12.1 The cosmological principle

Cosmology is concerned with the dynamics of the universe as a whole. Nowadays it is taken almost for granted that the universe is not static, but until the early 20th century the opposite seemed a more natural assumption. With hindsight, the first hint that the universe cannot be static was the observation by the astronomer Olbers in the 1820's that if the universe was static and infinitely extended in all directions, then the whole sky would be uniformly as bright as the sun. To understand this imagine a very narrow solid angle centered at the observer and ending on the surface of a star. The energy flux coming from the star in this solid angle is independent of the distance of the star (as long as the star is not so distant that it fails to fill the whole solid angle). This is because the energy coming to us from a fixed surface element of the star decreases with the square of the distance, but the surface spanned by the solid angle increases with the square of the distance. In an infinite, static, homogeneous universe, every line of sight would end on the surface of a star and by the argument above, the energy flux would be the same in all directions. The simple fact that the sky is dark at night is therefore a sign that the universe cannot be infinite, static and homogeneous. Olbers tried to argue that the light from distant stars would be absorbed by intervening dust or gas, but this argument is not correct, because in a static situation the dust would warm up until it has the same temperature of the radiation bath, and then would reemit the absorbed energy with a black body spectrum of the same temperature. Thus, Olbers failed to draw the correct conclusion, namely that the universe is not static. This had to wait for Hubble’s redshift measurements in the 1920’s.

When we accept that the universe is not static we have to describe its evolution by some physical law. Is Newtonian gravity sufficient, or do we need to use general relativity? To answer this question, recall from chapter 13 that Newtonian physics is a good approximation when $R$, the size of the body, is much greater than its Schwarzschild radius $MG/c^2$. In the case of stars, the general relativistic effects become sizable only when the body is compressed to extremely high density. However, for larger bodies, this need not be the case. If we describe the body as a uniform fluid of a given density $\rho$, the condition for the relativistic effects to be important is $R \approx R^3 \rho G/c^2$, or $R \approx c/\sqrt{\rho G}$. Thus, no matter how low the density, there is always a size above which general relativistic effects become important. In the case of the visible universe, its density is of the order of $10^{-31}$g/cm$^3$, and the critical size is then of the order of $10^{29}$cm. Since objects of comparable distance have been observed, General Relativity is necessary to give an accurate description of the universe.

Some simplifying assumptions have to be made to describe the large scale structure of the universe. The distribution of matter looks the same in all directions: it is isotropic around us. Furthermore, observations support the conclusion that on the largest scales it is also homogeneous in space. This implies that it will look isotropic also to any other observer. Since in general relativity the geometry is determined, via Einstein’s equations, by the distribution of matter, it is reasonable to expect that a homogeneous and isotropic distribution of matter through space will lead to a geometry that is itself homogeneous and isotropic. This assumption is known as the cosmological principle.

Notice that we are not assuming that spacetime is homogeneous and isotropic, only that the surfaces of constant time are. Stated more precisely, spatial homogeneity means that there exist a time coordinate $t$ such that the surface $\Sigma_t$ defined by $t=\text{const}$ is homogeneous, in the sense that there is a group of isometries acting transitively on it (see section 5.4). Spatial isotropy means that there exists a congruence of curves $\gamma_x$, labelled by $x \in \Sigma$, with tangent vector $u$, such that for any pair of vectors $v, w$ at $P$ and orthogonal to $u$ there exists an isometry preserving $P$ and mapping $v$ to $w$. Altogether, the surfaces $\Sigma_t$ are maximally symmetric and therefore fall into one of the cases discussed in section 10.2.

At every spacetime point $P$ there can be at most one observer for which the distribution of the velocities of surrounding matter appears isotropic: any other observer in motion relative to the first would see a distribution of velocities that is necessarily anisotropic. We call an observer that sees an isotropic distribution of matter an “isotropic observer”. The lines $\gamma_x$ are the worldlines of such isotropic observers.

If the universe is both spatially homogeneous and isotropic, the surfaces $\Sigma_t$ must be orthogonal to the curves $\gamma_x$. If they were not, then at a point $P = (t, x)$, the intersection of the tangent space to $\Sigma_t$ with the normal to the curve $\gamma_x$ would identify a class of directions with special properties, in contrast to the assumption of isotropy.

The preceding discussion implies that choosing $t$ and $x^i$ as coordinates of spacetime, the metric must have the form

$$
\begin{pmatrix}
g_{tt}(t) & 0 \\
0 & \delta_{ij}(t, x)
\end{pmatrix}
$$

(12.1.1)
where \( \tilde{h}_{ij}(t, x) \) is a homogeneous and isotropic metric in \( \Sigma_t \).

Recall from section 12.5 that the conditions of homogeneity and isotropy are the maximal possible symmetry and determine the metric entirely, up to an overall scale. We can thus write
\[
\tilde{h}_{ij}(t, x) = a^2(t) h_{ij}(x),
\]
where \( a^2 \) is a positive, time dependent scale factor and the three–dimensional line element \( d\ell^2 = h_{ij}dx^idx^j \) is one of the following: the \( SO(4) \)– invariant metric on the unit sphere \( S^3 \)
\[
d\ell^2 = d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{12.1.2}
\]
the Euclidean metric on \( R^3 \)
\[
d\ell^2 = dx^2 + dy^2 + dz^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{12.1.3}
\]
or the \( SO(3, 1) \)–invariant metric on the unit hyperboloid \( H^3 \)
\[
d\ell^2 = d\chi^2 + \sinh^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2). \tag{12.1.4}
\]
These three cases can be described in a unified fashion by defining a new coordinate \( r = \sin \chi \) for \( S^3 \), \( r = \chi \) for \( R^3 \) and \( r = \sinh \chi \) for \( H^3 \). Then
\[
d\ell^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{12.1.5}
\]
where \( K = 1 \) for \( S^3 \), \( K = 0 \) for \( R^3 \) and \( K = -1 \) for \( H^3 \). The final form of the spacetime metric is then
\[
d s^2 = -dt^2 + a^2(t)d\ell^2 \tag{12.1.6}
\]
and is called the Friedmann–Robertson–Walker metric. The Christoffel symbols of this metric are
\[
\Gamma_0^\mu_0 = 0 \tag{12.1.7a}
\]
\[
\Gamma_0^0_\mu = 0 \tag{12.1.7b}
\]
\[
\Gamma_i^0_j = a\dot{a}h_{ij} \tag{12.1.7c}
\]
\[
\Gamma_0^i_j = \frac{\dot{a}}{a} \delta^i_j \tag{12.1.7d}
\]
\[
\Gamma_j^i_k = (3) \Gamma_j^i_k \tag{12.1.7e}
\]
where an overdot denotes derivative with respect to \( t \) and \( \Gamma^{(3)}_j^i_k \) are the Christoffel symbols of the three dimensional metric \( h_{ij} \). From here one can compute the Riemann tensor and the Ricci tensor and Ricci scalar:
\[
R_{00} = -3\frac{\ddot{a}}{a} \tag{12.1.8a}
\]
\[
R_{0i} = 0 \tag{12.1.8b}
\]
\[
R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2K)h_{ij} \tag{12.1.8c}
\]
\[
R = 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} \right) \tag{12.1.8d}
\]

12.2 Matter content

The universe contains matter and radiation. The visible matter is concentrated in galaxies, which can be described as a perfect fluid of density \( \rho_0 \) and vanishing pressure. The most important component of radiation is the cosmic microwave background (CMB), a photon gas with a thermal spectrum at approximately \( 2.7^0 K \). The CMB can be used to give a precise definition of the isotropic observers. Observations show that the
temperature of the CMB is the same in all directions to approximately one part in $10^4$. This is only true for a particular class of observers, because an observer in motion relative to these would see the spectrum blueshifted in the direction of motion and redshifted in the opposite direction. By definition, then, the isotropic observers are those that see an isotropic CMB spectrum.

Before the discovery of the CMB the isotropic observers were defined using the velocity distribution of galaxies. This is a much less precise definition, but it agrees with the one based on the CMB.

The typical velocities of galaxies with respect to the CMB is quite low, of the order of a few hundreds of km/s. Thus to a good approximation the observers sitting on a galaxy are isotropic observers.

Notice that the four velocity of such an observer, in FRW coordinates, has components $u^\mu = (1, 0, 0, 0)$; then using (12.1.7) one sees that

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 ,$$

that is, the world lines of the isotropic observers are also geodesics and the time coordinate $t$ coincides with the proper time of these observers. For this reason the coordinates $(t, x^i)$ are also called comoving coordinates.

Even though the typical galaxies can be regarded as being at rest in the comoving coordinates, the relative proper distance between them changes with time. If we choose $r = 0$ at our location, the distance between us and a galaxy at coordinate $r$, measured in the three dimensional surface $\Sigma_t$, is

$$d(r, t) = \int_0^r dr' \sqrt{g_{rr}(t, r')} = a(t) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = a(t) \chi = a(t) \times \begin{cases} \arcsin r & \text{for } K = 1 , \\ r & \text{for } K = 0 , \\ \arcsinh r & \text{for } K = -1 . \end{cases}$$

While geometrically intuitive, such measurements at constant $t$ do not correspond to the way in which distances are measured physically, because the signals we receive from other galaxies were emitted at an earlier time. This will be discussed further in section 12.5.

Let us now discuss the consequences of the conservation equations in the FRW cosmology. We assume a general perfect fluid energy momentum tensor of the form (9.1.13). In comoving coordinates

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & p a^2 h_{ij} \end{pmatrix} ; \quad T^{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & \frac{p}{a^2} h^{ij} \end{pmatrix} .$$

Using (12.1.7) the time component of the conservation equation yields

$$0 = \nabla_\mu T^{\mu0} = \partial_0 T^{00} + \Gamma^i_{00} T^{0i} + \Gamma^i_{ij} T^{ij} = \partial_t \rho + 3 \frac{\dot{a}}{a} \rho + 3 \frac{\dot{a}}{a} p$$

Multiplying by $a^3$ this can be rewritten as

$$0 = \frac{d}{dt}(\rho a^3) + p \frac{da^3}{dt} ,$$

which can be regarded as a restatement of the first law of thermodynamics for the cosmic perfect fluid. To proceed further one need to specify the equation of state, i.e. the relation between pressure and energy density. This is usually assumed to be a simple proportionality

$$p = w \rho ,$$

with $w = 0$ for dust and $w = \frac{1}{3}$ for radiation. Equation (12.2.4) can be rewritten in the form

$$0 = \frac{d}{da}(\rho a^3) + 3 a^2 p ,$$

or equivalently

$$0 = \frac{d \rho}{da} + 3 a^2 (\rho + p) ,$$

(12.2.6)
which for a given equation of state can be used to solve for \( \rho \) varies as a function of \( a \):

\[
\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+w)} .
\] (12.2.7)

Thus for \( w = 0 \) we find \( \rho \approx a^{-3} \), whereas for radiation \( \rho \approx a^{-4} \). A more exotic possibility is \( w = -1 \), which corresponds to a cosmological constant, in which case \( \rho \) is independent of \( a \).

### 12.3 Qualitative aspects of the time evolution

From (12.1.8) one finds the following nonvanishing components for the Einstein tensor:

\[
R_{00} - \frac{1}{2} g_{00} R = 3 \left( \frac{a}{\dot{a}} \right)^2 \frac{K}{a^2}
\] (12.3.1.a)

\[
R_{ij} - \frac{1}{2} g_{ij} R = -(2a\ddot{a} + \dot{a}^2 + K) h_{ij}
\] (12.3.1.b)

Therefore Einstein’s equations (with vanishing cosmological constant) read

\[
3 \left( \frac{\dot{a}}{a} \right)^2 \frac{K}{a^2} = 8\pi G \rho
\] (12.3.2a)

\[
2\ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 \frac{K}{a^2} = -8\pi G p
\] (12.3.2b)

Using (12.3.2.a) equation (12.3.2.b) can be rewritten in the form

\[
\ddot{a} = -\frac{4\pi G}{3} a (\rho + 3p) .
\] (12.3.3)

This equation can also be obtained by deriving (12.3.2.a) and using (12.2.4). This is due to the fact that the equations are not functionally independent, being related by the Bianchi identities.

Before using the equation of state, one can draw some general conclusions on the qualitative behaviour of the solutions just by inspection. From equation (12.3.3), assuming that \( \rho > 0 \) and \( p > 0 \) we find \( \ddot{a} < 0 \). Furthermore, observations show that \( \dot{a} > 0 \). There follows that \( a = 0 \) a finite time ago. The following figure clarifies the point:

If we call \( t_0 \) the present time and \( a_0 \) the present value of the scale factor, the singularity in the past must have occurred no more that \( T_0 \) ago, where \( T_0 = \frac{1}{H_0} \) and \( H_0 = \frac{\dot{a}_0}{a_0} \) is called the Hubble parameter. This limiting case corresponds to \( \dddot{a} = 0 \) (the straight line in the figure). For \( \dddot{a} < 0 \) the singularity must be closer to the present.

Thus we have a striking prediction of a singularity in the past, called the “big bang”. One may worry that the singularity is an artifact of the high degree of symmetry that we have postulated. It is in fact conceivable that if we allowed the initial conditions to be a little less symmetric (for example by superimposing small fluctuations), then the singularity may not occur. This is not the case: there are powerful general “singularity
theorems” which imply that quite independent of any symmetries, if the energy momentum tensor of matter satisfies some reasonable conditions a cosmological singularity will occur (1).

Concerning the evolution in the future, the situation depends on the value of the constant $K$. From (12.3.2.a) we see that if $K = -1$ or $K = 0$, $\dot{a}^2$ is always positive, so the universe will keep expanding indefinitely. For $K = 1$ equation (12.3.2.a) can be written in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{1}{a^2}.$$ 

For $w > -1/3$ we see from (12.3.7) that as $a$ increases, the first term on the r.h.s. decreases faster than the second, so $\dot{a}^2$ must also decrease until it becomes eventually zero. There are then two possibilities: either this happens at some finite value of $t$ or it happens asymptotically for $t \to \infty$. The latter possibility is ruled out since in this case we would have $\ddot{a} \to 0$, with finite and positive values of $a$, $\rho$ and $p$, but this is excluded by equation (12.3.3). Therefore $\dot{a}$ must reach a zero at finite $t$ in the future and then become negative. Then following the same arguments given above, but with $\dot{a} < 0$, we conclude that the universe recollapses into a future singularity.

### 12.4 Some exact solutions

The Friedmann equation (12.3.2) can be solved for $a(t)$ in some simple cases. Let us begin by assuming that the cosmic fluid has the equation of state of dust, $w = 0$. Then we can substitute $\rho \approx a^{-3}$ in the Friedmann equation, which becomes

$$\dot{a}^2 + K - \frac{C}{a} = 0$$

with $C = \frac{8\pi G}{3}\rho_0 a_0^3$. If $K = 0$ this can be solved and we find

$$a(t) = \left(\frac{9C}{4}\right)^{1/3} t^{2/3}. \quad (12.4.1)$$

When $K = 1$ the solution is a cycloid, which can be written in parametric form

$$t = \frac{1}{2} C(\eta - \sin \eta) ; \quad a = \frac{1}{2} C(1 - \cos \eta). \quad (12.4.2)$$

As $\eta$ grows from zero to $2\pi$, $t$ grows from zero (the big bang) to $C\pi$, at which point $a = 0$ (the future singularity, or big crunch). For $K = -1$ the solution, also in parametric form, is

$$t = \frac{1}{2} C(\sinh \eta - \eta) ; \quad a = \frac{1}{2} C(\cosh \eta - 1). \quad (12.4.3)$$

These solutions are plotted in the following figure ($K = -1$, 0 and 1 from top to bottom), rescaled so that $C = 1:

\[\text{Diagram of solutions for different values of } K\]

---

When the cosmic fluid has the equation of state of radiation, \( w = \frac{1}{3} \) and we can substitute \( \rho \approx a^{-4} \) in the Friedmann equation, which becomes

\[
\dot{a}^2 + K - \frac{C}{a^2} = 0.
\]

If \( K = 0 \) this can be solved to give

\[
a(t) = (4C)^{1/4} t^{1/2},
\]

so the solution expands forever, but at a slightly slower rate than for a pressureless fluid. For \( K = 1 \)

\[
a = \sqrt{C} \sqrt{1 - (1 - \frac{t}{\sqrt{C}})^2}.
\]

This is a solution that reaches a maximum expansion \( a_{\text{max}} = \sqrt{C} \) at \( t = \sqrt{C} \), and then recollapses at \( t = 2\sqrt{C} \). For \( K = -1 \)

\[
a = \sqrt{C} \sqrt{\left(1 + \frac{t}{\sqrt{C}}\right)^2 - 1}.
\]

These solutions are plotted in the following figure (\( K = -1, 0 \) and 1 from top to bottom), rescaled so that \( C = 1 \):

![Graph of solutions](image)

Let us now consider some solutions with \( \Lambda \neq 0 \). For simplicity we will only consider vacuum solutions \( (T_{\mu\nu} = 0) \). Equation (12.3.2a) becomes in this case

\[
3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} \right) = \Lambda.
\]

Let us consider first the case \( K = 1 \). The solution is then given by

\[
a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t.
\]

This is the de Sitter metric (10.2.14), where we have redefined \( \tau = t/r \) and the field equation has given \( r = \sqrt{3/\Lambda} \). For \( K = 0 \) (12.4.4) reduces to

\[
\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}},
\]

and therefore the solution expands exponentially

\[
a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t}.
\]

The complete line element can be written in the form

\[
ds^2 = -dt^2 + e^{2\sqrt{\frac{\Lambda}{3}}}(dx^2 + dy^2 + dz^2).
\]
This is again the de Sitter metric but written in a different coordinate system. One can verify that it is obtained from the flat five dimensional line element (10.2.11) by choosing

\[ \hat{t} = r \log \frac{x^4 + x^5}{r}, \quad \hat{x} = \frac{rx^1}{x^4 + x^5}, \quad \hat{y} = \frac{rx^2}{x^4 + x^5}, \quad \hat{z} = \frac{rx^3}{x^4 + x^5}. \]

This coordinate system, which is analogous to (10.4.7) for anti-de Sitter space, covers only half of the de Sitter hyperboloid.

### 12.5 The cosmological red shift

Astronomers came to the conclusion that the universe is expanding by studying the red shift of the light of distant galaxies. We show here that in a FRW metric the light coming from other galaxies is red- or blue-shifted, depending on the behaviour of the scale factor. In section 11.2 the gravitational red- or blue-shift in the Schwarzschild metric was derived by appealing to the existence of a timelike Killing vector. The general FRW metric (12.1.5) does not have a timelike Killing vector (unless \( a \) is constant), but we can use to the same effect one of the Killing vectors related to spatial homogeneity or isotropy.

First, consider a null geodesic \( x^\mu(\lambda) \), with affine parameter \( \lambda \) and tangent vector \( k^\mu \), describing the world line of a photon in the approximation of geometric optics (section 8.6). In the comoving coordinate system we can write \( x^\mu(\lambda) = (x^0(\lambda), x^i(\lambda)) \) and it is easy to verify, using (12.1.7d-e) that \( x^i(\lambda) \) is a geodesic in the three dimensional space \( \Sigma \), relative to the metric \( h_{ij} \) (but note that \( \lambda \) is no longer an affine parameter for this projected geodesic.)

Now let \( u = \partial_t \) be the four-velocity field of the isotropic observers. At each point \((t, x^i)\) we can decompose \( k = \omega u + \hat{k} \), where \( \omega = g(k, u) = -k^0 \) is the frequency of the photon, as measured by the isotropic observer at that point and \( \hat{k} \) is tangent to \( \Sigma_t \). Since \( k \) is a null vector, we have

\[ \omega^2 = a^2 h(\hat{k}, \hat{k}). \tag{12.5.1} \]

On the other hand, there exist a Killing vector \( K \) which is parallel to \( \hat{k} \). For example, if \( \Sigma \) is flat as in (12.1.3), and the light ray propagates in the \( x \) direction, the space translation generator \( \partial_x \) is such a Killing vector. If \( \Sigma = S^3 \) as in (12.1.2), the geodesics are great circles and each great circle can be seen as the orbit of a Killing vector. The components of a Killing vector in the comoving coordinates are independent of \( t \), so \( g(K, K) = a^2 h(K, K) \), and \( h(K, K) \) is time-independent. The conditions above define \( K \) up to a constant factor. Now note that \( h(K, K) \) is a function of \( x^i \) in general, but it must be constant on the orbit of \( K \), because

\[ K(h(K, K)) = \mathcal{L}_K h(K, K) + h(\mathcal{L}_K K, K) + h(K, \mathcal{L}_K K) = 0. \]

We will assume that \( K \) is normalized such that \( h(K, K) = 1 \) on the projected geodesic.

From equation (12.5.1) we then see that \( \frac{\hat{k}}{K} = K \omega/a \). We also know from Noether’s theorem (section 10.1) that \( g(k, K) = (\omega/a)g(K, K) = \omega a \) is constant along the geodesic. Thus, if a photon is generated by one observer at time \( t_1 \) and received by another observer at time \( t_0 \), the frequencies measured by the observers are related by

\[ \frac{\omega_1}{\omega_0} = \frac{a(t_0)}{a(t_1)}. \tag{12.5.2} \]

If the universe has expanded during the trip of the photon, the frequency will be redshifted while if the universe has contracted the frequency will be blueshifted. Note that the result depends only on the initial and final scale factors, not on the history of the scale factor. The ratio (12.5.2) is commonly denoted \( 1 + z \), where \( z \) is called the redshift: it is positive for redshift and negative (\( > -1 \)) for blueshift. The fact that the light we receive from distant galaxies is redshifted is therefore a sign that the universe is expanding.

Hubble’s original observation is that the redshift is proportional to the distance. To clarify the meaning of this statement, recall the definition of \( r \) as a function of \( \chi \) in (12.1.5), where \( \chi \) is a constant for comoving matter. Putting \( \chi = 0 \) at our position, \( \chi \) is called the comoving distance of an object located at coordinate \( \chi \). Equation (12.2.2) says that the proper distance, measured on the constant-time surface \( \Sigma_t \), is \( d(t) = a(t)\chi \). Differentiating this with respect to \( t \) we find

\[ \dot{d} = a \chi = H d, \tag{12.5.3} \]
where \( H(t) = \dot{a}(t)/a(t) \) is the Hubble parameter, already defined in section 12.4. This equation, written for the present time, tells us that \( \dot{d} \), the recessional velocity of the galaxies, is proportional to their present proper distance. Physically, distance measurements are not made at constant time \( t \), but rather on our past light cone: the light we see coming from any object was emitted at some time in the past. So the interpretation of the redshift as due to the recessional velocity \( \dot{d} \) is only correct for sufficiently small \( d \), such that the expansion of the universe can be neglected during the travel time of the photon.

In general, the comoving distance \( \chi \) of a galaxy is determined as follows. The photons we receive from a galaxy travel on null geodesics, with \( dt = a(t)d\chi \) (the other angular coordinates remain constant). Integrating we find

\[
\chi(t) = \int_t^{t_0} \frac{dt'}{a(t')}, \tag{12.5.4}
\]

where \( t_0 \) is the present time and \( t \) is the time at which the photon was emitted. This formula is of limited use, because \( t \) is not observable. However, we can rewrite this in terms of the redshift, which is observable. The relation between the redshift and the time of emission is

\[
z(t) = \frac{a(t_0)}{a(t)} - 1,
\]

which can be inverted to give \( t(z) \). Differentiating we find that

\[
\frac{dt}{a(t)} = -\frac{1}{a(t_0)} \frac{dz}{z},
\]

Therefore we can rewrite (12.5.4) as

\[
\chi(z) = \frac{1}{a(t_0)} \int_0^z \frac{dz'}{H(z')} \tag{12.5.5}
\]

The comoving distance of an object can in this way be expressed as a function of its redshift. For small \( z \) that we can consider \( H \) constant, in which case (12.5.5) can be integrated to \( z = H_0 d \), with \( d = a_0 \chi \), which is Hubble’s original relation.