\theta = 0 \]

The coefficient of \( dr \wedge d\varphi \) must vanish, and this implies that \( L = -\frac{\sin \theta}{\sqrt{h}} \). The coefficient of \( d\theta \wedge d\varphi \) must vanish, and this implies that \( G = 0 \). The coefficient of \( d\theta \wedge d\varphi \) must vanish, and this implies that \( M = -\cos \theta \).

Altogether we have found, with remarkably little calculational effort,
\[ \Gamma^t_r = -\frac{f'}{2\sqrt{h}f} \, dt ; \quad \Gamma^r_\theta = 0 ; \quad \Gamma^r_\varphi = 0 ; \quad \Gamma^\theta_r = -\frac{1}{\sqrt{h}} \, d\theta ; \quad \Gamma^\theta_\varphi = -\frac{\sin \theta}{\sqrt{h}} \, d\varphi ; \quad \Gamma^\varphi_r = -\cos \theta \, d\varphi . \]

(11.2.6)

From here by using (11.2.3) and then extracting the coefficient of \( \theta^a \wedge \theta^b \) one reads off the independent vierbein components of the Riemann tensor (one per generator of the Lorentz algebra):
\[ R^a_{tbt} = \frac{1}{2r h f} \frac{d}{dr} \frac{f'}{\sqrt{hf}} ; \quad R^a_{t\theta t} = R^a_{t\varphi t} = \frac{f'}{2r h f} ; \quad R^a_{r \theta r} = R^a_{r \varphi r} = \frac{h'}{2r h^2} ; \quad R^a_{\theta \varphi \varphi} = \frac{1}{r^2} \left( 1 - \frac{1}{h} \right) . \]

(11.2.7)
Contracting the first and third index of $R_{abcd}$ with $\eta^{ac}$ one gets the vierbein components of the Ricci tensor. Only the diagonal components are nonzero, and the $\theta\theta$ and $\varphi\varphi$ are equal. Thus the independent Einstein equations in vacuum reduce to

$$R_{tt} = \frac{1}{2\sqrt{hf}} \frac{d}{dr} \sqrt{hf} \frac{f'}{rh} = 0 ,$$  

(11.2.8a)

$$R_{rr} = -\frac{1}{2\sqrt{hf}} \frac{d}{dr} \sqrt{hf} \frac{f'}{rh^2} + \frac{h'}{rh^2} = 0 ,$$  

(11.2.8b)

$$R_{\theta\theta} = R_{\varphi\varphi} = -\frac{f'}{2r^2h} + \frac{h'}{2r^2h^2} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right) = 0 ,$$  

(11.2.8c)

Summing the first two equations one obtains

$$\frac{f'}{f} + \frac{h'}{h} = 0$$

whose solution is $f = \frac{K}{h}$ for some integration constant $K$. One can use the freedom of rescaling $t \rightarrow t/\sqrt{K}$ to set $K = 1$. Reinserting in (11.2.8c) one obtains a single differential equation for $f$, which reduces to

$$\frac{d}{dr}(rf) = 1 .$$

The solution of this equation is $f = 1 + \frac{C}{r}$, where $C$ is a constant of integration. This constant can be determined by comparison with (9.2.2):

$$-\left( 1 + \frac{C}{r} \right) \rightarrow -1 + \frac{2MG}{r}$$

which implies $C = 2MG$. To summarize, the static spherically solution of Einstein’s equations in vacuum can be written in the form

$$ds^2 = -\left( 1 - \frac{2MG}{r} \right) dt^2 + \left( 1 - \frac{2MG}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$  

(11.2.9)

This is called the Schwarzschild solution in “standard” form. It has an singularity at $r = 0$, that one would have expected on the basis of our Newtonian understanding, but also a singularity at $r = 2MG$ that one could not have anticipated. We shall discuss the meaning if this (apparent) singularity later on.

Exercise 11.2.1. For $r > 2MG$ define a new radial coordinate $r'$ by $r = r' \left( 1 + \frac{MG}{2r} \right)^2$ (the transformation becomes singular at $r = 2MG$). Show that the metric assumes the “isotropic” form

$$ds^2 = -\left( 1 - \frac{MG}{r} \right) dt^2 + \left( 1 + \frac{MG}{2r} \right)^4 \left[ d\theta^2 + \varphi^2 (2 \sin^2 \theta d\varphi^2) \right]$$

The coordinate singularity now occurs at $r' = MG/2$.

Exercise 11.2.2. Construct harmonic coordinates for the Schwarzschild metric. (See Weinberg equation (8.2.15).)

Exercise 11.2.3. From (11.2.7), using the explicit solution for $f$ and $h$. compute the tetrad components of the Riemann tensor. Compare with the calculation in exercise 8.3.1. The components that were called $R_{0z0z}$ and $R_{0\theta\theta}$ correspond to $R_{\theta\theta\theta}$ and $R_{\varphi\varphi\varphi}$, while $R_{0z0z}$ corresponds to $R_{rrrr}$.

In the preceding derivation we have assumed that the body producing the gravitational field is a point particle. The gravitational fields that we can observe are all produced by macroscopic bodies such as planets or stars, which have a finite spatial extension. In Newton’s theory, Birkhoff’s theorem guarantees that the gravitational field outside a spherical body is the same as that produced by a point particle located in the
are parallel to \( K \) that \( k \). Using the Schwarzschild solution we find, a timelike Killing vector \( K \) two observers located at two fixed radii \( r \) shift/blueshift, that we made in section 1.4 based just on the equivalence principle. To this effect, imagine the surface. solutions for the interior of planets and stars and match them continuously to the Schwarzschild solution at the second observer? Let us call \( k \) with four–velocity \( u \) \( \sqrt{|g_{00}|} \) is constant along the null geodesic traced by the photon. The four-velocities of the two observers are parallel to \( K \) but have different norm: \( g(u, u) = -1 \) whereas \( g(K, K) = g_{00} \), therefore \( K^\mu = \sqrt{|g_{00}|} u^\mu \). Then we have \( k_\mu u^\mu \sqrt{|g_{00}|} = \text{constant} \). But \( k_\mu u^\mu \) is the frequency of the photon as measured by the observer with four–velocity \( u \) (section 10.6). Thus

\[
\omega_1 \sqrt{|g_{00}(r_1)|} = \omega_2 \sqrt{|g_{00}(r_2)|}
\]

Using the Schwarzschild solution we find,

\[
\frac{\omega_2}{\omega_1} = \frac{\sqrt{1 - \frac{2MG}{r_1}}}{\sqrt{1 - \frac{2MG}{r_2}}}.
\]  

(11.4.1)

We see that a red shift occurs when the photon travels outwards and a blueshift occurs when the photon travels inwards. In particular, when the second observer is at \( r_2 = \infty \) the frequency is red-shifted by a factor \( \sqrt{1 - \frac{2MG}{r_1}} \). This factor increases as \( r_1 \) decreases and becomes infinite when \( r_1 = 2MG \). Therefore \( r = 2MG \) is a surface of infinite redshift. As discussed in section 1.4, one can see the redshift as the energy loss of the photon as it climbs out from the potential well; then the infinite redshift means that a photon generated just outside \( r = 2MG \) must spend all its energy to climb the potential well. This suggests that a photon generated just inside \( r = 2MG \) will not be able to climb the potential well at all. We shall see later that this is indeed the case: anything that is inside the surface \( r = 2MG \) cannot escape, and for this reason this surface will appear to an external observer as a black hole.

When both \( r_1 \) and \( r_2 \) are much larger than the Schwarzschild radius \( 2MG \), we can approximate

\[
\frac{\omega_2}{\omega_1} \approx 1 - \frac{MG}{r_1} + \frac{MG}{r_2}.
\]  

(11.4.2)

and the fractional frequency shift is

\[
\frac{\omega_2 - \omega_1}{\omega_1} \approx MG \frac{r_1 - r_2}{r_1 r_2}.
\]  

(11.4.3)

It would be possible to measure this effect for photons coming from the sun, were it not for the fact that the thermal widening of the spectral lines is much greater than the predicted frequency shift. As mentioned in section 1.4, it has been possible to observe this effect directly in the laboratory using gamma ray photons.

11.3 Geodesics

The first experimental tests of Einstein’s theory came from accurate measurements of the motion of planets or light rays in the gravitational field of the Sun. This requires understanding the geodesics of the Schwarzschild metric.
The solution is greatly simplified by exploiting the symmetries of the problem. Due to the existence of four Killing vectors, there are four conserved quantities along any geodesic, which can be roughly associated to energy and angular momentum. As in Newtonian mechanics, two of the three conserved quantities associated to spherical symmetry imply that the motion occurs in a plane. Without loss of generality we can assume this to be the plane $\theta = \pi/2$. The remaining two Killing vectors are then $\partial_t$ and $\partial\phi$. Consider first the case of a massive particle. If we call $u$ the vector tangent to the geodesic (the four-velocity) we define $g(\partial_t, u) = -E$ and $g(\partial\phi, u) = L$. The four-momentum $p$ is related to $u$ by $p = mu$, so we find that the covariant components $p_0 = -mE$ and $p_\phi = mL$ are constant. The other components of the momentum are $p^\theta = 0$ (because the motion is in a plane) and $p^r = m\frac{dr}{d\tau}$. The equation $g(p, p) = -m^2$ then tells us that

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V_{\text{eff}}(r),$$

(11.3.1)

where

$$V_{\text{eff}}(r) = \left(1 - \frac{2MG}{r}\right) \left(1 + \frac{L^2}{r^2}\right).$$

(11.3.2)

For a massless particle $p_0 = -E$ and $p_\phi = L$ are constant. The previous argument leads again to equation (11.3.1) but now with

$$V_{\text{eff}}(r) = \left(1 - \frac{2MG}{r}\right) \frac{L^2}{r^2}.$$ 

(11.3.3)

In both cases the qualitative features of the orbits can be gleaned from these equations, considered as the energy conservation for a particle with mass 2 and total energy $E^2$, moving in the potential $V_{\text{eff}}$. The potential for a massive particle is plotted in the following figure for $L = 8, 6, 4, 2$ (from top to bottom), on two different scales.

The effective potential has a maximum and a minimum at

$$r_\pm = \frac{L^2}{2MG} \left(1 \pm \sqrt{1 - 12(MG)^2} \frac{L^2}{L^2}\right).$$

(11.3.4)

If $E^2$ is greater than the maximum, an incoming particle reaches $r = 0$, if it is less than the maximum, $r$ reaches a minimum value at the turning point. There are two circular orbit corresponding to the stationary points $r_+$ and $r_-$. The one at $r_+$ is stable, the one at $r_-$ unstable.

The stable orbit is given by $r(\tau) = r_+$. The function $\varphi(\tau)$ can be determined as follows:

$$\frac{d\varphi}{d\tau} = u^\varphi = \frac{1}{m} g^{\tau\varphi} p_\varphi = \frac{L}{r^2}.$$ 

(11.3.5)

Therefore $\varphi(\tau) = \frac{L}{r^2} \tau$ and the orbital period is $T = \int d\tau = \frac{2\pi^2}{L}$. Let us consider a small perturbation of this stable orbit. It will be given by a small radial oscillation with frequency

$$\omega_r^2 = \frac{1}{2} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_+} = \frac{MG(r_+-6MG)}{r_+^3(r_+-3MG)}.$$ 

(11.3.6)
On the other hand the frequency of the angular motion remains the same to first order. Using the identity 
\[ r^2 = \frac{L^2}{MG} (r - 3MG), \]
it can be written
\[
\omega^2 = \left( \frac{2\pi}{T} \right)^2 = \frac{L^2}{r^4} = \frac{MG}{r^2 (r - 3MG)}.
\] (11.3.7)

If the two frequencies were the same, then the orbits would closed curves, as happens in Newtonian gravity. But they are not the same: the ratio is
\[
\frac{\omega^2}{\omega_\phi^2} = 1 - \frac{6MG}{r_+}.
\] (11.3.8)

The difference \( \omega_P = \omega_\phi - \omega_r \) is called the precession. To first order it is given by
\[
\omega_P = 3 \frac{(MG)^{3/2}}{r_+^{5/2}}.
\] (11.3.9)

The precession of the perihelion of Mercury has been measured and is in accordance with the prediction.

Let us now consider null geodesics, the orbits of photons in the geometric optics approximation. The following figure shows the effective potential for \( L=8,6,4,2, \) from top to bottom.

![Potential diagram](image)

It has a maximum at \( r = 3MG \) and the value of the potential at the maximum is \( L^2/27MG \). It has a zero at \( r = 2MG \). There is only one circular orbit at \( r = 3MG \) and it is unstable. Every other null geodesic falls in one of the following two classes. If \( E^2 > L^2/27MG \), it comes in from \( r = \infty \) and ends at the singularity \( r = 0 \). If \( E^2 < L^2/27MG \), it comes in from \( r = \infty \), reaches a turning point and returns to \( r = \infty \). In both cases, the motion of the photon in the gravitational field of the particle in the origin can be viewed as a scattering problem. At large distances the geometry becomes Minkowskian so we can apply the standard notions of special relativity. The angular momentum \( L = pb \) where \( p = |\vec{p}| \) and \( b \) is the impact parameter, and \( E = p \). Thus the distinction between the two classes of orbits depends on whether \( b \) is larger or smaller than \( \sqrt{27MG} = 5.196MG \). If it is smaller it falls inside the horizon at \( r = 2MG \) and eventually reaches the singularity at \( r = 0 \). We can therefore say that the cross section for the capture of photons by the black hole is \( 27\pi M^2 G^2 \).

Consider now a photon that does escape to infinity. To calculate the scattering angle without integrating the equations of the geodesic, we can integrate \( \frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} \), where a dot indicates derivative with respect to an affine parameter. Using (11.3.1) and (11.3.5) we have
\[
\frac{d\phi}{dr} = \frac{L}{r^2} \left( E^2 - \left( 1 - \frac{2MG}{r} \right) \frac{L^2}{r^2} \right)^{-1/2} = \frac{1}{r \sqrt{\frac{L^2}{r^2} - 1 + \frac{2MG}{r}}}. \]

Therefore the scattering angle is
\[
\Delta \phi = 2 \int_{r_0}^{\infty} \frac{dr}{r} \frac{1}{r \sqrt{\frac{L^2}{r^2} - 1 + \frac{2MG}{r}}}, \] (11.3.10)

6
where \( r_0 \) is the turning point, defined by \( E^2 = V_\text{eff}(r_0) \). Using (11.3.3) it satisfies \( r_0^3 = b^2(r_0 - 2MG) \). Once again, observations are in agreement with the predicted deflection.

### 11.4 Resolving coordinate singularities

The theory of general relativity has certain peculiar features originating from its gauge symmetry which, in the metric gauge that we are using here, consists of coordinate transformations (or diffeomorphisms, when one uses the active point of view). A first consequence of this gauge symmetry is that when one finds a new solution, it is always possible that it is an already known solution written in a different coordinate system. There is no simple criterion that allows one to decide whether the new solution is really new, and one has to resort to detailed analyses on a case by case basis. A related consequence is that when one encounters a singularity it is not a priori clear whether it is a true physical singularity or the result of a bad choice of coordinates. A related point is this: when one set out to solve Einstein’s equations one does not know a priori what spacetime manifold it will describe. This will only emerge in the end, once one has understood the global properties of the solution.

All these features have analogs in Yang-Mills theories. When one finds a new solution, it may be an already known solution written in a different gauge. A singularity in the gauge potential may be due to a bad choice of local gauge and disappear when one chooses a different local gauge. And finally, when one sets out to solve the Yang-Mills equations, one does not know a priori what bundle they will describe (the base space and the fiber are given, but the geometry and even the topology of the total space are not given a priori).

Thus, having found the Schwarzschild solution (11.2.9) is not the end of the story. We still have to understand its global structure, and the nature of the singularities at \( r = 0 \) and \( r = 2MG \). But before addressing these issues in the next section, it will be convenient to study some simpler examples in two dimensions.

The first example is the following metric which a silly physicist may find solving Einstein’s equations in vacuum:

\[
\begin{align*}
    ds^2 &= -\frac{1}{t^4}dt^2 + dx^2.
\end{align*}
\]

It is singular at \( t = 0 \) and at \( t = \infty \); what is the nature of this singularity? A hint comes from the following observation. Consider the distance between two points located at \((t_1, x)\) and \((t_2, x)\), with the same value of \( x \). It is given by

\[
\int ds = \int_{t_1}^{t_2} \frac{dt'}{t'^2} = \frac{1}{t_2} - \frac{1}{t_1}.
\]

Thus, the distance from \( t_1 = 1 \) to \( t_2 = 0 \) is infinite, whereas the distance from \( t_1 = 1 \) to \( t_2 = \infty \) is one. Clearly the coordinate system does not reflect at all the metric relations in this spacetime. To make this more physical: curves of constant \( x \) are geodesics, and if one follows such a geodesic from \( t = 1 \) to \( t = \infty \) one arrives there in finite proper time. Where does one go next? When this happens one says that the geodesics are *incomplete*. Clearly the spacetime must continue beyond \( t = \infty \). The natural thing that comes to mind is to define \( t' = 1/t \), and indeed one has

\[
\begin{align*}
    ds^2 &= dt'^2 + dx^2.
\end{align*}
\]

A somewhat less trivial example is the Rindler metric

\[
\begin{align*}
    ds^2 &= -x^2dt^2 + dx^2,
\end{align*}
\]

with \(-\infty < t < \infty \) and \( 0 < x < \infty \). It is singular at \( x = 0 \). If one computes the curvature tensor one finds that it is actually zero, so this cannot be a physical singularity. Actually we know from section 1.4 that there must exist a coordinate system where the metric reduces to the Minkwskian form. But how do we find such a coordinate system? As in the previous example, a hint comes from studying geodesics. It will be sufficient to study null geodesics, which indicate the causal structure. In fact, we do not even need to know the parametrized form of the curves \((t(\lambda), x(\lambda))\): it will be enough to know \( x(t) \) or \( t(x) \). This is very simple: one just sets \( ds^2 = 0 \) and obtains the equation

\[
\frac{dt}{dx} = \pm \frac{1}{x},
\]
whose solutions are \( t = \pm \log x + \text{constant} \). They form a sort of grid, which suggests using the integrations constants as new coordinates. Define \( u = t - \log x \) and \( v = t + \log x \); \( u \) is constant on outgoing geodesics, \( v \) on incoming geodesics. The inverse coordinate transformation is \( t = (u + v)/2 \) and \( x = \exp((v - u)/2) \), and in the new coordinates

\[
ds^2 = -e^v - u \, du \, dv.
\]

The new coordinates range from \(-\infty\) to \(+\infty\) but still only cover the region \( x > 0 \). In fact the singularity at \( x = 0 \) corresponds to \( u \to \infty \) or \( v \to -\infty \).

Let us now ask ourselves if the null geodesics are complete or not. Instead of directly integrating the equation for the geodesics we use some shortcuts. The Rindler metric (11.4.4) has a timelike Killing vector \( \partial_t \). Calling \( k^\mu = \dot{x}^\mu \) the tangent vector to the geodesics, the quantity \( E = g(k, \partial_t) = x^2 \dot{t} \) is constant along the geodesics, by Noether’s theorem. From here we see that \( \lambda = x^2 \dot{t}/E \), and integrating on an outgoing null geodesic \( (u=\text{constant}) \) we find

\[
\lambda = \frac{1}{E} \int x^2(t) dt = \frac{1}{E} \int e^{v-u} dv = \frac{e^{-u}}{E} e^v,
\]

where we have set an integration constant to zero. So we see that \( e^v \) is an affine parameter on the outgoing geodesics, and by a similar argument \(-e^{-u}\) is an affine parameter on the incoming geodesics. Clearly this space is geodesically incomplete: an outgoing geodesic comes from \( v \to -\infty \), but its affine parameter there was finite, namely zero. One can continue such geodesics further in the past. Similarly an incoming geodesics arrives at \( u = \infty \) when the affine parameter is zero, and can be continued further into the future. These arguments show that the coordinates \( (u, v) \) cannot cover all the spacetime.

The transformation to flat coordinates is suggested by looking at (11.4.4): \( U = -e^{-u} \) and \( V = e^v \). In such coordinates \( ds^2 = -dU \, dV \), which is the Minkowski metric in null coordinates. In fact, defining further \( T = (U + V)/2 \) and \( X = (V - U)/2 \), the line element becomes \( ds^2 = -dT^2 + dX^2 \). The main point to stress is that the coordinate system \( (u, v) \) covers only the region \( U < 0 \) and \( V > 0 \), but there is no singularity in the metric at either \( U = 0 \) or \( V = 0 \), which correspond to the location of the apparent singularity in the \( (u, v) \) coordinates. Therefore spacetime can be extended beyond the original coordinate patch.

The complete transformation from the original Rindler coordinates to the Minkowskian coordinates is \( x = \sqrt{T^2 - X^2} \) and \( t = \text{arctanh}(T/X) \). Thus the lines \( t = \text{constant} \) correspond to straight lines through the origin in Minkowski space, and the lines \( x = \text{constant} \) correspond to hyperbolas. The following figure shows the Rindler coordinate patch and coordinate grid.

The following facts should be remarked. The lines \( x = \text{constant} \) are the worldlines of uniformly accelerated observers in Minkowski space, and the proper time of these observers is \( xt \). Therefore Rindler coordinates are somehow naturally well suited for such uniformly accelerated observers, that we shall call “Rindler observers”. A Rindler observer cannot be influenced by any event that occurs at \( U > 0 \) and cannot influence any event that occurs at \( V < 0 \). Therefore the boundary of the Rindler wedge (the line \( x = 0 \)) is an event horizon for the Rindler observer. The line \( V = 0, U < 0 \) is a past event horizon, meaning a surface across which the Rindler observer can see but can have no influence on. The line \( U = 0, V > 0 \) is a future
event horizon, meaning a surface across which the Rindler observer can never see but can have influence on. All the above is only true for an eternally accelerated observer: if the acceleration started at some finite time an lasted forever, then there would only be a future horizon and no past horizon, and if the acceleration had lasted forever in the past and stopped at some finite time, then there would only be a past horizon and no future horizon.

11.4 Kruskal coordinates

In order to appreciate the similarities between the Rindler metric and the Schwarzschild metric, let us discuss the acceleration four–vector along a worldline \( a = \nabla_a u \), where \( u \) is the four-velocity. The acceleration by definition vanishes on a geodesic, but here we will be interested in worldlines that are not geodesics.

In order to know \( a \) along some curve, in general one has to know the Christoffel symbols. However, we will only be interested in the case when the spacetime is static with a timelike Killing vector \( K \), and the four-velocity is everywhere parallel to the Killing vector. In other words we are interested in the case when the curve describes an observer at rest in the coordinates of (11.1.1). Then there is a simpler way to calculate \( a \). Let us define the positive scalar function \( V = -K_\lambda K^\lambda \). Since \( u \) is unit-normalized, \( u = K/\sqrt{V} \). Therefore

\[
a^\mu = \frac{K^\nu \nabla_\nu K_\mu}{V} - \frac{1}{2V^2} K_\mu K^\nu \nabla_\nu V.
\]

Using the Killing equation, it is easy to see that \( K^\nu \nabla_\nu V = 0 \). Therefore

\[
a^\mu = \frac{\nabla_\mu V}{2V}.
\]

In the case of a Rindler observer, \( V = -g_{00} = x^2 \) and therefore \( a = 1/x \). This confirms what we already knew, namely that the acceleration is constant on the lines \( x = \text{constant} \), and that it is inversely proportional to the distance of the observer from its own horizon. In particular, an observer at \( x = 0 \) would have infinite acceleration.

Compare with an observer at rest in Schwarzschild coordinates \((r, \theta, \varphi)\). Now \( V = 1 - \frac{2MG}{r} \), and the only component of the acceleration is \( a' = g'^{\nu} \partial_\nu V/2V = MG/r^2 \). The modulus of the acceleration is \( a = \sqrt{g_{rr}a'^2} = (1 - \frac{2MG}{r})^{-1/2} \frac{MG}{r^2} \). It becomes infinite at the surface \( r = 2MG \). It is not possible to remain at \( r = \text{constant} \) when \( r < 2MG \), because the coordinate \( r \) is timelike there.

**Exercise.** Compute the acceleration of a Schwarzschild observer by computing the covariant derivative of \( u \). Only the Christoffel symbol \( \Gamma_{0'0} \) is needed.

This suggests that the nature of the singularity at \( r = 2MG \) is very similar to that of the singularity at \( x = 0 \) in Rindler coordinates. The null geodesics have \( ds^2 = 0 \), so

\[
\frac{dt}{dr} = \pm \left(1 - \frac{2MG}{r}\right)^{-1}.
\]

Let us define the coordinate \( r_\ast \) by \( dr_\ast = \left(1 - \frac{2MG}{r}\right)^{-1} dr \). Then

\[
r_\ast = r + 2MG \log \left(\frac{r}{2MG} - 1\right).
\]

The Schwarzschild line element can be written

\[
ds^2 = \left(1 - \frac{2MG}{r}\right) (-dt^2 + dr_\ast^2) = - \left(1 - \frac{2MG}{r}\right) du dv,
\]

where \( u = t - r_\ast, \ v = t + r_\ast \). Unlike in the Rindler case, it is impossible to explicitly invert the relation between \( r \) and \( r_\ast \), so in the following \( r \) has to be regarded as a function of \( r_\ast \), or \( u \) and \( v \). From (11.4.2) one has

\[
e^{\frac{u - v}{2MG}} = e^{\frac{r - r_\ast}{2MG}} \frac{r}{2MG} \left(1 - \frac{2MG}{r}\right).
\]
and therefore the metric (11.4.3) can be rewritten in the form
\[ ds^2 = -\frac{2MG}{r}e^{-\frac{r}{2MG}}e^{\frac{u}{2MG}}du\,dv, \] (11.4.4)

The remarkable fact is that there is no singularity at \( r = 2MG \) anymore. As in the Rindler case we can define new coordinates \( U = -\exp(-u/4MG) < 0 \) and \( V = \exp(v/4MG) > 0 \), such that
\[ ds^2 = -\frac{32M^3G^3}{r}e^{-\frac{r}{2MG}}dU\,dV, \] (11.4.5)

and since there is no longer any singularity at \( r \neq 0 \), we can extend the solution to all values of \( U \) and \( V \). Finally, passing to coordinates \( T \) and \( X \)
\[ ds^2 = \frac{32M^3G^3}{r}e^{-\frac{r}{2MG}}(-dT^2 + dX^2), \] (11.4.6)

Tracing back all the transformations,
\[ \left( \frac{r}{2MG} - 1 \right)e^{\frac{r}{2MG}} = -UV = X^2 - T^2, \quad \frac{t}{2MG} = 2\arctanh \frac{T}{X}. \] (11.4.7)

Thus \( t \) is constant along lines through the origin and \( r \) is constant along hyperbolas, as in the Rindler case. Of course, the geometry is not flat here, but in the plane \((r, t)\) it is conformally flat, so the causal structure is very simple to visualize: the light cones are the same as in flat space. The maximal analytic extension of the Schwarzschild metric can now be represented by the following figure:

![Schwarzschild Metric Diagram](image)

The horizon at \( r = 2MG \) corresponds to the light cone through the origin. The singularity at \( r = 0 \) corresponds to the hyperbolas \( T^2 - X^2 = -1 \). The role of the surface \( r = 2MG \) as an event horizon for the observers at \( r > 2MG \) is now evident, and exactly parallels the Rindler case. The original Schwarzschild coordinates \( r \) and \( t \) with \( 2MG < r < \infty \) cover only the right quadrant, and with \( 0 < r < 2MG \) the upper quadrant. But the solution extends to a manifold which is actually double of the one originally described. The existence of the left and lower quadrant could not have been guessed beforehand.

In the Rindler case the existence of a past and future horizon corresponds to an eternally accelerating observer; likewise here it corresponds to the existence of an eternal black hole. As far as we know this is an unphysical idealization. Ethernal black holes do not exist for the simple reason that the universe itself is supposed to have emerged from a singularity a finite time ago. Any black holes that exist today must have formed through astrophysical processes of accretion and gravitational collapse. For example in the case of a star, the surface of the star is at constant \( r \) for a very long time. This corresponds to one of the hyperbolas in the right quadrant, let’s say for \( T < 0 \). Suppose the star begins to collapse at \( T = 0 \). Then the surface of the star describes a curve with decreasing \( r \), eventually crossing the Schwarzschild radius and falling into the singularity at \( r = 0 \). The point is that the interior solution, to the left of the surface, is not described by the Schwarzschild metric. Therefore all that lies to the left of this line is unphysical. A real black hole can only have a future horizon and a future singularity.