Experimental observation of planar elastica and comparison with theoretical results

Topics in continuum Mechanics

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SISSA

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Kinematics

Deformed rod in the plane

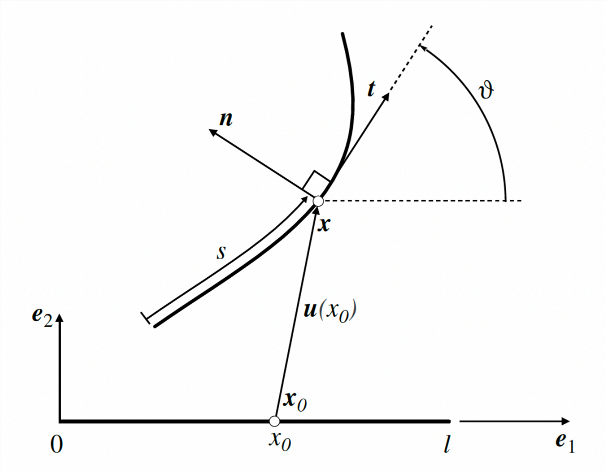

Figure 1: Elastic inextensible rod of length l, rectilinear in the reference configuration.

Kinematics description

Deformation

Smooth field $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, with $\mathbf{x} = \mathbf{g}(\mathbf{x}_0)$.

Displacement

$\mathbf{u}(\mathbf{x}_0) = u_1(\mathbf{x}_0)\mathbf{e}_1 + u_2(\mathbf{x}_0)\mathbf{e}_2 = \mathbf{g}(x_0\mathbf{e}_1) - x_0\mathbf{e}_1.$

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Displacement

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Define $\mathbf{F} = D\mathbf{g}(\mathbf{x_0})$, $\mathbf{Fe_1}$ is the tangent vector.

Inextensibility condition

$|\mathbf{Fe_1}| = |\mathbf{e_1}| \implies (u_1' + 1)^2 = 1 - (u_2')^2$.

Tangent vector

$\mathbf{t} = (u_1' + 1)\mathbf{e}_1 + u_2'\mathbf{e}_2 = \pm\sqrt{1 - (u_2')^2}\mathbf{e}_1 + u_2'\mathbf{e}_2$

Kinematics through θ

Rotation field

The angle of rotation θ is define through

$$\mathbf{t} = \cos\theta(s)\mathbf{e}_1 + \sin\theta(s)\mathbf{e}_2$$

Kinematics relations

$$\sin\theta(s) = x_2'(s) = u_2'(s), \quad \cos\theta(s) = x_1'(s) = (u_1' + 1)(s).$$

Signed curvature

$$\kappa = \theta'(s), \quad s \in [0, l].$$

Dynamics

Sketch of the system

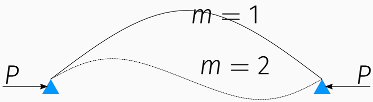

Figure 2: Doubly supported elastica

Sketch of the system

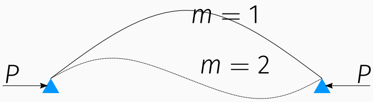

Figure 2: Doubly supported elastica

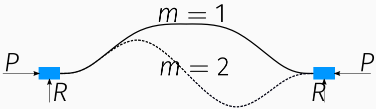

Figure 3: Doubly clamped elastica

Loading the elastica

Constitutive assumption

Shearing and normal forces are neglected

$$\mathsf{M}(s) = B\theta'(s), \quad B\colon \text{ bending stiffness.}$$

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$$\mathsf{M}(s) = B\theta'(s), \quad B\text{: bending stiffness.}$$

Equilibrium condition

$$\mathsf{M}(s) = \left[-Pu_2(s) + R(x_1(l) - x_1(s))\right]\mathsf{e}_3,$$

P: horizontal load
R: vertical reaction

Loading the elastica

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R: vertical reaction

Equilibrium equation

$$\theta''(s) + \frac{P}{B}\sin\theta(s) + \frac{R}{B}\cos\theta(s) = 0 \quad s \in [0, l].$$

Simply supported elastica

Description of the system

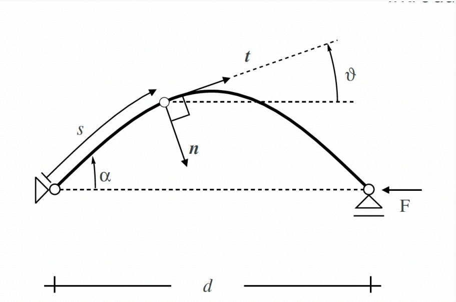


Figure 4: The kinematics of a simply supported elastic.

Note

The vertical reaction R at the supports is null, except when $d = 0$.

Complete non linear system

Define $\lambda^2 = \frac{P}{B}$

System of equations

$$
\begin{aligned}
&\theta''(s) + \lambda^2 \sin\theta(s) = 0 \quad s \in [0, l], && \text{Equilibrium eqn.}\\
&\theta'(0) = \theta'(l) = 0, && \text{Null moment at endpoints}\\
&u_1(0) = 0, && \text{Null displacement}\\
&u_2(0) = u_2(l) = 0, && \text{Null vert. displ.}\\
&u_1'(s) = \cos\theta(s) - 1 \quad s \in [0, l], && \text{Eqn. horiz. displ.}\\
&u_2'(s) = \sin\theta(s) \quad s \in [0, l]. && \text{Eqn. vert. displ.}
\end{aligned}
$$

The trivial solution $\theta = 0$ is always possible.

Linearized problem

Linearize the problem around $\theta = 0$, for which $u_1(s) = 0$

$$\theta''(s) + \lambda^2\theta(s) = 0 \quad s \in [0, l],$$
$$\theta'(0) = \theta'(l) = 0,$$
$$u_2'(s) = \theta(s) \quad s \in [0, l].$$

Solutions

$$\theta(s) = A_n \cos\left(\frac{n\pi s}{l}\right), \quad n = 0, 1, \dots$$

Euler's critical loads

Non trivial solutions arise if and only if

$$\lambda = \lambda_n = \frac{n\pi}{l} \quad \iff \quad P = P_n^{cr} = \frac{n^2\pi^2 B}{l^2}, \quad n = 1, 2, \dots$$

Simply supported elastica

Determination of equilibrium solutions

Back to the complete problem

By integration of the equilibrium equation

$$\frac{d}{ds}[\theta'(s) - \lambda^2 \cos(\theta(s))] = 0$$

follows

$$\theta'(s) = \lambda\sqrt{\cos(\theta) - \cos(\alpha)}, \quad \alpha = \theta(0).$$

Change of variables

$$k = sin(\alpha/2), \quad k\sin(\phi(s)) = \sin(\theta(s)/2),$$

$$\phi'(s) = \lambda\sqrt{1 - k^2\sin^2(\phi(s))},$$

$$\phi(0) = \frac{4h+1}{2}\pi, \quad \phi(l) = \frac{2j+1}{2}\pi, \quad h, j \in \mathbb{Z}.$$

Elliptic integral

By integration

$$s\lambda = \int_{\frac{4h+1}{2}\pi}^{\phi(s)} \frac{dt}{\sqrt{1-k^2\sin^2(t)}}.$$

Compute in $s = l$

$$l\lambda = \int_{\frac{4h+1}{2}\pi}^{\frac{2j+1}{2}\pi} \frac{dt}{\sqrt{1-k^2\sin^2(t)}} = 2mK(k), \quad m \in \mathbb{Z}.$$

Elliptic integral of the first kind

$$K(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2\sin^2(t)}}, \quad K(k) = K(\pi/2, k).$$

Bifurcation loads

Relation between P and α

$$l\lambda = 2mK(k) \quad \Longleftrightarrow \quad P = P(\alpha) = \frac{4m^2[K(\sin\frac{\alpha}{2})]^2}{l^2}B, \quad m = 1, 2, \ldots.$$

Note

Taylor expansions reveals that Euler's critical loads correctly determine the bifurcation points emanating from the trivial path.

Heading to the solutions

$$s\lambda + (4h+1)K(k) = \int_0^{\phi(s)} \frac{dt}{\sqrt{1-k^2\sin^2(t)}}, \quad h \in \mathbb{Z}$$

Explicit formula for ϕ

$$\phi(s) = \mathrm{am}(s\lambda + K(k), k) + 2h\pi, \quad h \in \mathbb{Z}$$

Implicit formula for θ

$$\sin(\theta/2) = k\,\mathrm{sn}(s\lambda + K(k), k).$$

Solutions

Differential equations

$$x_1'(s) = 1 - 2k^2 \mathrm{sn}^2(s\lambda + K(k), k)$$
$$x_2'(s) = 2k \mathrm{sn}(s\lambda + K(k), k) \mathrm{dn}(s\lambda + K(k), k).$$

Plane coordinates

$$x_1(s) = -s + \frac{2}{\lambda}\{E[\mathrm{am}(s\lambda + K(k), k), k] - E[\mathrm{am}(K(k), k), k]\}$$
$$x_2(s) = -\frac{2k}{\lambda} \mathrm{cn}(s\lambda + K(k), k).$$

Relative displacements

Horizontal and vertical

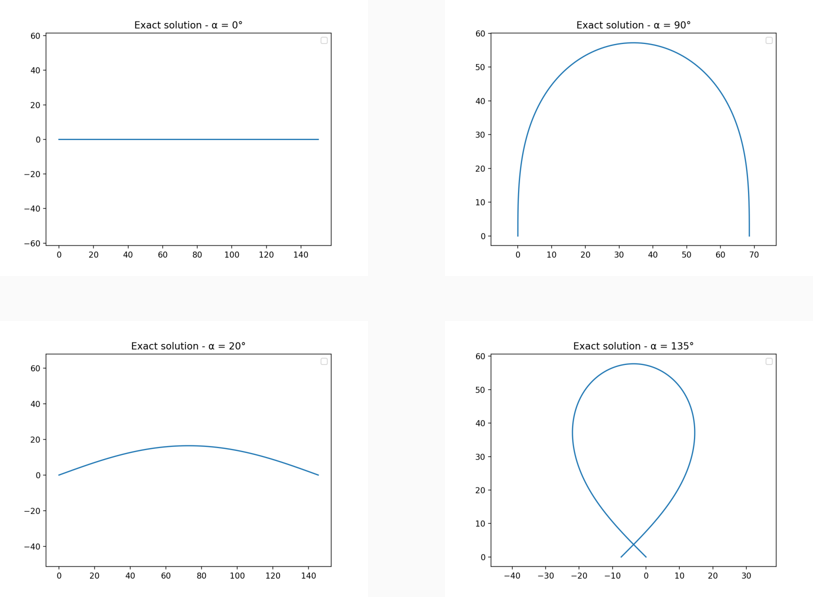
$$\frac{|u_1(l)|}{l} = 2 - 2\frac{E(k)}{K(k)}$$

$$\frac{|u_2(l/2)|}{l} = \frac{k}{mK(k)}, \quad m = 1, 3, 5, \dots$$

Simply supported elastica

Experimental results

Exact solutions



Experimental setting

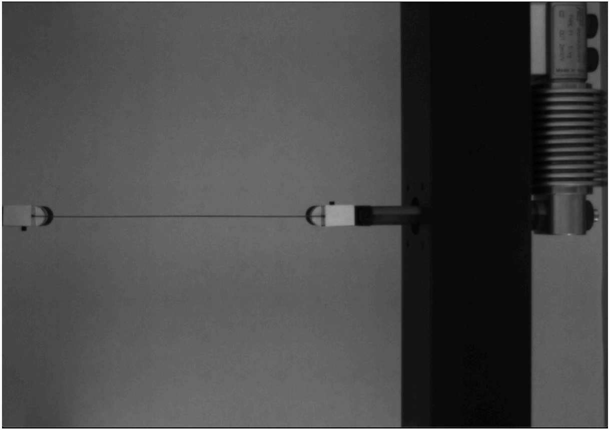


Figure 5: Rod in the initial configuration.

Raw results

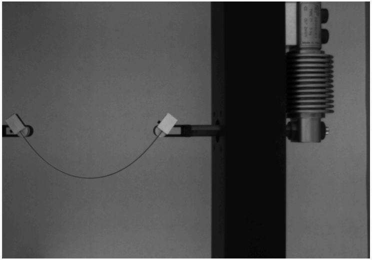


Figure 6: Photo of the experimental setup at $t = 200$s.

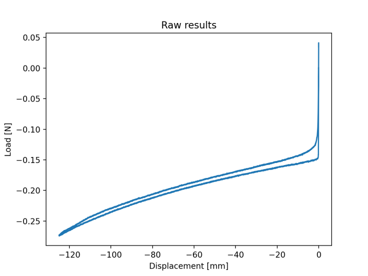
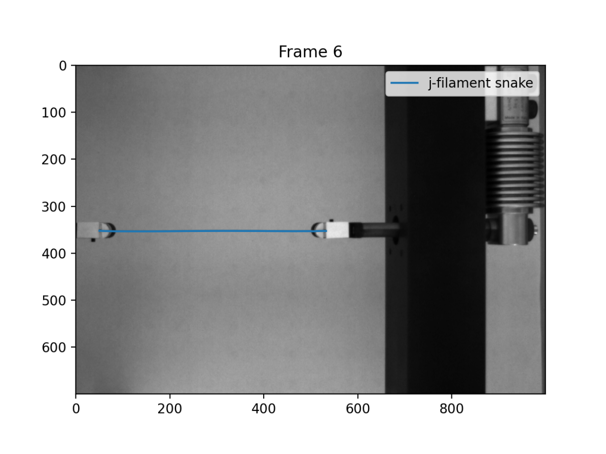
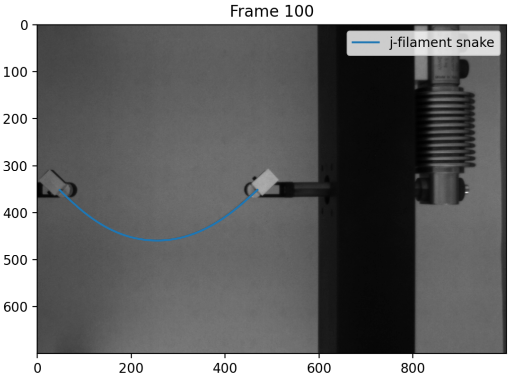
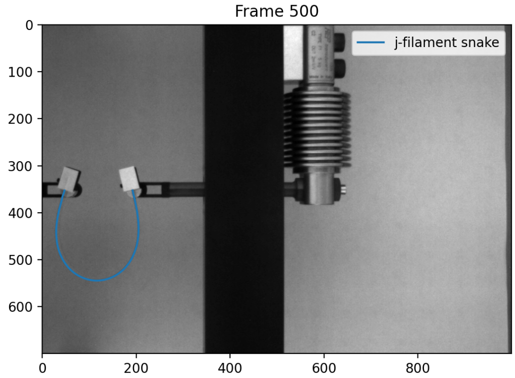


Figure 7: Measured load against measured displacement.

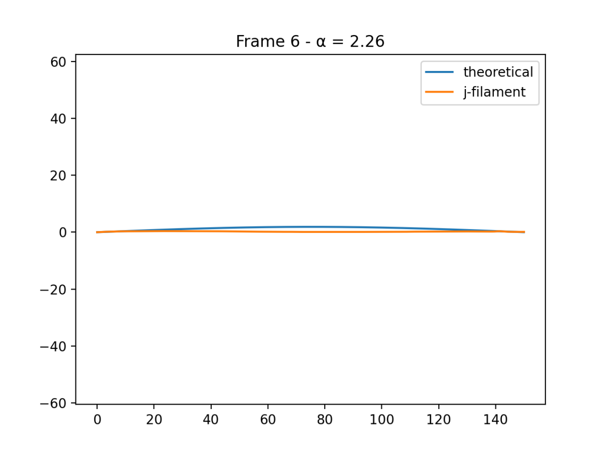
Jfilament tracking

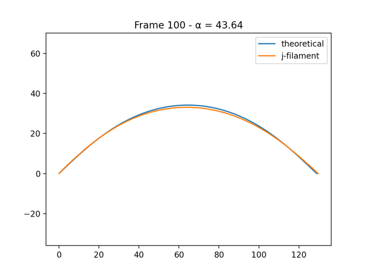


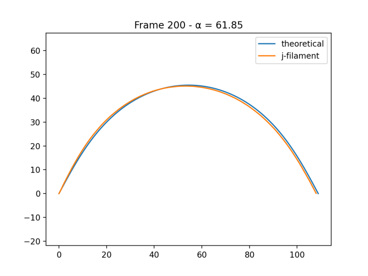

Frame 100
j-filament snake
0
100
200
300
400
500
600
0
200
400
600
800

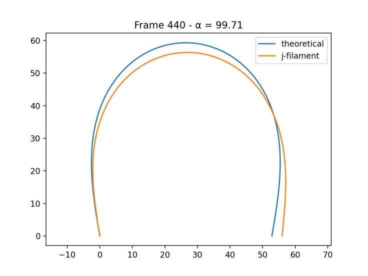

Frame 500
j-filament snake
0
100
200
300
400
500
600
0
200
400
600
800

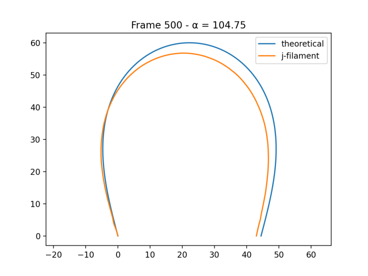
Comparison with theoretical curve




Frame 100 - α = 43.64
theoretical
j-filament
60
40
20
0
−20
0
20
40
60
80
100
120


Frame 200 - α = 61.85
theoretical
j-filament
60
50
40
30
20
10
0
−10
−20
0
20
40
60
80
100


Frame 440 - α = 99.71
theoretical
j-filament


Frame 500 - α = 104.75
theoretical
j-filament

Bending stiffness

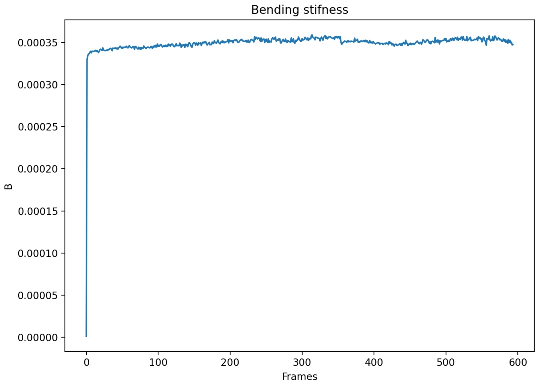


Figure 8: $B = B(k) = \dfrac{Pl^2}{4K^2(k)}$

Doubly clamped elastica

Description of the system

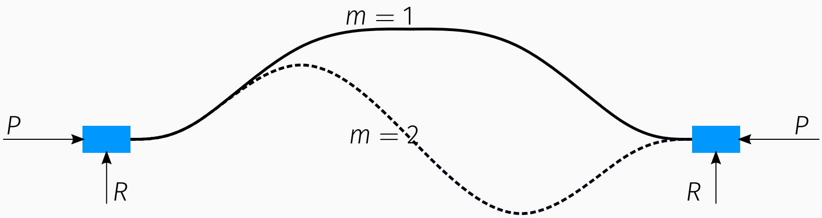


Figure 9: General scheme of the doubly clamped elastica

Equilibrium equations

Nonlinear eigenvalue problem

$$\theta''(s) + \frac{P}{B}\sin\theta(s) + \frac{R}{B}\cos\theta(s) = 0 \quad s \in [0, l], \qquad \text{Equilibrium eqn.}$$

$$\theta(0) = \theta(l) = 0, \qquad \text{Clamp constraint}$$

$$u_1(0) = 0, \qquad \text{Null displacement}$$

$$u_2(0) = u_2(l) = 0, \qquad \text{Null vert. displ.}$$

$$u_1'(s) = \cos\theta(s) - 1 \quad s \in [0, l], \qquad \text{Eqn. horiz. displ.}$$

$$u_2'(s) = \sin\theta(s) \quad s \in [0, l]. \qquad \text{Eqn. vert. displ.}$$

Always admits the trivial solution $\theta(s) = 0, \ \forall s \in [0, l]$.

Linearized equilibrium equations

Linear eigenvalue problem

$$\theta''(s) + \frac{P}{B}\theta(s) + \frac{R}{B} = 0 \quad s \in [0, l], \qquad \text{Equilibrium eqn.}$$

$$\theta(0) = \theta(l) = 0, \qquad \text{Clamp constraint}$$

$$\int_0^l \theta(s)\mathrm{d}s = 0. \qquad \text{Same vert. displ. ends}$$

Nontrivial solutions of linear eigenvalue problem

$$\theta(s) = \frac{R}{P}\left[\cos\left(\sqrt{\frac{P}{B}}s\right) + \frac{1 - \cos\left(\sqrt{\frac{Pl^2}{B}}\right)}{\sin\left(\sqrt{\frac{Pl^2}{B}}\right)}\sin\left(\sqrt{\frac{P}{B}}s\right) - 1\right],$$

Bifurcation loads for the doubly claped rod

The solution must satisfy

$$\int_0^l \theta(s)\mathrm{d}s = 0,$$

leading to the following

Characteristic equation

$$2\left(1 - \cos\sqrt{\frac{Pl^2}{B}}\right) = \sqrt{\frac{Pl^2}{B}} \sin\sqrt{\frac{Pl^2}{B}},$$

Critical loads

$$P_1^{\mathrm{cr}} = \frac{4\pi^2 B}{l^2}, \quad P_2^{\mathrm{cr}} = \frac{8.1830\pi^2 B}{l^2}, \quad P_3^{\mathrm{cr}} = \frac{16\pi^2 B}{l^2}, \ldots$$

Doubly clamped elastica

Symmetric buckling modes

Symetries of the odd buckling modes

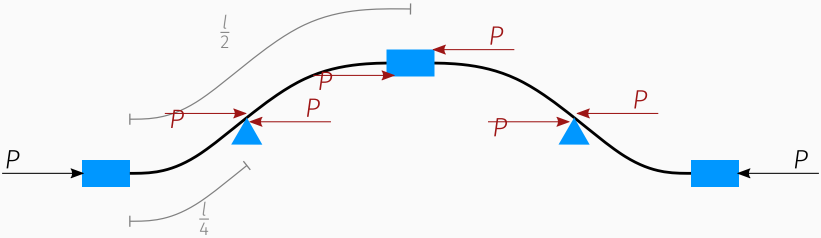


Figure 10: The symmetric configurations of the doubly clamped rod can be decomposed into 4 identical rods of length $l/4$ with zero slope θ at one end and zero moment θ' at the other.

Equilibrium of symmetric buckling modes

By equilibrium of vertical forces, the reaction term is $R = 0$. Let $\lambda^2 = \frac{P}{B}$

Equilibrium equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{2}(\theta'(s))^2 - \lambda^2 \cos\theta(s)\right] = 0,$$

$$\theta(0) = \theta'(l/4) = 0,$$

So integration over $[s, l/4]$ results in

Equation for the rotation field

$$\theta'(s) = \pm\lambda\sqrt{2\left(\cos\theta(s) - \cos\hat{\theta}\right)}, \quad \hat{\theta} = \theta(l/4).$$

Equilibrium of symmetric buckling modes continued

$$k = \sin\left(\frac{\hat{\theta}}{2}\right), \quad k \sin\phi(s) = \sin\left(\frac{\hat{\theta}}{2}\right),$$

Equation for the transformed rotation field

$$\begin{aligned} &\frac{\mathrm{d}\phi(s)}{\mathrm{d}s} = \lambda\sqrt{1 - k^2 \sin^2\phi(s)}, \quad && s \in [0, l/4], \\ &\phi(0) = h\pi, && \\ &\phi\left(\frac{l}{4}\right) = \frac{2j+1}{2}\pi, && h, j \in \mathbb{Z}, \end{aligned}$$

Implicit expression for $\phi(s)$

$$s\lambda = \int_{h\pi}^{\phi(s)} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \quad h \in \mathbb{Z}.$$

Critical loads

$$\frac{l}{4}\lambda = \int_{h\pi}^{\frac{2j+1}{2}\pi} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad h, j \in \mathbb{Z};$$

by periodicity and properties of the elliptic integral of the first kind, we can write

$$l\lambda = 2(m+1)K(k), \quad m = 1, 3, 5, \dots$$

which allows to determine

Critical loads

$$P_m^{cr} = \frac{(m+1)^2\pi^2 B}{l^2}\left[K\left(\sin\frac{\hat{\theta}}{2}\right)\right]^2, \quad m = 1, 3, 5, \dots.$$

Critical loads: comparison with linear analysis

For small $\hat{\theta}$, i. e. small $k = \sin\frac{\hat{\theta}}{2}$, a Taylor series expansion gives

$$P_m^{\text{cr}} \approx \frac{(m+1)^2\pi^2 B}{l^2}\left[K^2(0) + \cancelto{0}{K(0)K'(0)} + o\left(\hat{\theta}^2\right)\right],$$
$$= \frac{(m+1)^2\pi^2 B}{l^2}, \quad m = 1, 3, 5, \ldots$$

providing the odd critical values determined by the analysis of the linearized problem.

The elastic curve

Implicit expression for $\phi(s)$

$$s\lambda = \int_{h\pi}^{\phi(s)} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad h \in \mathbb{Z}.$$

The elastic curve

Explicit expression for $\phi(s)$

$$\phi(s) - h\pi = \mathrm{am}\,(s\lambda,\, k)$$

The elastic curve

Explicit expression for $\phi(s)$

$$\phi(s) - h\pi = \mathrm{am}\,(s\lambda,\, k)$$

Rotation field

$$\theta(s) = 2\arcsin\left(k\,\mathrm{sn}\,(s\lambda,\, k)\right), \quad s \in [0, l].$$

The elastic curve

Explicit expression for $\phi(s)$

$$\phi(s) - h\pi = \mathrm{am}\,(s\lambda,\, k)$$

Rotation field

$$\theta(s) = 2\arcsin\left(k\,\mathrm{sn}\,(s\lambda,\, k)\right), \quad s \in [0, l].$$

Kinematic boundary conditions

$$\begin{aligned} x_1'(s) &= \cos\theta(s), \\ &= 1 - 2k^2\,\mathrm{sn}^2(s\lambda,\, k), \\ x_2'(s) &= \sin\theta(s), \\ &= 2k\,\mathrm{sn}\,(s\lambda,\, k)\,\sqrt{1 - k^2\,\mathrm{sn}^2(s\lambda,\, k)}; \end{aligned}$$

The elastic curve

Explicit expression for $\phi(s)$

$$\phi(s) - h\pi = \mathrm{am}\,(s\lambda,\ k)$$

Rotation field

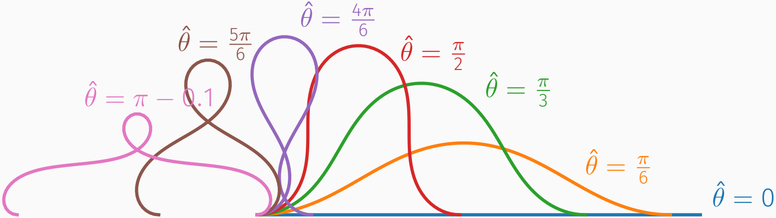
$$\theta(s) = 2\arcsin\left(k\,\mathrm{sn}\,(s\lambda,\ k)\right), \quad s \in [0, l].$$

Planar coordinates of the elastic curve

$$x_1(s) = -s + \frac{2}{\lambda}E\left[\mathrm{am}\,(s\lambda,\ k),\ k\right],$$

$$x_2(s) = \frac{2k}{\lambda}\left[1 - \ \mathrm{cn}\,(s\lambda,\ k)\right].$$

Doubly clamped elastic curve family



Displacement field

Horizontal displacement

$$u_1(l) = -2l + \frac{2}{\lambda} E\left[\operatorname{am}\left(sl,\ k\right),\ k\right],$$
$$= -2l + \frac{2(m+1)}{\lambda} E(k),$$

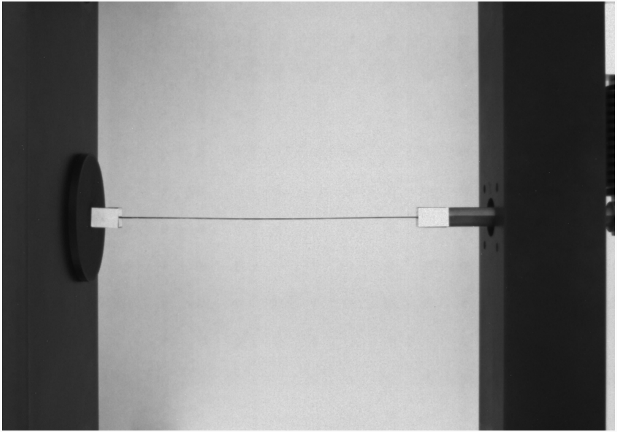
Relative horizontal displacement

$$\frac{|u_1(l)|}{l} = -2\left(\frac{E(k)}{K(k)} - 1\right),$$

Doubly clamped elastica

Experimental observations

Experimental setup



Raw results

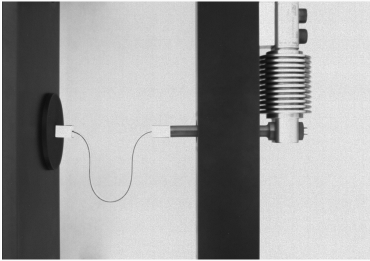


Figure 11: Photo of the experimental setup at $t = 400$s.

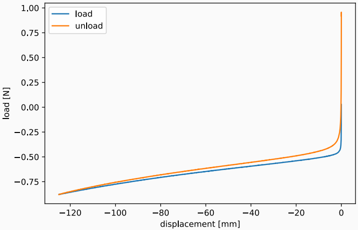


Figure 12: Measured load against measured displacement.

Comparison with theoretical load curve

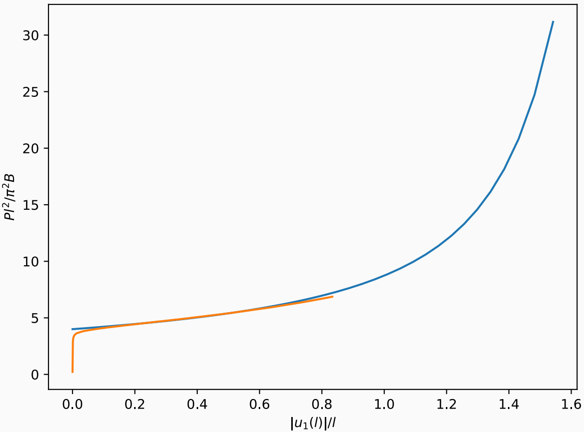


Figure 13: Dimensionless load versus dimensionless displacement.

Experimental determination of B

B from P

$$B = \frac{Pl^2}{4(\underbrace{m}_{1} + 1)^2 K^2(k)}$$

Experimental value of B

$$\text{mean}(B) = 2.916\,678 \times 10^{-4}\text{Nm}^2$$
$$\text{var}(B) = 1.538\,657 \times 10^{-11}$$

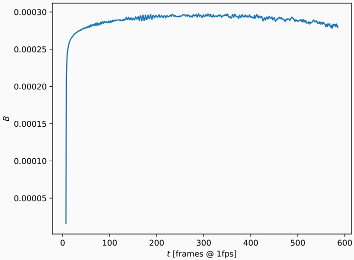


Figure 14: Experimental values of B during the experiment.

Expected value of B

Bending stiffness

$$B = EI,$$

E Young modulus, I moment of inertia relative to the normal direction of the inflection plane

Young modulus of PETG

$$E \in [1.9, 2.5]\text{GPa}$$

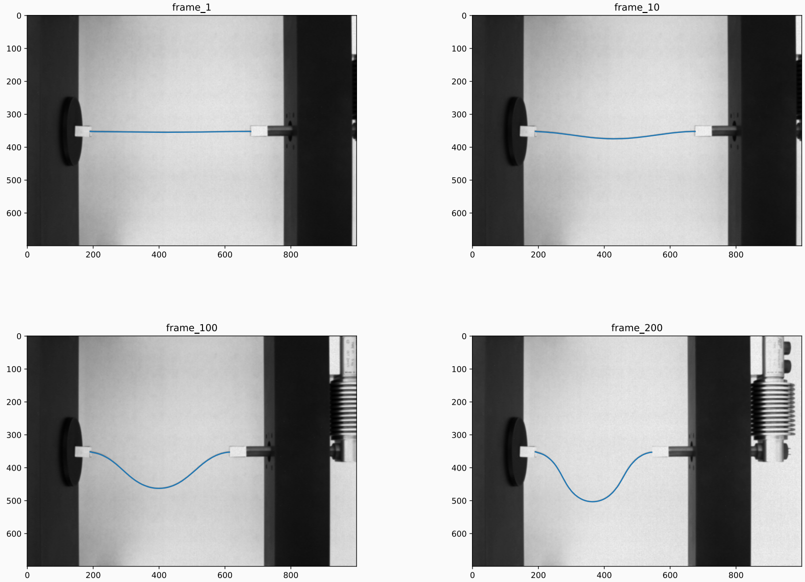
Moment of inertia

$$I = \frac{(0.5 \times 10^{-3}\text{m})^3 \times 10^{-2}\text{m}}{12} \approx 1.041\,666 \times 10^{-13}\text{m}^4$$

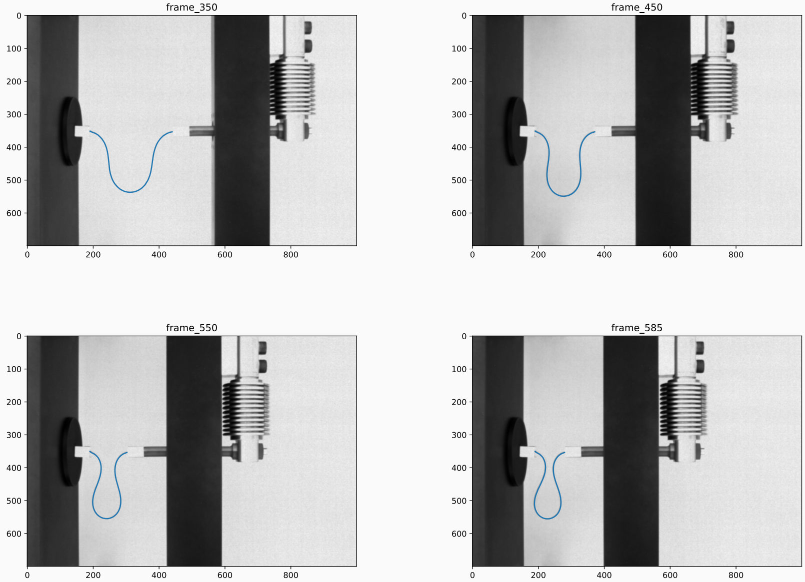
Expected bending stiffness

$$B \in [1.9 \times 10^{-4}, 2.6 \times 10^{-4}]\text{Nm}^2$$

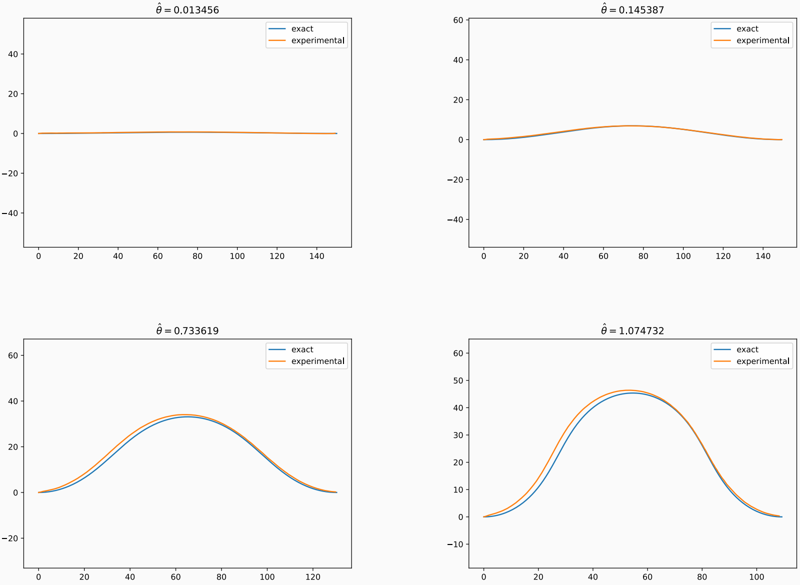
Traking of the rod



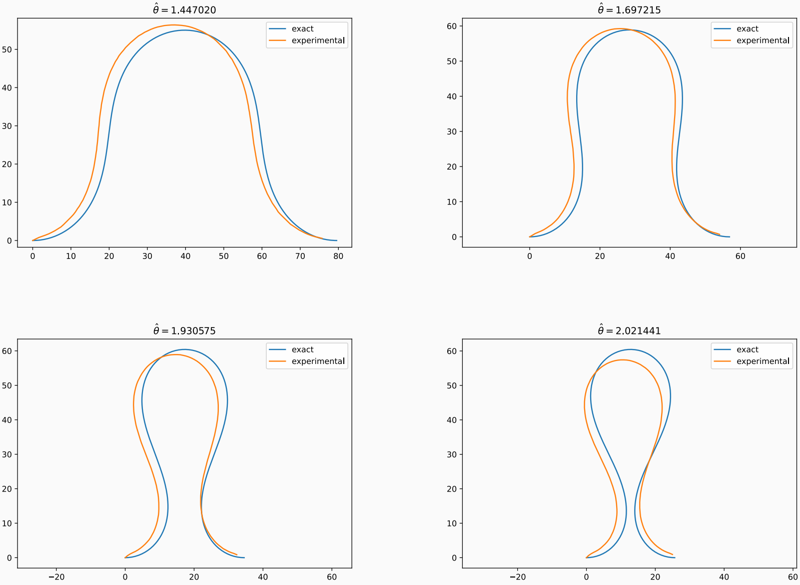
Traking of the rod



Comparison with theoretical curves



Comparison with theoretical curves



Thank you for your attention