Black body and thermal radiation



- Black body and Kirchhoff's law
- Thermal radiation
- Black body spectrum
- Properties of the Planck's law
- Thermal radiation from real bodies

• Rybicki, G., and Lightman, A., 1979, Radiation processes in astrophysics, Chapter 1. John Wiley and Sons.

Blackbody radiation

Because the absorption and emission processes have not been specified, α_v and j_v seem to be independent. However, they are not independent in full thermodynamic equilibrium (TE). In TE, matter and radiation are in equilibrium at the same temperature T.

Consider a container bounded by opaque walls (no transmission out of the container, but not totally reflective). Keep the enclosure at temperature T until equilibrium is achieved. Since photons are massless, they can be created and destroyed in arbitrary number by the walls of the container. The non conservation photon number will adjust their number to an equilibrium value, dependent on the temperature T. The radiation inside the cavity gets to equilibrium. Piercing the cavity with a small hole, we can measure the radiation inside without disturbing the equilibrium. The intensity and spectrum of the radiation emerging from the hole is **independent of** the wall shape, material (e.g., painted wood, shiny copper, gray concrete, etc.) and any absorbing material (e.g., gas, dust, fog, etc.) that may be inside the cavity, as proven by the Kirchhoff's thought experiment.

It invokes two cavities in thermodynamic equilibrium connected through a filter that passes radiation in the narrow frequency range v to v+dv. The cavities may be made of different materials and shapes and contain different emitting/absorbing particles. In equilibrium at any temperature, radiation can transfer no net power from one cavity to the other, to not violate the 2th law of thermodynamics. Therefore I'_v = I_v universal function of T and $v \equiv B_v(T)$. A corollary is that it is also isotropic. $B_v(T)$ is called the **Planck function**.





Furthermore, from the radiative transfer equation,

$$\frac{\mathrm{d}\mathbf{I}_{\mathbf{v}}}{\mathrm{d}\mathbf{s}} = \mathbf{0} = -\alpha_{\mathbf{v}}B_{\mathbf{v}}(T) + \mathbf{j}_{\mathbf{v}}$$

$$\frac{j_{\nu}(T)}{\alpha_{\nu}(T)} = B_{\nu}(T)$$

Kirchhoff's law for a system in TE

Thermal radiation

Remarkable because it connects the properties j_{ν} and α_{ν} of any kind of *matter* to the single universal spectrum $B_{\nu}(T)$ of equilibrium *radiation*. It was derived for a system in TE. However:

- $B_{\nu}(T)$ is *independent* of the properties of the radiating/absorbing material. Describes the spectrum of photon's equilibrium at temperature T.
- j_{v} and α_{v} depends *only* on the materials in the cavity and on T: they do not depend on the ambient radiation field or its spectrum.
- → Kirchhoff's law also applies whenever the radiating/absorbing material is in thermal equilibrium, in *any* radiation field. If the emitting/absorbing material is in thermal equilibrium at a well-defined temperature T, it is said to be in local thermodynamic equilibrium (LTE) even if it is not in equilibrium with the radiation field.

Blackbody radiation is itself in thermal equilibrium: its spectrum is given by $I_{\nu}(T) = B_{\nu}(T)$ (Planck function).

Thermal radiation is radiation emitted by matter in thermal equilibrium with itself : in this case, then, the source function equals the Planck function, $S_{\nu}(T)=B_{\nu}(T)$

Thermal radiation (=by a medium of matter particles in thermal equilibrium) becomes blackbody radiation (=has a Planck spectrum) only for **optically thick** media, for which the radiation specific intensity I_v , according to the transfer equation, propagates in that medium approaching the source function of the medium.

Thermal radiation

Why thermal radiation becomes blackbody radiation only for large optical depths?

From the transfer equation,

 $\frac{dI_{\nu}}{ds} = j_{\nu} - \alpha_{\nu}^{abs} I_{\nu} = \alpha_{\nu}^{abs} S_{\nu} - \alpha_{\nu}^{abs} I_{\nu} = dE/dA ds d\Omega dt d\nu , \text{ whose formal solution for constant source function is:}$ $I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}} + S_{\nu}(1 - e^{-\tau_{\nu}}) = S_{\nu} + e^{-\tau_{\nu}} (I_{\nu}(0) - S_{\nu}) ; \text{ and, for matter in LTE at temperature T, } S_{\nu} = B_{\nu}(T).$ $I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}} + B_{\nu}(T)(1 - e^{-\tau_{\nu}}) , \text{ which approaches } B_{\nu}(T) \text{ for large optical depth.}$

 $I_{\nu}(0)$ is the unabsorbed spectrum, dominating in optically thin medium, or for small optical depths.

It may be any incident, external radiation field; or, it may be light emitted by the LTE matter **itself**, before being absorbed and re-emitted by other particles (**self-absorption**), like in the figure below: this radiation is not coming from outside, but inner to the LTE particle cloud.

 $I_{\nu}(0)$ emitted by the LTE matter elementary volume, can have different shapes, depending on the mechanism creating LTE (e.g. dipole oscillations of charges inside atoms and molecules, as for any body at non-zero temperature, or by Coulombian interactions between free charges, as in free-free emisssion). Remember that the brighness due to spontaneous emission is $I_{\nu} = j_{\nu}$ ds, in absence of absorption (or before being absorbed).



Blackbody spectrum

To derive the spectrum, assume that:

- Photons are bosons, i.e. more than one photon per phase cell is allowed;
- Photons are in **thermodynamical equilibrium** at all frequencies (as in a blackbody enclosure): $\mu = 0$;

And then:

Step 1: Compute the average energy per photon state;

Step 2: Compute the number density of space-cells (photon states) as a function of frequency.

Step1: What is the average energy of the state having frequency v? Each state may contain n photons of energy hv, where n=0,1,2..... Thus, the energy may be $E_n = n h v$. According to statistical mechanics, the probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, with $\beta = 1/kT$. The average energy is

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = \frac{h\nu}{\exp(\frac{h\nu}{kT}) - 1}$$

Since hv is the energy of one photon of frequency v, the «occupation number» (average number of photons of frequency hv) is $n_v = \left[\exp\left(\frac{hv}{kT}\right) - 1 \right]^{-1}$ (Bose-Einstein statistics with a limiteless number of particles, chemical potential = 0).

Blackbody spectrum

Step 2: Compute the number density of photon states for a given frequency. Consider a photon of frequency v propagating in direction \hat{d} inside a box. The photon is represented as a standing wave in the box. The wave vector of the photon is $\mathbf{k} = \left(\frac{2\pi}{\lambda}\right) \hat{d} = \left(\frac{2\pi\nu}{c}\right) \hat{d}$. The number of «states» (distinguishable photons at the same frequency) = number of photons with same $|\mathbf{k}|$.

For stationary waves, the **number of nodes** is the integral numer of wavelenghts in a given direction, e.g. $n_x = \frac{L_x}{\lambda} = \frac{k_x L_x}{2\pi}$

In a wavenumber interval, $\Delta n_{\rm x} = \frac{L_{\rm x} \Delta k_x}{2\pi}$

For large number of states, we can go to the continuum limit,

 $\Delta \mathbf{N} = \Delta \mathbf{n}_{\mathbf{x}} \Delta n_{\mathbf{y}} \Delta n_{\mathbf{z}} = \frac{V d^3 k}{(2\pi)^3} \qquad d^3 k = k^2 dk \ d\Omega = \frac{(2\pi)^3}{c^3} \nu^2 d\nu \ d\Omega$

Then, the **density of states** (number of states per solid angle, volume and frequency) is :

 $\rho_s = \frac{\Delta N}{d\nu d\Omega} = 2 \frac{\nu^2}{c^3}$ where the factor 2 is from spin (2 polarization states).

The **average energy density** per solid angle, volume, frequency is therefore

$$\overline{E} \rho_{s} = u_{\nu} (\Omega) \, \mathrm{dV} \, \mathrm{d} \, \nu \, \mathrm{d\Omega} = \frac{2h \, \nu^{3}}{c^{3}} \, \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$



Blackbody spectrum

For an isotropic radiator, $u_v(\Omega) = I_v / c = B_v / c$. In CGS units, B_v is in erg cm⁻² s⁻¹ sr⁻¹ Hz⁻¹

The intensity spectrum is then the **Planck function**:









M. Planck and A. Einstein

Properties of the Planck's law

• Rayleigh-Jeans Law:

In the case $h\nu << kT$, expanding the exponential,

 $B_{\nu}^{RJ}(T) = \frac{2\nu^2}{c^2} \,\mathrm{kT}$

Notice that: it doesn't contain the Planck constant, it was originally derived assuming $\overline{E} = kT$ (classical equipartition value for an electromagnetic wave). It corresponds to the straight line in the log-log plot. Applied to all frequencies, it would lead to the *ultraviolet catastrophe*.

• Wien Law:

In the limit hv >> kT, we must account for the discrete nature of photons.

 $B_{\nu}^{W}(T) = \frac{2h\nu^{3}}{c^{2}} \exp\left(-\frac{h\nu}{kT}\right)$

• Monotonicity with temperature

Of two blackbody curves, the one with higher temperature lies entirely above the other. The partial derivative $\frac{\partial B_{\nu}(T)}{\partial T}$ is indeed always positive. At any frequency, an **increase of the temperature increases the brightness**.

Also, $B_{\nu} \rightarrow 0$ as $T \rightarrow 0$ and $B_{\nu} \rightarrow \infty$ as $T \rightarrow \infty$

Properties of the Planck's law

• Wien Displacement Law

The peak frequency of $B_{\nu}(T)$ occurs at $h\nu_{max} = 2.82 \text{ kT}$ linear in T! (it is sufficient solve $\frac{\partial B_{\nu}(T)}{\partial T} | \nu_{max} = 0$), or $\frac{\nu_{max}}{T} = 5.88 \times 10^{10} \text{ Hz deg}^{-1}$. Similarly, the derivative w.r.t. λ gives the peak of $B_{\lambda}(T)$ at $\lambda_{max}T = 0.290 \text{ cm deg}$.

Note that $\lambda_{max} \neq c / \nu_{max}$! «spectral paradox»

Problem 2. Explain why the peak of frequency in the Planck spectrum does not correspond to the inverse of the peak wavelength multiplied by c.

Hint: $B_{\lambda}(T)$ is the amount of energy (erg) put out each second (s) in a wavelength range (Angstrom) which is radiated by a surface area (cm²) into a solid angle of space (steradian).

Total brightness of a black body

$$B(T) = \int_0^\infty B_\nu(T) d\nu = \sigma \frac{T^4}{\pi} \qquad \text{where} \quad \sigma = \frac{2\pi^5 k^4}{15c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{deg}^{-4} \text{ s}^{-1}$$

The emergent flux from an isotropically emitting surface (such a blackbody) is $\pi \times$ brightness, so

$$F = \int F_{\nu} d\nu = \pi \int B_{\nu} d\nu = \pi B(T) \longrightarrow F = \sigma T^{4}$$
Stefan-Boltzmann law

Remember that the flux at a surface of uniform brightness is $\pi B!$

Example. The CMB

The Cosmic Microwave Background (CMB) is a fossil radiation out of the Big Bang. At the beginning the Universe was optically thick. As the expansion advanced, there was a transition to optically thin. The CMB is the footprint of that transition time (*recombination epoch*).



The microwave sky: the CMB observed by the ESA Planck satellite

In the Big Bang cosmological model,

the primordial universe was filled with an opaque fog of dense, hot plasma of sub-atomic particles. As the universe expanded, this plasma cooled to the point where protons and electrons combined to form neutral atoms of mostly hydrogen. Unlike the plasma, these atoms could not scatter thermal radiation by Thomson scattering, and the universe became transparent.

The energy density of the CMB is 0.260 eV/cm^3 (4.17×10⁻¹⁴ J/m³) which yields about 411 photons/cm³.

Example. The CMB

Its spectrum is a perfect blackbody with a temperature of 2.725 K, so the frequency peak is in the microwave, around 160 GHz, corresponding to a wavelength of 1.9 mm and to a photon energy of about 6.626×10^{-4} eV.



Problem 3. If the background blackbody radiation scales with cosmological redshift as $T(z)=T_0(1+z)$, and if the recombination occurred at a redshift $z \sim 1100$, determine the temperature of the CMB at the recombination epoch and its peak frequency at that time. In which band of the electromagnetic spectrum was the CMB emission peak?

Example. The color of the stars







With a surface temperature of T = 5778 K the Solar frequency spectrum reaches a maximum at $v_{max} = 3.6 \times 10^{14}$ Hz = 360 THz, which translates to a wavelength of 831 nm.

In terms of wavelength, peak solar radiation occurs at about 500 nm, in the range of human vision.

Problem 4. What is the temperature at the solar surface? Use both the intensity of radiation on Earth (from Problem 1) and that the spectrum peaks about 500 nm to get answers.

Thermal radiation from real bodies

In a **real medium**, the light of an incoming beam with intensity Iv at a given frequency can be absorbed, but also partially reflected or transmitted.



For energy conservation, Power in = Power out \rightarrow Incident $P_v = a P_v + r P_v + t P_v \rightarrow$

a + r + t = 1, having defined r, a, t, the **fraction** of incident power that went into these different channels: reflectivity r, transmissivity t, absorptivity a (quantities integrated over the solid angle).

Let us define the normalized emissivity $e = \frac{1}{4\pi} \int \frac{j_v}{j_v^{BB}} d\Omega$:

e = a = 1 - r - t (Kirchoff's law for real bodies)

For a blackbody, r=t=0, so a=e=1. The blackbody is the more efficient radiator system.

	Absorptivity (α_{λ})	Reflectivity (ρ_{λ})	Transmissivity (τ_{λ})	Emissivity (ε_{λ})
Perfect Absorption	I	0	0	0-1
Perfect Reflection	0	I	0	0-1
Perfect Transparency	0	0		0-1
Black Body	l (at all λ)	0 (at all λ)	0 (at all λ)	l (total)
Gray Body	0-1	0-1	0-1	0-1 (same for all λ)

Table I. Surface Properties Involved in Radiative Heat Transfer.Unless otherwise specified, each of these properties is a function of wavelength.



Thermal radiation from real bodies

Example. Temperature differences across the GBT reflector caused by differential solar heating can deform the surface and degrade its performance. A special paint that is white at visible wavelengths, black in the mid-infrared, and transparent at radio wavelengths keeps the surface cool and does not harm performance at radio wavelengths by absorbing incoming radio waves or emitting radio noise.



Figure 8.1: The 100-m Green Bank Telescope (GBT). Image credit: NRAO/AUI/NSF.

The special paint exploits Kirchhoff's law to perform three separate functions simultaneously:

- It is opaque (no transmission) and white (no absorption, just reflection) in the visible portion of the spectrum to reflect sunlight (T≈5800 K).
- It is **black** in the mid-infrared so that the GBT (T≈300 K) can cool itself efficiently by reradiation (absorption and emission).
- It is **transparent** at radio wavelengths so that it **neither absorbs** incoming radio waves **nor emits** (and no absorption!) thermal noise in the radio.

https://www.cv.nrao.edu/~sransom/web/Ch2.html#E47

Modified blackbodies

What makes blackbody so different?

Remember the dfinition of asorption coefficient, for spherical absorbers of radius *a* :

 $\alpha_{v}^{abs} = n \sigma_{v}^{abs} \equiv n Q_{abs} \pi a^{2}$ where πa^{2} is the geometrical cross section of the sphere.

So Q_{abs} is intepreted as the fraction of geometrical cross section which is actually absorbing photons.

For the blackbody, this fraction is $Q_{abs} = 1$, independent on frequencies: the cross section for absorption, in a blackbody, is just its geometrical cross section. In real bodies, Q_{abs} is generally dependent on λ , on the shape and composition of the absorbing material. In any case, $Q_{abs} \leq 1$. We can define a cross section for scattering $\alpha_{\nu}^{scat} = n \sigma^{scat} = n Q_{scat} \pi a^2$, where Q_{scat} is =0 for a blackbody: it measures the fraction of incident

light which is scattered. For a single scatterer, Q_{scat} can also be >1 , (scattering area larger than the geometrical cross section) because of diffraction! (see figure).

In any case, in LTE, (thermalized matter at temperature T), Kirchhoff's law guarantees the validity of the monodirectional relation:

 $j_{\nu} = \alpha_{\nu}^{abs} B_{\nu}(T)$.

When, in LTE, $Q_{abs} = Q_{abs}(\lambda)$, we end up with a *modified blackbody*. This is noticeably useful to explain the thermal emission from dust!



Problem 5

Show that, defining the emission efficiency as $Q_{em} \equiv \frac{j_{\nu}}{j_{\nu}^{BB}}$ and the absorption efficiency as $Q_{abs} \equiv \frac{\alpha_{\nu}}{n\pi a^2}$ (with n = number density of absorbers, and πa^2 = geometric section of the absorber) in LTE $Q_{em} = Q_{ab}$.

(I omitted the index $\nu~$ in Q_{em} and Q_{abs} , but they are generally function of $\nu)$.