Quench analysis of the Ising model

Final work for the course of Complex Systems
Prof. Alessandro Vezzani

Ariel S. Boiardi

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Dipartimento di Scienze Matematiche, Fisiche e Informatiche
Università degli Studi di Parma

Outline

The Ising model

Simulation

Dynamics

Quench

Anisotropic lattices and disorder

The Ising model

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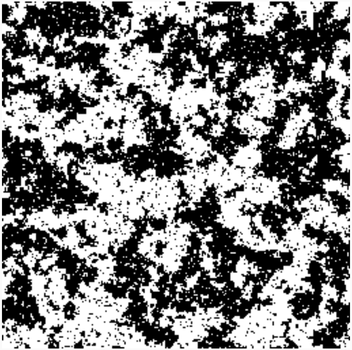
  - Found explicit formulas of the partition function and free energy for the 2D model $\Rightarrow$ all thermodynamic quantities
  - A rigorous proof of the existence of a phase transition was later given by Lee and Young [4, 3]

The Ising model / 2D lattice



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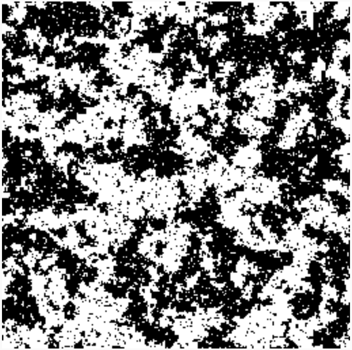
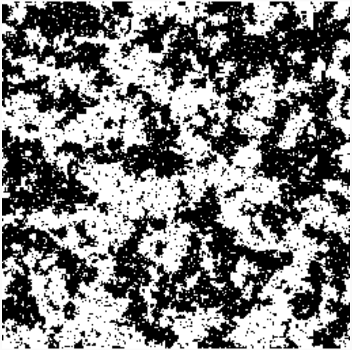


H

The Ising model / 2D lattice



$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

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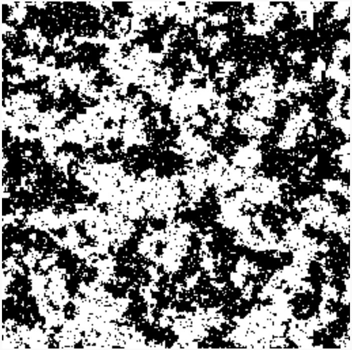

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_j s_j. \qquad (1)$$

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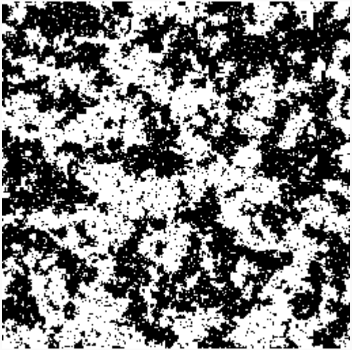

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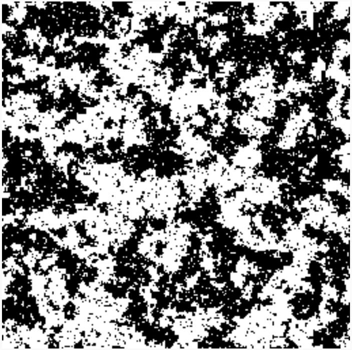

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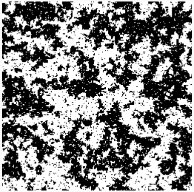
Susceptibility

$$\chi = \frac{\partial M}{\partial h} = N\left[\left\langle M_\Lambda^2\right\rangle - \langle M_\Lambda\rangle^2\right] \tag{6}$$

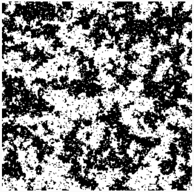
Simulation of the Ising model

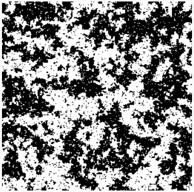
Simulation / Transition probability

s



s s'

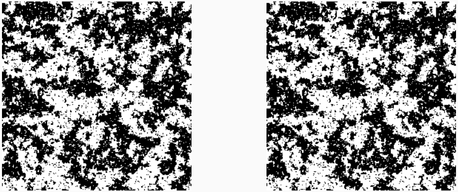




Simulation / Transition probability

$W(\mathsf{s} \to \mathsf{s}')$

s → s'



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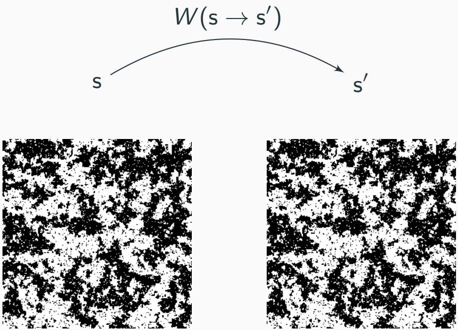

Metropolis

$$W(s \to s') = \begin{cases} 1 & \text{if } \delta H \leq 0 \\ e^{-\beta \delta H} & \text{if } \delta H > 0. \end{cases} \tag{7}$$

Simulation / Equilibrium phase transition

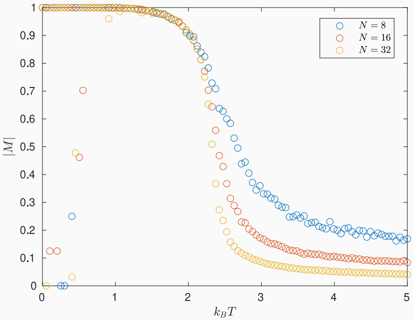

Figure 1: Cumulated magnetization

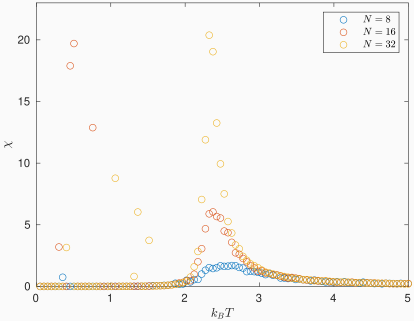

Figure 2: Magnetic susceptibility

Simulation / Numerical determination of the critical point

$$U_L = \frac{1}{2}\left(3 - \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}\right)$$

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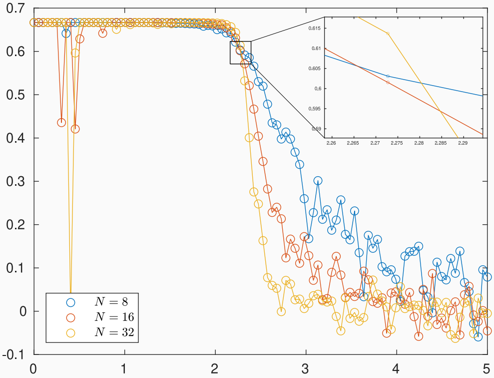
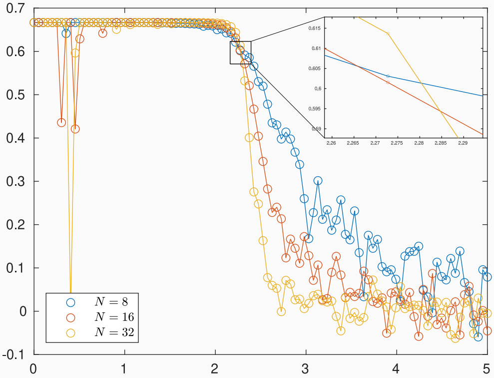


Figure 3: Binder's cumulant for three system sizes.

$$U_L = \frac{1}{2}\left(3 - \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}\right)$$

$$T_C \approx 2.275 \pm 0.01$$



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Dynamic Ising model

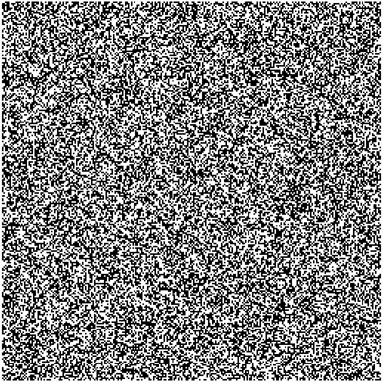
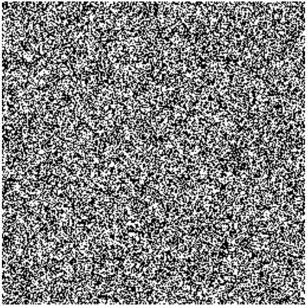


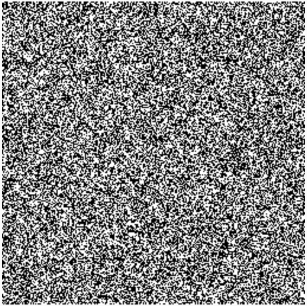
Figure 4: Critical dynamic of the Ising model

Quench

Quench / Supercritical



(a) Supercritical dynamic at $kT = 4$



(a) Supercritical dynamic at $kT = 2.5$

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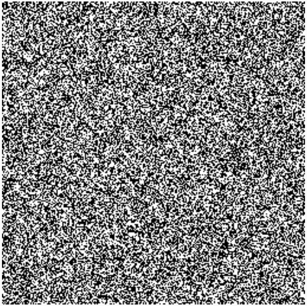

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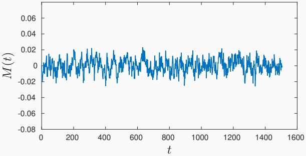
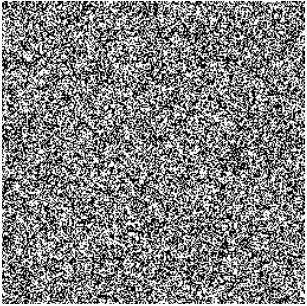


(b) Magnetization



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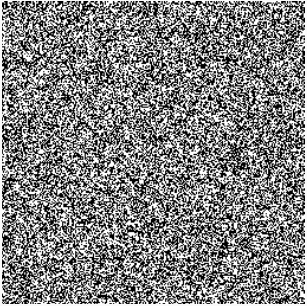

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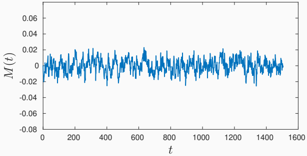

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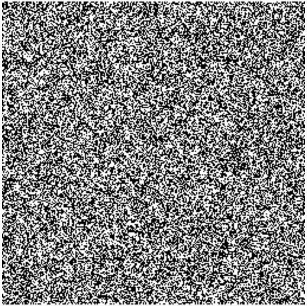

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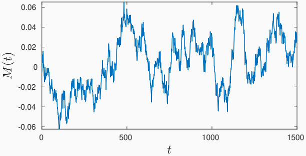

(b) Magnetization

Time displaced autocorrelation

$$\langle s(t_W)s(t)\rangle = \left\langle \frac{1}{N}\sum_{j=1}^{N} s_j(t_W)s_j(t) \right\rangle \tag{8}$$

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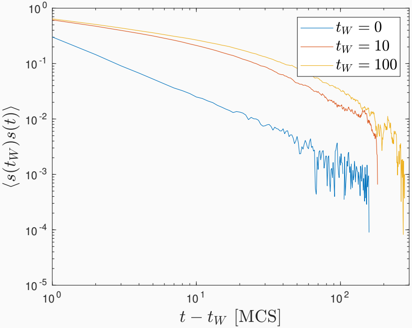
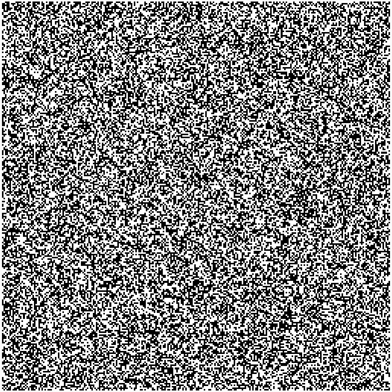


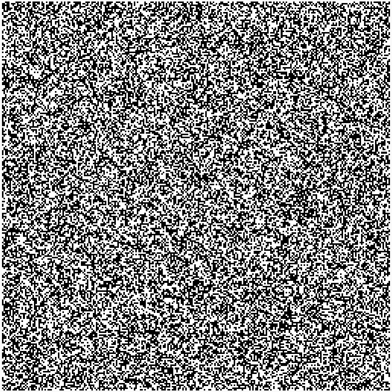
Figure 7: Time delayed autocorrelation function with respect to states at t_W. The random initial configuration is quenched at $T = 2.5$

Quench / Critical



(a) Critical dynamic at $kT = 2.27$

Quench / Critical



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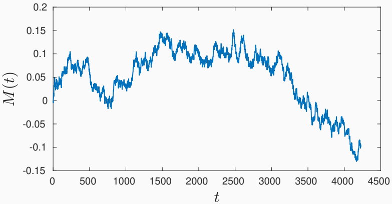


(b) Magnetization

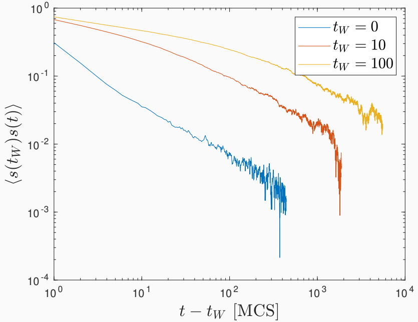
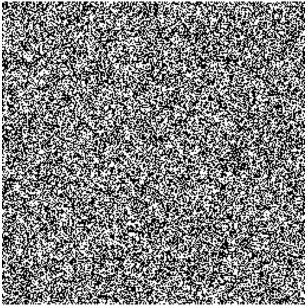


Figure 9: Time delayed autocorrelation function with respect to states at t_W. The random initial configuration is quenched at $T = T_C$. System linear size $L = 128$.

Quench / Subcritical



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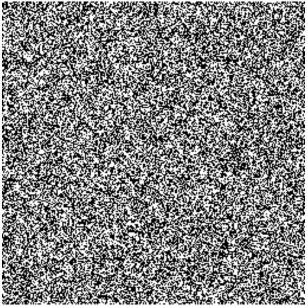
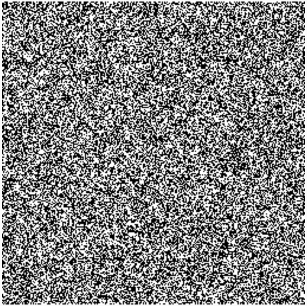


(a) Subcritical dynamic at $kT = 1$

Quench / Subcritical



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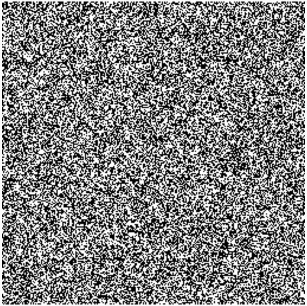

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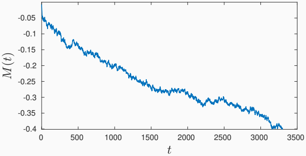
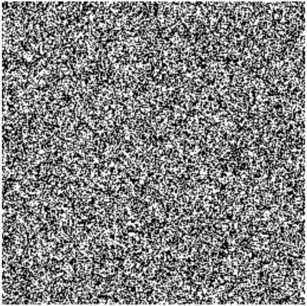


(b) Magnetization



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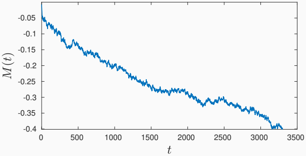

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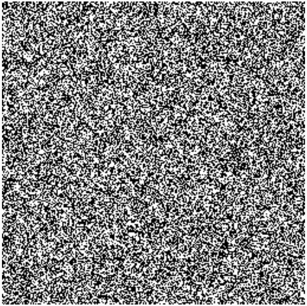

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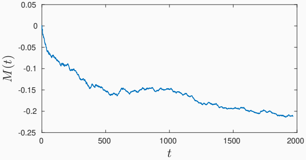

(b) Magnetization

Anisotropic lattices and disorder

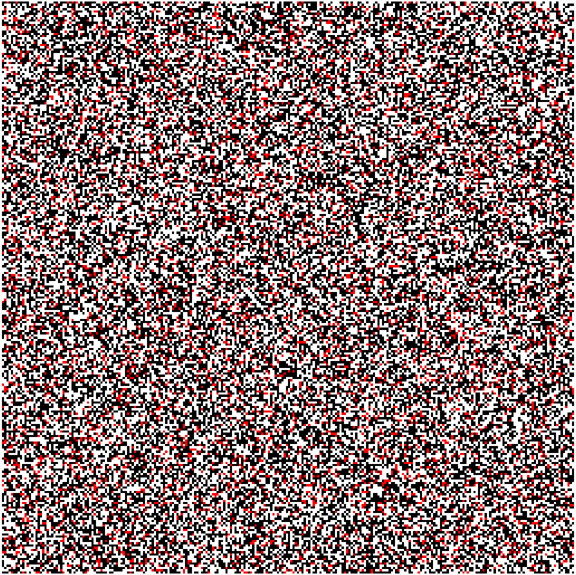

Figure 12: Disordered lattice, $kT = 1$.

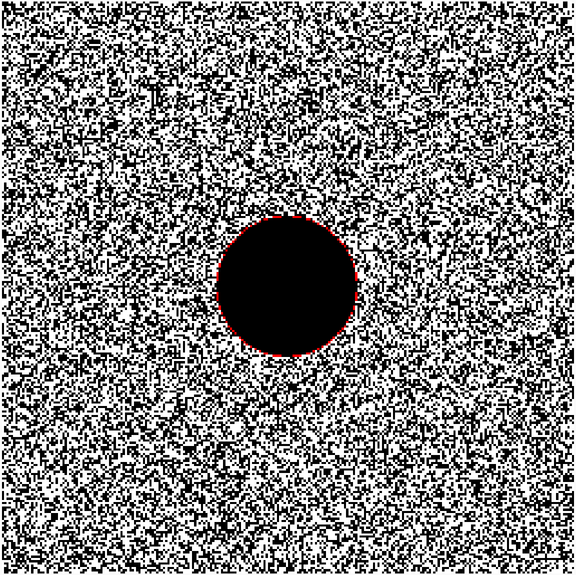

Figure 13: Disordered lattice, $kT = 1$.

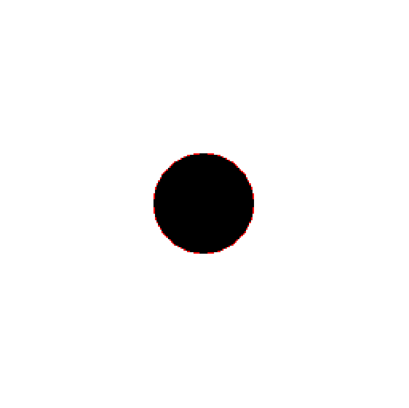

Figure 14: Process activated by thermal energy $kT = 2.26$.

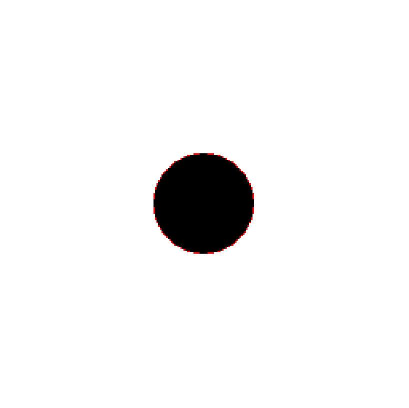

Figure 15: Process activated by external magnetic field $h = -0.5$.

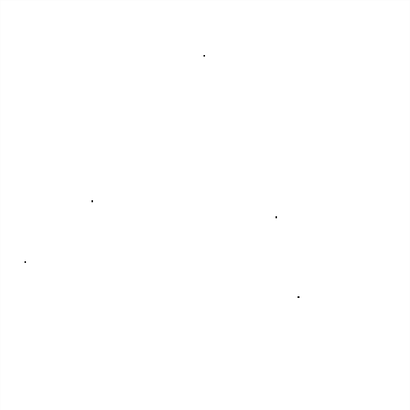

Figure 16: Nucleation at $kT = 1$, the system is prepared at a magnetization and evolved under a field in the opposite direction with $|h| = 0.5$, then the filed is flipped...

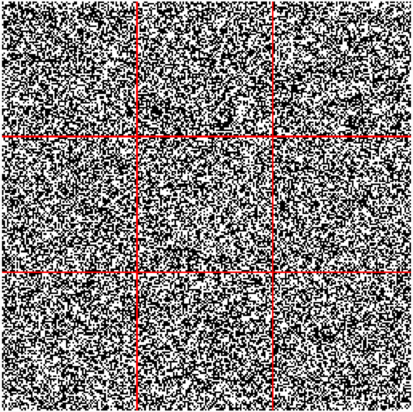

Figure 17: Split system at subcritical temperature $kT = 1$

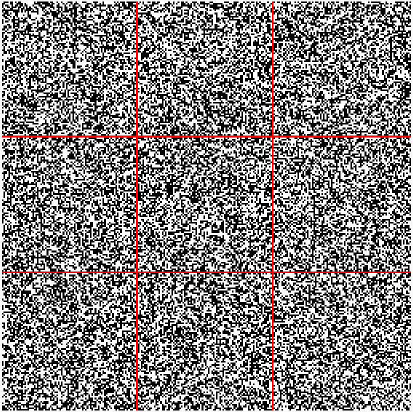

Figure 18: Split system at critical temperature

Thank you for your attention

References

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