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Mimetic Finite Differences and Virtual Element Methods for diffusion problems on polygonal meshes

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Abstract

We show that the nodal mimetic finite difference method for the discretization of diffusion problems on unstructured polygonal meshes can be recast in a generalised finite element framework herein denoted Virtual Element Method. First, we introduce a family of low-order virtual element spaces, i.e. finite element spaces that contain the linear polynomials plus other basis functions that are never defined explicitly; then, we show that each particular choice of the virtual basis functions yields an instance of the mimetic finite difference method. The case of quadrilateral meshes is analysed in details and some numerical examples are given.

Key words: Diffusion problem, Poisson equation, mimetic finite difference method, polygonal mesh, generalized mesh

1. Introduction

Mimetic discretizations incorporate important physical and mathematical properties of models based upon partial differential equations into the discrete framework. The list of investigated properties includes conservation laws, solution symmetries, positivity and monotonicity. The mimetic approach is in many ways a development of the pioneering work on the numerical treatment of diffusion problem with heterogeneous permeabilities on meshes with highly distorted cells carried out in [15, 21, 22]. Among the most recent developments, the Mimetic Finite Difference (MFD) method for elliptic problems in mixed [3, 5, 6, 8, 10, 11, 18] and nodal [4, 7] form combines the flexibility of the mesh that characterizes the finite volume methods with the solid mathematical foundation of the finite element method.

In the MFD method we consider a discrete variational problem that is formulated directly in terms of the degrees of freedom, while the underlying basis functions that would make a finite element interpretation possible are not specified explicitly. This approach allows us to use general polygonal and polyhedral meshes, even with non-matching and non-convex elements. The nodal MFD method was proved to be convergent and first-order accurate in a mesh-dependent energy-like norm in [7]. Here, we complete the analysis of the nodal MFD method by establishing a finite element type interpretation. First, we introduce a *virtual* element space, i.e. a finite element space whose basis functions are never defined explicitly but such that it contains the linear polynomials; then, we interpret the mimetic method as a Virtual Element Method with a particular choice of the basis functions. This approach constitutes a new framework for the generalisation of conforming linear elements from triangular to polygonal meshes; see also [1] where an arbitrary order Virtual Element Method is introduced. Analogous extensions of the finite element method to general meshes of polygons and polyhedra can be found in [19, 23–25], while connections between the MFD method and other schemes such as, e.g., the finite volume methods are established in [2, 13].

The paper is organized as follows. In Section 2 we introduce the model problem, and in Section 3 we briefly present the family of nodal MFD methods. In Section 4 we define the virtual element space and we introduce the finite element interpretation of the MFD method, while in Section 5 we analyze the inverse problem, i.e. to give conditions under which we can reproduce a given MFD method with a virtual element space. In Section 6 we show that the relationship between the two approaches is independent of the particular matrix representation used to analyse them. In Section 7 we discuss the particular but very important case of quadrilateral cells. In Section 8, we assess the robustness of the MFD method by studying how the accuracy of the numerical solution depends on the mimetic stabilization parameters. In Section 9, we offer our final remarks, perspectives for future developments, and conclusions.

2. Model problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Gamma = \partial\Omega$ its boundary. The diffusion of the scalar variable u in Ω is governed by the Poisson problem

$$-\mathrm{div}(\mathsf{K}\nabla u) = f \qquad \text{in } \Omega, \tag{1}$$

$$u = g \qquad \text{on } \Gamma, \tag{2}$$

where K is the diffusion tensor, f is the source term, and g is the Dirichlet boundary data. We assume that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma) \cap C^0(\Gamma)$. We also assume that K is a uniformly elliptic symmetric positive definite tensor in $\left(W^{1,\infty}(\Omega)\right)^{2\times 2}$. From this assumption it follows immediately that $\mathsf{K}(\mathbf{x})$ is a non-singular matrix for every $\mathbf{x} \in \Omega$, and that $\mathsf{K}^{-1}(\mathbf{x})$ is also a symmetric and positive definite matrix for every $\mathbf{x} \in \Omega$. We consider the affine subspace of $H^1(\Omega)$

$$\mathcal{V}_g = \left\{ v \in H^1(\Omega) \mid v_{|\Gamma} = g \right\},$$

and the linear subspace $\mathcal{V}_0$ for $g = 0$. The variational form of problem (1)–(2) reads

find $u \in \mathcal{V}_g$ such that

$$\int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV = \int_\Omega f v \, dV \qquad \forall v \in \mathcal{V}_0(\Omega). \tag{3}$$

The existence and uniqueness of the weak solution follows by proving that the bilinear form in (3) is continuous and coercive, see, for instance, [16].

3. The family of nodal MFD schemes

3.1. *Notation and technicalities*

A mesh, denoted by Ω_h, will be a polygonal partition of Ω in $\mathbb{R}^2$. Note that the polygons in Ω_h are not necessarily convex. A mesh is labeled by its diameter h, which is defined by $h = \max_{\mathsf{P}\in\Omega_h}\{h_\mathsf{P}\}$, where h_P is the diameter of the polygon P. We denote a generic *mesh vertex* by v and its positional vector by $\mathbf{x}_\mathsf{v}$; a generic *cell interface* or *boundary edge* by e and its length by $|\mathsf{e}|$; the area of the *polygonal cell* P by $|\mathsf{P}|$, its center of gravity by $\mathbf{x}^c_\mathsf{P}$ and its boundary by $\partial\mathsf{P}$. We further denote by m_P the number of vertices (end edges) of P and, for any edge $\mathsf{e} \subset \partial\mathsf{P}$, we denote by $\mathbf{n}_{\mathsf{P},\mathsf{e}}$ the unit normal vector pointing out of P along e.

Subscripts referring to local numberings will be omitted: the vertices and edges of a given $\mathsf{P} \in \Omega_h$ will be respectively denoted by v_i and e_i, for $i = 1, \ldots, m_\mathsf{P}$, with the edge e_i connecting the vertices v_i and v_{i+1}. Further, we assume that items indexed by $j + m_\mathsf{P}$ and j coincide for all j. We also denote the positional vector of v_i by $\mathbf{x}_i$, instead of $\mathbf{x}_{\mathsf{v}_i}$. The local indexed notation is shown in Figure 1.

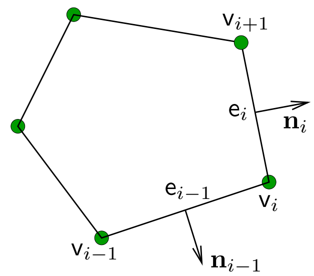

Fig. 1. Polygonal cell notation. The degrees of freedom are associated with the vertices (circles).

We generally denote the restrictions of a set of mesh items, e.g., vertices or edges, to a given geometrical object Q by the subscript $(\,\cdot\,)_Q$.

3.2. *Mesh regularity*

We assume that each mesh in the mesh family $\{\Omega_h\}_h$ is conformal, i.e., the intersection of any two distinct polygons P_1 and P_2 of a given mesh Ω_h is either empty, or the union of mesh points, or the union of mesh edges (two adjacent polygons may share more than one edge). Furthermore, we assume that for each mesh Ω_h there exists a sub-partition obtained by decomposing each polygon in a uniformly bounded number of triangles, whose union is a conformal and regular mesh in the sense of Ciarlet [12]. Mesh regularity assumptions are derived as a restriction to the two-dimensional case of the three-dimensional assumptions considered in [7] and formally stated below.

Assumption 3.1 (*HG - Shape-regularity*) *There exist two positive real numbers $\mathcal{N}^s$ and ρ_s such that every mesh Ω_h admits a sub-partition S_h into shape-regular triangles such that*

– *(**HG1**) every polygonal cell $\mathsf{P} \in \Omega_h$ admits a decomposition $\mathsf{S}_h|_{\mathsf{P}}$ made of less than $\mathcal{N}^s$ triangles;*

– *(**HG2**) the shape-regularity of the triangles $\mathsf{T} \in \mathsf{S}_h$ is defined as follows: the ratio between the radius r_{T} of the inscribed circle and the diameter h_{T} is bounded from below by the constant ρ_s:*

$$\frac{r_{\mathsf{T}}}{h_{\mathsf{T}}} \geq \rho_s > 0. \tag{4}$$

These minimal assumptions imply some restrictions on the elemental shape in order to avoid some pathological situations as $h \to 0$. Nonetheless, the meshes of $\{\Omega_h\}_h$ may contain very general elements including non-convex or "singular" elements, as the ones that may occur in adaptive mesh refinement strategies [20].

3.3. *Degrees of freedom and mimetic discretization*

Let $\mathcal{V}_h$ be the linear space of the *nodal grid functions* $v_h = \{v_{\mathsf{v}}\}$ that associate one real number v_{v} with each mesh vertex v. These numbers are the *degrees of freedom* of the numerical method and thus the dimension of the linear space $\mathcal{V}_h$ equals the number of mesh vertices. The restriction of the nodal function v_h to the cell P, i.e. $v_{h,\mathsf{P}} := v_{h|\mathsf{P}}$, belongs to the local set $\mathcal{V}_{h,\mathsf{P}} := \mathcal{V}_{h|\mathsf{P}}$ with dimension equal to the number of vertices of P. To ease the notation, the subscript $(\cdot)_{\mathsf{P}}$ will be dropped whenever possible. Figure 1 illustrates the degrees of freedom of a pentagonal cell.

We use the grid functions space $\mathcal{V}_h$ to represent scalar fields. More precisely, given $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$, we denote by $v^{\mathtt{I}} \in \mathcal{V}_h$ the grid function obtained evaluating v at the grid nodes, that is

$$v^{\mathtt{I}}_{\mathsf{v}} = v^{\mathtt{I}}{}_{|\mathsf{v}} := v(\mathbf{x}_{\mathsf{v}}) \quad \forall \mathsf{v} \in \Omega_h. \tag{5}$$

Further, the restriction of $v^{\mathtt{I}}$ on $\mathsf{P} \in \Omega_h$ is denoted by $v^{\mathtt{I}}_{\mathsf{P}}$, thus $v^{\mathtt{I}}_{\mathsf{P}} = v^{\mathtt{I}}{}_{|\mathsf{P}} = \{v^{\mathtt{I}}_{\mathsf{v}}\}_{\mathsf{v}\in\partial\mathsf{P}} \in \mathcal{V}_{h,\mathsf{P}}$.

The idea of the nodal mimetic methods is to approximate the left-hand side of (3) by a discrete bilinear form $\mathcal{A}_h : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ such that

$$\mathcal{A}_h\big(u^{\mathrm{I}}, v^{\mathrm{I}}\big) \approx \int_{\Omega} \mathsf{K}\nabla u \cdot \nabla v \, dV,$$

and the right-hand side of (3) by a linear functional $\mathcal{L}_h : \mathcal{V}_h \to \mathbb{R}$ such that

$$\mathcal{L}_h(v^{\mathrm{I}}) \approx \int_{\Omega} f v \, dV.$$

The construction of the bilinear form $\mathcal{A}_h$ and the linear functional $\mathcal{L}_h$ is fully detailed in the next subsections.

We treat the Dirichlet boundary conditions of non-homogeneous type by seeking the mimetic solution u_h in $\mathcal{V}_{h,g} \subset \mathcal{V}_h$, the subset of the discrete fields v_h such that:

$$v_{\mathsf{v}} = g(\mathbf{x}_{\mathsf{v}}) \quad \forall \mathsf{v} \in \Gamma^D.$$

The subspace $\mathcal{V}_{h,0}$ is defined by setting $g = 0$ in the definition of $\mathcal{V}_{h,g}$. The mimetic finite difference method for the approximation of the variational problem (3) reads as:

Find $u_h \in \mathcal{V}_{h,g}$ *such that:*

$$\mathcal{A}_h\big(u_h, v_h\big) = \mathcal{L}_h(v_h) \qquad \forall v_h \in \mathcal{V}_{h,0}. \tag{6}$$

3.4. *The mimetic bilinear form* $\mathcal{A}_h$

Let $v_{h,\mathsf{P}} \in \mathcal{V}_{h,\mathsf{P}}$ be the degrees of freedom of the nodal grid function $v_h \in \mathcal{V}_h$ restricted to the polygonal cell P. The mimetic bilinear form $\mathcal{A}_h$ assembles the *local* bilinear forms $\mathcal{A}_{h,\mathsf{P}} : \mathcal{V}_{h,\mathsf{P}} \times \mathcal{V}_{h,\mathsf{P}} \to \mathbb{R}$ defined for each mesh cell P, which acts on the local degrees of freedom:

$$\mathcal{A}_h\big(u_h, v_h\big) = \sum_{\mathsf{P}\in\Omega_h} \mathcal{A}_{h,\mathsf{P}}\big(u_{h,\mathsf{P}}, v_{h,\mathsf{P}}\big) \qquad \forall u_h, v_h \in \mathcal{V}_h. \tag{7}$$

The bilinear form $\mathcal{A}_{h,\mathsf{P}}$ is required to satisfy a *stability* and a *linear consistency* condition. In view of defining the stability condition we endow the space $\mathcal{V}_{h,0}$ with the H^1_0-like mesh-dependent norm

$$||v_h||^2_{1,h} = \sum_{\mathsf{P}\in\Omega_h} ||v_h||^2_{1,h,\mathsf{P}}, \tag{8}$$

where the local norm is given by

$$||v_h||^2_{1,h,\mathsf{P}} = |\mathsf{P}| \sum_{\mathsf{e}=(\mathsf{v},\mathsf{v}')\subset\partial\mathsf{P}} \frac{\big|v_{\mathsf{v}'} - v_{\mathsf{v}}\big|^2}{|\mathsf{e}|^2}. \tag{9}$$

Here, $\mathsf{e} = (\mathsf{v}, \mathsf{v}')$ is the edge that connects the vertices v and v'. The two above mentioned conditions read:

(**S1**) *stability*: there exists two positive constants σ_* and σ^* independent of P such that for every $v_{h,\mathsf{P}} \in \mathcal{V}_{h,\mathsf{P}}$ there holds:

$$\sigma_* ||v_{h,\mathsf{P}}||^2_{1,h,\mathsf{P}} \leq \mathcal{A}_{h,\mathsf{P}}\big(v_{h,\mathsf{P}}, v_{h,\mathsf{P}}\big) \leq \sigma^* ||v_{h,\mathsf{P}}||^2_{1,h,\mathsf{P}};$$

(**S2**) *linear consistency*: for every $p \in \mathbb{P}_1(\mathsf{P})$ and every $v_h \in \mathcal{V}_h$ there holds:

$$\mathcal{A}_{h,\mathsf{P}}\big(v_{h,\mathsf{P}}, p^{\mathrm{I}}\big) = \mathsf{K}_{\mathsf{P}}\nabla p \cdot \sum_{\mathsf{e}=(\mathsf{v},\mathsf{v}')\subset\partial\mathsf{P}} \mathbf{n}_{\mathsf{P},\mathsf{e}} \frac{|\mathsf{e}|}{2} \big(v_{\mathsf{v}} + v_{\mathsf{v}'}\big),$$

where K_{P} is a suitable constant approximation of K within P.

The constant tensor K_{P} appearing in (S2) can be obtained by averaging K over P. Since the diffusion tensor K is uniformly elliptic, K_{P} is symmetric and positive definite and this is sufficient to ensure the well-posedness of the numerical formulation. The regularity assumption on K stated in Section 2 ensures the following property:

$$\max_{i,j=1,2} \sup_{x\in\mathsf{P}} |(\mathsf{K}_{\mathsf{P}})_{ij} - \mathsf{K}_{ij}(x)| \leq C^*_{\mathsf{K}} h_{\mathsf{P}}, \tag{10}$$

where C^*_{K} is a non-negative constant independent of h_{P} and P. Inequality (10) is used to prove the convergence of the method, see [7]. Another possibility is to take $\mathsf{K}_{\mathsf{P}} = \mathsf{K}(\mathbf{x}^c_{\mathsf{P}})$ where $\mathbf{x}^c_{\mathsf{P}}$ is the centroid of P.

Remark 3.1 *To explain condition (S2) we note that, for any* $p \in \mathbb{P}_1(\mathsf{P})$ *and* $v \in H^1(\mathsf{P})$*, an integration by parts yields*

$$\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla p \cdot \nabla v \, dV = \int_{\partial \mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla p \cdot \mathbf{n}_{\mathsf{P}} \, v \, dS = \mathsf{K}_{\mathsf{P}} \nabla p \cdot \sum_{\mathsf{e} \subset \partial \mathsf{P}} \mathbf{n}_{\mathsf{P},\mathsf{e}} \int_{\mathsf{e}} v \, dS. \tag{11}$$

If we approximate the right-hand side integral on the edge e *in* (11) *by the trapezoidal rule, which uses only the values of* v *at the end points of the edge* e*, we obtain the right-hand side of (S2). Thus, the bilinear form* $\mathcal{A}_{h,\mathsf{P}}$ *is an approximation of the bilinear form associated with the continuous problem.*

The construction of bilinear forms satisfying assumptions (S1)-(S2) is detailed in Subsection 3.7.

3.5. *Discretization of the load term*

According to [7], the discretization of the load term $\mathcal{L}_h$ is based on an integration rule that is exact on constants. Let $\{\omega_{\mathsf{P},\mathsf{v}}\}_{\mathsf{v}\in\partial\mathsf{P}}$ be a set of non-negative weights, associated with the vertices of $\mathsf{P} \in \Omega_h$, such that

$$\sum_{\mathsf{v}\in\partial\mathsf{P}} \omega_{\mathsf{P},\mathsf{v}} = |\mathsf{P}| \,. \tag{12}$$

Let $\bar{f}_{\mathsf{P}}$ be either the cell average of f on P, or, when f is sufficiently regular, the pointwise value $f(\mathbf{x}^{\mathsf{c}}_{\mathsf{P}})$. Using the weights $\omega_{\mathsf{P},\mathsf{v}}$, we approximate the forcing term by using the linear functional $\mathcal{L}_h : \mathcal{V}_h \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}_h(v_h) = \sum_{\mathsf{P}\in\Omega_h} \big[f, v_h\big]_{\mathsf{P}} \qquad \text{where} \qquad \big[f, v_h\big]_{\mathsf{P}} = \bar{f}_{\mathsf{P}} \sum_{\mathsf{v}\in\partial\mathsf{P}} v_{\mathsf{v}} \omega_{\mathsf{P},\mathsf{v}}. \tag{13}$$

3.6. *Convergence theorem*

For completeness, we recall the convergence result in the mesh-dependent norm defined by (8)-(9) in the theorem below. The proof is found in [7].

Theorem 3.2 *Let* $u \in H^2(\Omega) \cap H^1_0(\Omega)$ *be the solution of problem* (1)-(2) *(with* $g = 0$*) in variational form and* $u^I = (u(\mathbf{x}_{\mathsf{v}}))_{\mathsf{v}\in\mathcal{V}}$ *the restriction of* u *to the mesh vertices in* Ω_h*. Let* u_h *be the numerical solution provided by* (6) *where* $\mathcal{A}_h$ *is built in accordance with (S1)-(S2) and* $\mathcal{L}_h$ *is provided by* (13)*. Let* h *be the mesh size parameter and assume that the mesh family* $\{\Omega_h\}$ *satisfies (HG1)-(HG2). Then, there exists a positive constant* C *independent of* h *such that*

$$\|u_h - u^I\|_{1,h} \leq Ch \left(|u|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$

3.7. *Construction of the local stiffness matrix*

Let P be a polygon in Ω_h. As it is standard in the presentation of the MFD methods, we define the two $m_{\mathsf{P}} \times 3$ matrices N and R which are used to give an explicit formula of the local mimetic stiffness matrix M^{m} associated to P.

We recall that $\mathbf{x}_i$, for $i = 1, \ldots, m_{\mathsf{P}}$, is the coordinate vector of the i-th vertex of cell P, e_i is the i-th edge connecting $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, $|\mathsf{e}_i|$ is its length and $\mathbf{n}_i$ is its unit normal vector pointing out of P. The $m_{\mathsf{P}} \times 3$ matrices N and R are given by

$$\mathsf{N} = \big[\mathbb{1} \ \widehat{\mathsf{N}}\big] \quad \text{and} \quad \mathsf{R} = \big[0 \ \widehat{\mathsf{R}}\big], \tag{14}$$

where $\mathbb{1} = [1 \ 1 \ \ldots \ 1]^T$ and $\widehat{\mathsf{N}}$ and $\widehat{\mathsf{R}}$ are the two $m_{\mathsf{P}} \times 2$ submatrices defined as:

$$\widehat{\mathsf{N}} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_{m_\mathsf{P}}^T \end{bmatrix} \quad \text{and} \quad \widehat{\mathsf{R}} = \frac{1}{2} \begin{bmatrix} |\mathsf{e}_{m_\mathsf{P}}|\,\mathbf{n}_{m_\mathsf{P}}^T + |\mathsf{e}_1|\,\mathbf{n}_1^T \\ |\mathsf{e}_1|\,\mathbf{n}_1^T + |\mathsf{e}_2|\,\mathbf{n}_2^T \\ \vdots \\ |\mathsf{e}_{m_\mathsf{P}-1}|\,\mathbf{n}_{m_\mathsf{P}-1}^T + |\mathsf{e}_{m_\mathsf{P}}|\,\mathbf{n}_{m_\mathsf{P}}^T \end{bmatrix} \mathsf{K}_\mathsf{P}. \tag{15}$$

The following algebraic identities hold (see [7]):

$$\mathsf{R}^T\mathsf{N} = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{\mathsf{R}}^T\widehat{\mathsf{N}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & |\mathsf{P}|\,\mathsf{K}_\mathsf{P} \end{bmatrix}. \tag{16}$$

It is shown in [7, 17] that assumption (S2) is equivalent to the algebraic condition $\mathsf{M}^\mathsf{m}\mathsf{N} = \mathsf{R}$. Matrix M^m can be decomposed as

$$\mathsf{M}^\mathsf{m} = \mathsf{M}_0^\mathsf{m} + \mathsf{M}_1^\mathsf{m} \tag{17}$$

where the matrices M_0^m and M_1^m are defined as follows. Matrix M_0^m is given by

$$\mathsf{M}_0^\mathsf{m} = \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{R}}^T, \tag{18}$$

and is such that $\mathsf{M}_0^\mathsf{m}\mathsf{N} = \mathsf{R}$ thanks to (16). Matrix M_0^m satisfies (S2) but has clearly rank 2 so that (except possibly for triangles) cannot satisfy (S1), since (S1) implies that the rank of M^m is $m_\mathsf{P} - 1$. Hence, M_0^m has to be corrected with an appropriate matrix M_1^m that must satisfy the condition $\mathsf{M}_1^\mathsf{m}\mathsf{N} = 0$. The role of this matrix in the splitting of (17) is, thus, to guarantee that the stability property (S1) of the mimetic scheme is satisfied. For this reason, we will refer to such term as *the mimetic stabilization term*. A stability analysis of the MFD method is beyond the scope of this paper, which is focused on establishing the relation between the mimetic and the virtual discretizations. More details are found in [7].

The choice of M_1^m is not unique. As discussed in [9], the family of matrices M_1^m with the right rank in view of (S1) and such that $\mathsf{M}_1^\mathsf{m}\mathsf{N} = 0$ can be constructed as

$$\mathsf{M}_1^\mathsf{m} = \mathsf{D}\mathsf{U}\mathsf{D}^T, \tag{19}$$

where D is an $m_\mathsf{P} \times (m_\mathsf{P} - 3)$ matrix such that $\mathsf{N}^T\mathsf{D} = 0$ and the columns of N and D span $\mathbb{R}^{m_\mathsf{P}}$, i.e., $\mathsf{span}\{\mathsf{N}, \mathsf{D}\} = \mathbb{R}^{m_\mathsf{P}}$; U is any $(m_\mathsf{P} - 3) \times (m_\mathsf{P} - 3)$ symmetric and positive definite matrix of parameters.

A popular choice that leads to a single parameter family of schemes is given by

$$\mathsf{U} = \mu\mathsf{I}_{m_\mathsf{P}-3}, \tag{20}$$

where μ is a real positive parameter.

Alternatively, we can set

$$\mathsf{U} = \nu(\mathsf{D}^T\mathsf{D})^{-1},$$

where ν is a real positive parameter, and we obtain $\mathsf{M}_1^\mathsf{m} = \nu\mathsf{D}(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T$. Now, we observe that

$$\mathsf{D}(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T + \mathsf{N}(\mathsf{N}^T\mathsf{N})^{-1}\mathsf{N}^T = \mathsf{I}_{m_\mathsf{P}}, \tag{21}$$

since $\mathsf{span}\{\mathsf{N}, \mathsf{D}\} = \mathbb{R}^{m_\mathsf{P}}$ by construction, and D is orthogonal to the columns of N. Thus, in this case

$$\mathsf{M}_1^\mathsf{m} = \nu\left(\mathsf{I}_{m_\mathsf{P}} - \mathsf{N}(\mathsf{N}^T\mathsf{N})^{-1}\mathsf{N}^T\right). \tag{22}$$

The matrices D and U, and, consequently, the parameters μ and ν depend on P and have to be chosen in such a way that assumption (S1) is satisfied.

4. Virtual element setting

We now introduce a finite element framework, herein named *virtual element method*, for the family of MFD methods described in the previous section. We show in this section that to each virtual element space there corresponds a MFD method, while in the next section we analyze the inverse problem, i.e. given an MFD method, find the associated virtual element space.

4.1. *The virtual element space*

Let us consider a set of functions $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ belonging to $H^1(\mathsf{P})$ and such that:

($\mathbf{P1}^{\chi}$) $\chi_{i|\mathsf{e}}$, the restriction of χ_i to $\mathsf{e} \subset \partial\mathsf{P}$, is a linear polynomial on each edge e;
($\mathbf{P2}^{\chi}$) $\chi_i(\mathbf{x}_j) = \delta_{ij}$;
($\mathbf{P3}^{\chi}$) the monomials 1, x, y belong to $\mathsf{span}\{\chi_1, \chi_2, \ldots, \chi_{m_{\mathsf{P}}}\}$.

We denote the local space generated by the linear combinations of these functions by $V_{h,\mathsf{P}}$, i.e., $V_{h,\mathsf{P}} = \mathsf{span}\{\chi_1, \chi_2, \ldots, \chi_{m_{\mathsf{P}}}\}$. The spaces $V_{h,\mathsf{P}}$ can be joined together to yield a conforming finite element space $V_h \subset H^1(\Omega)$. Note that, since any element of V_h can be identified with its nodal values, the dimension of V_h equals the dimension of the discrete space $\mathcal{V}_h$ introduced in Section 3.

Remark 4.1 *The functions χ_i that satisfy conditions (P1)-(P3) are also known as* barycentric coordinates *and are a possible basis for the polygonal finite element method, see, for instance, [24].*

As the construction and properties of V_h are local, we shall concentrate on a single cell P and consider the local space $V_{h,\mathsf{P}}$.

Our goal is to investigate the relation between the mimetic matrix M^{m} and the local stiffness matrix M^{χ} associated with the functions χ_i, whose ij-th component is

$$\left(\mathsf{M}^{\chi}\right)_{ij} = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\chi_i \cdot \nabla\chi_j \, dV. \tag{23}$$

To this end, let us first investigate the properties of such functions.

A straightforward consequence of $(\mathsf{P2}^{\chi})$ is that $\left(\chi_i\right)^{\mathsf{I}}$ is the i-th vector of the canonical basis of $\mathbb{R}^{m_{\mathsf{P}}}$. This fact implies that the functions $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ are linearly independent and thus $\mathsf{dim}\left(V_{h,\mathsf{P}}\right) = m_{\mathsf{P}}$. Moreover, every function v of $V_{h,\mathsf{P}}$ has the representation:

$$v(\mathbf{x}) = \sum_{i=1}^{m_{\mathsf{P}}} v(\mathbf{x}_i)\chi_i(\mathbf{x}) \quad a.e.\ \mathbf{x} \in \mathsf{P}. \tag{24}$$

In particular, by taking $v = 1$, $v = x$, and $v = y$ in (24) we obtain the decompositions:

$$\sum_{i=1}^{m_{\mathsf{P}}} \chi_i = 1, \quad \sum_{i=1}^{m_{\mathsf{P}}} x_i\chi_i = x, \quad \text{and} \quad \sum_{i=1}^{m_{\mathsf{P}}} y_i\chi_i = y, \quad \text{with} \quad \mathbf{x}_i = (x_i, y_i),\ i = 1, \ldots, m_{\mathsf{P}}. \tag{25}$$

The first identity in (25) is the *partition of unity* and together with the other identities expresses the *linear completeness* of $V_{h,\mathsf{P}}$, *cf.* $(\mathsf{P3}^{\chi})$.

If P is a triangle, we actually have that $V_{h,\mathsf{P}}$ coincides with $\mathbb{P}_1(\mathsf{P})$. Otherwise, properties $(\mathsf{P1}^{\chi})$-$(\mathsf{P3}^{\chi})$ do not determine uniquely the space $V_{h,\mathsf{P}}$. Indeed, $(\mathsf{P1}^{\chi})$–$(\mathsf{P2}^{\chi})$ determine the behaviour of the trace of each function χ_i on each cell edge of $\partial\mathsf{P}$, while the behaviour inside P depends on the specific set of functions. However, these functions exist and a very special and important case is provided by the solutions of the harmonic problem on P where the boundary conditions on $\partial\mathsf{P}$ are determined by $(\mathsf{P1}^{\chi})$-$(\mathsf{P2}^{\chi})$, cf. [1].

Since any $v \in V_{h,\mathsf{P}}$ is linear along any edge of P, its integral there is completely determined by the end points values. It follows that the integral of ∇v over P only depends on the value of v at the vertices of P. Indeed, introducing the vector $\mathbf{g}_i$ defined as the cell average of $\nabla\chi_i$, we get

$$\mathbf{g}_i = \frac{1}{|\mathsf{P}|}\int_{\mathsf{P}} \nabla\chi_i \, dV = \frac{1}{|\mathsf{P}|}\int_{\partial\mathsf{P}} \mathbf{n}_{\mathsf{P}}\chi_i \, dS = \frac{1}{|\mathsf{P}|}\sum_{j=1}^{m_{\mathsf{P}}} \mathbf{n}_j \int_{\mathsf{e}_j} \chi_i \, dS = \frac{1}{2\,|\mathsf{P}|}\left(\mathbf{n}_{i-1}\,|\mathsf{e}_{i-1}| + \mathbf{n}_i\,|\mathsf{e}_i|\right), \tag{26}$$

as the trace of χ_i along the cell boundary $\partial\mathsf{P}$ is different from zero only along the edges e_i and e_{i-1}. The average gradient of any $v \in V_{h,\mathsf{P}}$ can now be obtained by linearity.

By comparison with (14)-(15), it follows that

$$\widehat{\mathsf{R}}^T = |\mathsf{P}|\,\mathsf{K}_\mathsf{P}\big[\mathbf{g}_1\ \mathbf{g}_2\ \ldots\ \mathbf{g}_{m_\mathsf{P}}\big]. \tag{27}$$

It also holds that $\widehat{\mathsf{R}}^T\mathbb{1} = 0$; indeed, using equations (26) and (27), rearranging the argument of the summation and the integral, and using the partition of unity (25) yield:

$$\widehat{\mathsf{R}}^T\mathbb{1} = |\mathsf{P}|\,\mathsf{K}_\mathsf{P}\sum_{i=1}^{m_\mathsf{P}}\mathbf{g}_i = \mathsf{K}_\mathsf{P}\sum_{i=1}^{m_\mathsf{P}}\int_\mathsf{P}\nabla\chi_i\,dV = \mathsf{K}_\mathsf{P}\int_\mathsf{P}\nabla\Big(\sum_{i=1}^{m_\mathsf{P}}\chi_i\Big)\,dV = \mathsf{K}_\mathsf{P}\int_\mathsf{P}\nabla(1)\,dV = 0. \tag{28}$$

Now, we can easily prove the following lemma.

Lemma 4.1 *Let $\widehat{\mathsf{R}}$ be the matrix defined by (15), or, equivalently, by (27). For every function $v \in V_{h,\mathsf{P}}$ it holds that*

$$\widehat{\mathsf{R}}^T v^I = \int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla v\,dV. \tag{29}$$

Proof. The representation of v given in (24) implies that

$$\int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla v\,dV = \sum_{i=1}^{m_\mathsf{P}} v(\mathbf{x}_i)\int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla\chi_i\,dV.$$

To complete the proof, we note that K_P is constant on P and we express the integral of $\nabla\chi_i$ using the formula for matrix $\widehat{\mathsf{R}}$ provided by (27):

$$\int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla\chi_i\,dV = \mathsf{K}_\mathsf{P}\int_\mathsf{P}\nabla\chi_i\,dV = \mathsf{K}_\mathsf{P}\,\mathbf{g}_i\,|\mathsf{P}| = i\text{-th column of }\widehat{\mathsf{R}}^T \quad \text{for } i = 1,\ldots,m_\mathsf{P}. \tag{30}$$

□

Now, it is convenient to introduce another set of basis functions that also generate the functional space $V_{h,\mathsf{P}}$. This basis is provided by m_P linearly independent functions $\phi_1,\phi_2,\ldots,\phi_{m_\mathsf{P}}$ defined on P as follows:

($\mathbf{P1}^\phi$) $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = y$,
($\mathbf{P2}^\phi$) ϕ_i for $i = 4,\ldots,m_\mathsf{P}$ are linearly independent combinations of the functions χ_i such that

$$\int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla\phi_i\cdot\nabla\phi_j\,dV = 0 \qquad i = 1,\ldots,3 \ \text{ and } \ j = 4,\ldots,m_\mathsf{P}. \tag{31}$$

The advantage offered by the set of basis functions $\{\phi_i\}_{i=1,\ldots,m_\mathsf{P}}$ is that they permit to separate the part of the stiffness matrix that reflects the inclusion of the linear polynomials in $V_{h,\mathsf{P}}$ from the part that depends on the other functions generating $V_{h,\mathsf{P}}$. Both properties ($\mathsf{P1}^\phi$) and ($\mathsf{P2}^\phi$) are crucial to this purpose as they provide a structured form of the stiffness matrix of the functions ϕ_i, here denoted by M^ϕ. This topic will be the subject of the next section.

The existence of the functions ϕ_j, $j = 4,\ldots,m_\mathsf{P}$, satisfying ($\mathsf{P2}^\phi$) can be proved as follows. Let us first decompose $V_{h,\mathsf{P}}$ as:

$$V_{h,\mathsf{P}} = \mathsf{span}\{1\}\oplus\mathsf{span}\{x,y\}\oplus S. \tag{32}$$

Then, we note that the bilinear form

$$u,v \mapsto \int_\mathsf{P}\mathsf{K}_\mathsf{P}\nabla u\cdot\nabla v\,dV \tag{33}$$

is a scalar product on $\mathsf{span}\{x,y\}\oplus S$. Hence, we choose the subspace S' as the orthogonal complement of $\mathsf{span}\{x,y\}$ in $\mathsf{span}\{x,y\}\oplus S$ with respect to such scalar product. Moreover, since there must hold that

$$V_{h,\mathsf{P}} = \mathsf{span}\{1\}\oplus\mathsf{span}\{x,y\}\oplus S', \tag{34}$$

we have that $\mathsf{dim}\big(S'\big) = \mathsf{dim}\big(V_{h,\mathsf{P}}\big) - \mathsf{dim}\big(\mathsf{span}\{1\}\big) - \mathsf{dim}\big(\mathsf{span}\{x,y\}\big) = m_{\mathsf{P}} - 3$, and every basis of the subspace S' is a proper choice for the $m_{\mathsf{P}} - 3$ functions ϕ_j, $j = 4, \ldots, m_{\mathsf{P}}$.

Arguing as in Remark 3.1, we note that the integral in ($\mathsf{P2}^{\phi}$) depends only on the degrees of freedom of the functions $\phi_4,\ldots,\phi_{m_{\mathsf{P}}}$ and, in particular, is independent of their internal values. This fact allows us to determine the stiffness matrix transformation between the basis $\{\phi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ and $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$.

Let $\mathsf{A} = (a_{ij})$ and $\mathsf{B} = (b_{ij})$ be the matrices transforming the basis $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ into $\{\phi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ and viceversa so that

$$\phi_i = \sum_{j=1}^{m_{\mathsf{P}}} a_{ij}\chi_j \qquad \text{and} \qquad \chi_i = \sum_{j=1}^{m_{\mathsf{P}}} b_{ij}\phi_j. \tag{35}$$

By definition, the matrices A and B are non-singular and such that $\mathsf{A} = \mathsf{B}^{-1}$. Moreover, the stiffness matrices associated with the functions $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ and $\{\phi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ are transformed in accordance with the formulas:

$$\mathsf{M}^{\phi} = \mathsf{A}\mathsf{M}^{\chi}\mathsf{A}^T \qquad \text{and} \qquad \mathsf{M}^{\chi} = \mathsf{B}\mathsf{M}^{\phi}\mathsf{B}^T. \tag{36}$$

By setting $v = \phi_i$ in (24) and comparing with the left-most expression in (35) we obtain that $a_{ij} = \phi_i(\mathbf{x}_j)$. Then, we reformulate the relations in (25) as

$$\phi_1 = \sum_{i=1}^{m_{\mathsf{P}}} \chi_i, \quad \phi_2 = \sum_{i=1}^{m_{\mathsf{P}}} x_i\chi_i, \quad \phi_3 = \sum_{i=1}^{m_{\mathsf{P}}} y_i\chi_i, \tag{37}$$

and we note that the first three rows of the transformation matrix A coincide with the matrix N^T defined in (14). The remaining rows of A contain the coefficients of ϕ_j for $j = 4, \ldots, m_{\mathsf{P}}$ with respect to the basis functions $\{\chi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ and are denoted by $\mathbf{a}_i^T$ for $i = 4, \ldots, m_{\mathsf{P}}$. Thus, collecting the latters into the matrix $\widehat{\mathsf{A}} = \big[\mathbf{a}_4 \ \ldots \ \mathbf{a}_{m_{\mathsf{P}}}\big]$, the matrix A can be written as

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \ldots & 1 \\ x_1 & x_2 \ \ldots & x_{m_{\mathsf{P}}} \\ y_1 & y_2 \ \ldots & y_{m_{\mathsf{P}}} \\ a_{41} & a_{42} \ \ldots & a_{4m_{\mathsf{P}}} \\ \vdots & \vdots & \vdots \\ a_{m_{\mathsf{P}}1} & a_{m_{\mathsf{P}}2} \ \ldots & a_{m_{\mathsf{P}}m_{\mathsf{P}}} \end{bmatrix} = \begin{bmatrix} \mathsf{N}^T \\ \mathbf{a}_4^T \\ \ldots \\ \mathbf{a}_{m_{\mathsf{P}}}^T \end{bmatrix} = \begin{bmatrix} \mathsf{N}^T \\ \widehat{\mathsf{A}}^T \end{bmatrix}. \tag{38}$$

Lemma 4.2 *The orthogonality condition (*$\mathsf{P2}^{\phi}$*) is equivalent to* $\widehat{\mathsf{R}}^T\widehat{\mathsf{A}} = 0$.

Proof. As noted before, each vector $\mathbf{a}_j$ collects the degrees of freedom of the corresponding function ϕ_j. As $\phi_2 = x$ and $\phi_3 = y$, we have that $\nabla\phi_2 = [1\ 0]^T$ and $\nabla\phi_3 = [0\ 1]^T$. Now, Lemma 4.1 implies that

$$0 = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}}\nabla\phi_2 \cdot \nabla\phi_j\, dV = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}}\nabla\phi_j\, dV = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \widehat{\mathsf{R}}^T\mathbf{a}_j$$

and similarly for ϕ_3. The orthogonality condition $\widehat{\mathsf{R}}^T\widehat{\mathsf{A}} = 0$ follows by considering all such relations for $j = 4, \ldots, m_{\mathsf{P}}$. □

Remark 4.2 *Conditions (*$\mathsf{P1}^{\phi}$*)-(*$\mathsf{P2}^{\phi}$*) do not determine uniquely the set of functions* $\{\phi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$*. In fact, a unique set of "*ϕ*" functions is determined as linear combinations of the "*χ*" functions once a couple of non-singular matrices* A *and* $\mathsf{B} = \mathsf{A}^{-1}$ *are fixed provided that* A *has the structure shown in* (38) *and its submatrix* $\widehat{\mathsf{A}}$ *satisfies the orthogonality condition of Lemma 4.2.*

The basis transformation given by the matrices A and B is fully characterized by the following lemma. This lemma also shows that the matrix D^{ϕ} introduced in the lemma to define B has the same properties of the matrix D used in (19) for the mimetic stabilization term.

Lemma 4.3 *Let* N *and* R *be the matrices defined in* (14) *with submatrices* $\widehat{\mathsf{N}}$ *and* $\widehat{\mathsf{R}}$ *such that* (16) *holds. Let* $\widehat{\mathsf{A}}$ *be a full rank* $(m_{\mathsf{P}}-3)\times m_{\mathsf{P}}$ *matrix, whose columns are linearly independent of the columns of* N *and satisfying the orthogonality condition* $\widehat{\mathsf{R}}^T\widehat{\mathsf{A}}=0$*. Let* A *be the matrix defined in* (38) *and* $\mathsf{B}=\mathsf{A}^{-1}$*. Then, the matrix* B *can be written in the following way:*

$$\mathsf{B}=\left[\mathbf{b}_1,\widehat{\mathsf{B}},\mathsf{D}^{\phi}\right] \tag{39}$$

with

$$\widehat{\mathsf{B}}=\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|},\qquad \mathsf{D}^{\phi}=\left(\mathsf{I}_{m_{\mathsf{P}}}-\mathbf{b}_1\mathbb{1}^T-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\widehat{\mathsf{A}}\left(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}}\right)^{-1}. \tag{40}$$

Furthermore, D^{ϕ} *is an* $m_{\mathsf{P}}\times(m_{\mathsf{P}}-3)$ *matrix such that* $\mathsf{N}^T\mathsf{D}^{\phi}=0$ *and the columns of* N *and* D^{ϕ} *together are a basis of* $\mathbb{R}^{m_{\mathsf{P}}}$*.*

Proof. We derive the formulas in (40) by writing explicitly that B is the inverse of A:

$$\mathsf{BA}=\mathbf{b}_1\mathbb{1}^T+\widehat{\mathsf{B}}\widehat{\mathsf{N}}^T+\mathsf{D}^{\phi}\widehat{\mathsf{A}}^T=\mathsf{I}_{m_{\mathsf{P}}}. \tag{41}$$

To derive the expression for matrix $\widehat{\mathsf{B}}$, let us multiply (41) from the right by $\widehat{\mathsf{R}}$:

$$\mathbf{b}_1\,\mathbb{1}^T\widehat{\mathsf{R}}+\widehat{\mathsf{B}}\,\widehat{\mathsf{N}}^T\widehat{\mathsf{R}}+\mathsf{D}^{\phi}\,\widehat{\mathsf{A}}^T\widehat{\mathsf{R}}=\widehat{\mathsf{R}}$$

As $\mathbb{1}^T\widehat{\mathsf{R}}=0$, $\widehat{\mathsf{N}}^T\widehat{\mathsf{R}}=\mathsf{K}_{\mathsf{P}}\,|\mathsf{P}|$, and $\widehat{\mathsf{A}}^T\widehat{\mathsf{R}}=0$, we obtain that

$$\widehat{\mathsf{B}}\,\mathsf{K}_{\mathsf{P}}\,|\mathsf{P}|=\widehat{\mathsf{R}},$$

from which the expression for $\widehat{\mathsf{B}}$ follows. To derive the expression for matrix D^{ϕ}, we substitute $\widehat{\mathsf{B}}$ into (41) and multiply the resulting equation from the right by $\widehat{\mathsf{A}}$:

$$\mathbf{b}_1\,\mathbb{1}^T\widehat{\mathsf{A}}+\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\,\widehat{\mathsf{N}}^T\widehat{\mathsf{A}}+\mathsf{D}^{\phi}\,\widehat{\mathsf{A}}^T\widehat{\mathsf{A}}=\widehat{\mathsf{A}}. \tag{42}$$

As $\widehat{\mathsf{A}}$ is full rank, so is matrix $\widehat{\mathsf{A}}^T\widehat{\mathsf{A}}$, which is, thus, nonsingular. We multiply (42) from the left by the inverse matrix $\left(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}}\right)^{-1}$ and rearrange the terms to obtain the expression for D^{ϕ} stated in (40).

To prove the last part of the lemma, we use the relation $\mathsf{AB}=\mathsf{I}_{m_{\mathsf{P}}}$. A straightforward calculation yields

$$\mathsf{AB}=\begin{bmatrix}\mathbb{1}^T\\ \widehat{\mathsf{N}}^T\\ \widehat{\mathsf{A}}^T\end{bmatrix}\left[\mathbf{b}_1,\widehat{\mathsf{B}},\mathsf{D}^{\phi}\right]=\begin{bmatrix}\mathbb{1}^T\mathbf{b}_1 & \mathbb{1}^T\widehat{\mathsf{B}} & \mathbb{1}^T\mathsf{D}^{\phi}\\ \widehat{\mathsf{N}}^T\mathbf{b}_1 & \widehat{\mathsf{N}}^T\widehat{\mathsf{B}} & \widehat{\mathsf{N}}^T\mathsf{D}^{\phi}\\ \widehat{\mathsf{A}}^T\mathbf{b}_1 & \widehat{\mathsf{A}}^T\widehat{\mathsf{B}} & \widehat{\mathsf{A}}^T\mathsf{D}^{\phi}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \mathsf{I}_2 & 0\\ 0 & 0 & \mathsf{I}_{m_{\mathsf{P}}-3}\end{bmatrix}=\mathsf{I}_{m_{\mathsf{P}}}, \tag{43}$$

from which we have that $\mathsf{N}^T\mathsf{D}^{\phi}=(\mathbb{1}^T,\widehat{\mathsf{N}}^T)\mathsf{D}^{\phi}=0$. Therefore, the three columns of N and the $(m_{\mathsf{P}}-3)$ columns of D^{ϕ} generate two linearly independent and orthogonal subspaces of $\mathbb{R}^{m_{\mathsf{P}}}$ and are such that $\mathsf{span}\big\{\mathsf{N}\big\}\oplus\mathsf{span}\big\{\mathsf{D}^{\phi}\big\}=\mathbb{R}^{m_{\mathsf{P}}}$. □

We complete the characterization of the inverse of matrix A by providing an explicit formula for vector $\mathbf{b}_1$. Let us multiply equation (41) from the right by $\mathbb{1}$:

$$\mathbf{b}_1\,\mathbb{1}^T\mathbb{1}+\widehat{\mathsf{B}}\,\widehat{\mathsf{N}}^T\mathbb{1}+\mathsf{D}^{\phi}\,\widehat{\mathsf{A}}^T\mathbb{1}=\mathbb{1}. \tag{44}$$

Using the expressions of matrices $\widehat{\mathsf{B}}$ and D^{ϕ} in (40) yield

$$\mathbf{b}_1\mathbb{1}^T\mathbb{1}+\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T+\left(\mathsf{I}_{m_{\mathsf{P}}}-\mathbf{b}_1\mathbb{1}^T-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\widehat{\mathsf{A}}\left(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}}\right)^{-1}\widehat{\mathsf{A}}^T\mathbb{1}=\mathbb{1}. \tag{45}$$

Then, we rearrange the terms and we get

$$\mathbf{b}_1\left(\mathbb{1}^T\mathbb{1}-\mathbb{1}^T\widehat{\mathsf{A}}^T(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T\mathbb{1}\right)+\left(\mathsf{I}_{m_{\mathsf{P}}}-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}\right)\widehat{\mathsf{A}}^T(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T=\mathbb{1}-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T, \tag{46}$$

from which we readily obtain

$$\mathbf{b}_1 = \frac{1}{\mathbb{1}^T\big(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{A}}(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T\big)\mathbb{1}} \left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{A}}(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T\right)\mathbb{1}. \tag{47}$$

Substituting (47) in the formula of D^ϕ in (40) gives an explicit expression for D^ϕ:

$$\mathsf{D}^\phi = \left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\left[\mathsf{I}_{m_\mathsf{P}} - \frac{1}{\mathbb{1}^T\left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{A}}(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T\right)\mathbb{1}}\left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{A}}(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}\widehat{\mathsf{A}}^T\right)\mathbb{1}\mathbb{1}^T\right]\widehat{\mathsf{A}}(\widehat{\mathsf{A}}^T\widehat{\mathsf{A}})^{-1}. \tag{48}$$

4.2. *Stiffness matrices*

Since $\phi_1 = 1$, $\phi_2 = x$, and $\phi_3 = y$, from condition ($\mathsf{P1}^\phi$) and due to the orthogonality condition ($\mathsf{P2}^\phi$), the stiffness matrix associated with the functions ϕ_i has the form:

$$\mathsf{M}^\phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & |\mathsf{P}|\,\mathsf{K}_\mathsf{P} & 0 \\ 0 & 0 & \widehat{\mathsf{M}}^\phi \end{bmatrix} \quad \text{with} \quad (\widehat{\mathsf{M}}^\phi)_{i-3,j-3} = \int_\mathsf{P} \mathsf{K}_\mathsf{P}\nabla\phi_i\cdot\nabla\phi_j\,dV, \quad i,j = 4\ldots,m_\mathsf{P}. \tag{49}$$

Proposition 4.1 *The matrix block* $\widehat{\mathsf{M}}^\phi$ *in* (49) *is a symmetric and positive definite matrix and can be reformulated as*

$$\widehat{\mathsf{M}}^\phi = \widehat{\mathsf{A}}^T\mathsf{M}^\chi\widehat{\mathsf{A}}. \tag{50}$$

Proof. The matrix formulation (50) is a consequence of (36) and the matrix partitioning of A introduced in (38). Matrix $\widehat{\mathsf{M}}^\phi$ is symmetric and non-negative definite by (49); furthermore, it is nonsingular because the functions $\phi_4,\ldots,\phi_{m_\mathsf{P}}$ are linearly independent of $\phi_1 = 1$, which spans its null space. □

Now, we split the matrix M^ϕ in (49) as

$$\mathsf{M}^\phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & |\mathsf{P}|\,\mathsf{K}_\mathsf{P} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathsf{M}}^\phi \end{bmatrix} = \mathsf{M}_0^\phi + \mathsf{M}_1^\phi \tag{51}$$

and use the basis transformation (36) to define the matrices

$$\mathsf{M}_0^\chi = \mathsf{B}\mathsf{M}_0^\phi\mathsf{B}^T \quad \text{and} \quad \mathsf{M}_1^\chi = \mathsf{B}\mathsf{M}_1^\phi\mathsf{B}^T. \tag{52}$$

With the following proposition, we show that M_0^χ coincides with the mimetic matrix M_0^m defined in (18) and that M_1^χ is an admissible mimetic stabilization matrix, i.e, it can be taken as matrix M_1^m in (17), cf. Subsection 3.7.

Proposition 4.2 *The identity* $\mathsf{M}_0^\chi = \mathsf{M}_0^m$ *holds. Moreover,* $\mathsf{M}_1^\chi = \mathsf{D}^\phi\mathsf{M}_1^\phi\big(\mathsf{D}^\phi\big)^T$ *and this matrix is an admissible mimetic stabilization term.*

Proof. We use the expression of matrix B given in (39) with the definition of $\widehat{\mathsf{B}}$ in (40) and we exploit the structural pattern of matrix M_0^ϕ to see that

$$\mathsf{M}_0^\chi = \mathsf{B}\mathsf{M}_0^\phi\mathsf{B}^T = \widehat{\mathsf{B}}\mathsf{K}_\mathsf{P}\,|\mathsf{P}|\,\widehat{\mathsf{B}}^T = \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\,\mathsf{K}_\mathsf{P}\,|\mathsf{P}|\,\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{R}}^T = \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{R}}^T = \mathsf{M}_0^\mathsf{m},$$

where the last step follows from (18).

We show that $\mathsf{M}_1^\chi = \mathsf{B}\mathsf{M}_1^\phi\mathsf{B}^T$ is an admissible mimetic stabilization term by considering the block partitioning $\mathsf{B} = \big[\mathbf{b}_1\ \widehat{\mathsf{B}}\ \mathsf{D}^\phi\big]$ of Lemma 4.3. The particular sparsity pattern of matrix M_1^ϕ implies that:

$$\mathsf{M}_1^\chi = \mathsf{B}\mathsf{M}_1^\phi\mathsf{B}^T = \mathsf{D}^\phi\widehat{\mathsf{M}}^\phi\big(\mathsf{D}^\phi\big)^T, \tag{53}$$

which has the same structure of (19). Indeed, the matrix D^{ϕ} provided by Lemma 4.3 has the properties required by the formulation of $\mathsf{M}_1^{\mathsf{m}}$ in (19). Moreover, $\widehat{\mathsf{M}}^{\phi}$ is a symmetric and positive definite matrix of dimension $(m_{\mathsf{P}}-3)\times(m_{\mathsf{P}}-3)$ in view of Proposition 4.1, and, thus, can play the role of the matrix of parameters U in (19). □

Remark 4.3 *The matrix $\mathsf{M}_0^{\chi}=\mathsf{M}_0^{m}$ can be expressed in terms of the vectors $\mathbf{g}_i$ as follows:*

$$\left(\mathsf{M}_0^{\chi}\right)_{ij}=\left(\mathsf{M}_0^{m}\right)_{ij}=\left(\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{R}}^T\right)_{ij}=\left(|\mathsf{P}|\,\mathsf{K}_{\mathsf{P}}\mathbf{g}_i\right)^T\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\,|\mathsf{P}|\,\mathsf{K}_{\mathsf{P}}\mathbf{g}_j=\mathbf{g}_i^T\,\mathsf{K}_{\mathsf{P}}\mathbf{g}_j\,|\mathsf{P}|\,.$$

Moreover, matrix M_0^{m} approximates the stiffness matrix of the "χ" functions by taking the cell average of their gradients:

$$\left(\mathsf{M}_0^{m}\right)_{ij}=\mathbf{g}_i^T\mathsf{K}_{\mathsf{P}}\mathbf{g}_j\,|\mathsf{P}|=\int_{\mathsf{P}}\mathsf{K}_{\mathsf{P}}\Big(\frac{1}{|\mathsf{P}|}\int_{\mathsf{P}}\nabla\chi_i\,dV\Big)\cdot\Big(\frac{1}{|\mathsf{P}|}\int_{\mathsf{P}}\nabla\chi_j\,dV\Big)\,dV\approx\int_{\mathsf{P}}\mathsf{K}_{\mathsf{P}}\nabla\chi_i\cdot\nabla\chi_j\,dV.$$

5. The reconstruction problem

In this section we study if we can reproduce a given MFD scheme with a virtual element space. More precisely, given a full rank $m_{\mathsf{P}}\times(m_{\mathsf{P}}-3)$-sized matrix D such that $\mathsf{N}^T\mathsf{D}=0$ and $\mathsf{span}(\mathsf{N},\mathsf{D})=\mathbb{R}^{m_{\mathsf{P}}}$, and a $(m_{\mathsf{P}}-3)\times(m_{\mathsf{P}}-3)$-sized symmetric and positive matrix U we study if there exists a set of functions χ_i satisfying properties $(\mathsf{P1}^{\chi})$–$(\mathsf{P3}^{\chi})$ such that the local stiffness matrix M^{χ} is equal to $\mathsf{M}_0^{\chi}+\mathsf{DUD}^T$, where M_0^{χ} is defined in (52). If this is the case, since by Proposition 4.2 we have $\mathsf{M}_0^{\chi}=\mathsf{M}_0^{\mathsf{m}}$, the two methods coincide. We will see that the answer is positive if U is "not too small".

The role of matrix D will be investigated in Lemma 5.1, where we prove that we can always define a couple of transformation matrices A and $\mathsf{B}=\mathsf{A}^{-1}$ with A satisfying the orthogonality condition of Lemma 4.2 once a matrix D is properly assigned. The role of matrix U will be clarified by Theorem 5.1, where we investigate the conditions under which we can find a set of basis functions $\{\phi_i\}_{i=1,\dots,m_{\mathsf{P}}}$ that reproduce U as the submatrix $\widehat{\mathsf{M}}^{\phi}$.

Lemma 5.1 *Let N and R be the matrices defined in (14) with the matrices $\widehat{\mathsf{N}}$ and $\widehat{\mathsf{R}}$ defined in (15). Let D be a full rank $m_{\mathsf{P}}\times(m_{\mathsf{P}}-3)$ matrix such that $\mathsf{N}^T\mathsf{D}=0$ and $\mathsf{span}\{\mathsf{N},\mathsf{D}\}=\mathbb{R}^{m_{\mathsf{P}}}$. Then, the matrices*

$$\mathsf{A}=\begin{bmatrix}\mathbb{1}^T\\ \widehat{\mathsf{N}}^T\\ \widehat{\mathsf{A}}^T\end{bmatrix}\qquad\text{and}\qquad\mathsf{B}=\Big[\mathbf{b}_1,\widehat{\mathsf{B}},\mathsf{D}\Big]\tag{54}$$

are nonsingular and satisfy the conditions $\mathsf{AB}=\mathsf{BA}=\mathsf{I}_{m_{\mathsf{P}}}$ with $\widehat{\mathsf{R}}^T\widehat{\mathsf{A}}=0$ if and only if

$$\mathbf{b}_1=\frac{1}{m_{\mathsf{P}}}\left(\mathsf{I}_{m_{\mathsf{P}}}-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\mathbb{1}\;+\mathsf{D}\boldsymbol{\gamma},\tag{55}$$

$$\widehat{\mathsf{B}}=\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|},\tag{56}$$

$$\widehat{\mathsf{A}}^T=\left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T\left(\mathsf{I}_{m_{\mathsf{P}}}-\mathbf{b}_1\mathbb{1}^T-\widehat{\mathsf{R}}\frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\tag{57}$$

where $\boldsymbol{\gamma}$ is an arbitrary vector in $\mathbb{R}^{m_{\mathsf{P}}-3}$.

Proof. The proof will be organized as follows. First we show the "if" part, i.e. we show that if $\mathbf{b}_1$, $\widehat{\mathsf{B}}$, and $\widehat{\mathsf{A}}$ have the expression above, then $\mathsf{AB}=\mathsf{I}$. Then we show the "only if" part, i.e. we show that if $\mathsf{AB}=\mathsf{I}$, then $\mathbf{b}_1$, $\widehat{\mathsf{B}}$, and $\widehat{\mathsf{A}}$ must have the expression above.

"if" part.

We assume that $\mathbf{b}_1$, $\widehat{\mathsf{B}}$, and $\widehat{\mathsf{A}}$ have the expressions above, and we check that $\mathsf{BA}=\mathsf{I}$. This is equivalent to prove, after rearranging the terms in (41), that

$$
\begin{aligned}
0 &= \mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{B}}\widehat{\mathsf{N}}^T - \mathsf{D}\widehat{\mathsf{A}}^T \\
&= \mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T - \mathsf{D}\left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T \left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right) \\
&= \left(\mathsf{I}_{m_\mathsf{P}} - \mathsf{D}\big(\mathsf{D}^T\mathsf{D}\big)^{-1}\mathsf{D}^T\right)\left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right).
\end{aligned}
\tag{58}
$$

By hypothesis, the columns of N and the columns of D span all $\mathbb{R}^{m_\mathsf{P}}$ and the latters are orthogonal to the formers. In terms of the orthogonal projectors, as we have already observed in (21), we have:

$$
\mathsf{D}(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T + \mathsf{N}(\mathsf{N}^T\mathsf{N})^{-1}\mathsf{N}^T = \mathsf{I}. \tag{59}
$$

Using (59) in (58) we obtain the condition

$$
\mathsf{N}\big(\mathsf{N}^T\mathsf{N}\big)^{-1}\mathsf{N}^T\left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right) = 0,
$$

which is satisfied because it holds that

$$
\mathsf{N}^T\left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1 \mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right) = 0. \tag{60}
$$

To prove (60), we recall that $\mathsf{N}^T = [\mathbb{1}^T, \widehat{\mathsf{N}}^T]$, cf. (14), and we preliminarly check that $\mathbb{1}^T\mathbf{b}_1 = 1$ and $\widehat{\mathsf{N}}^T\mathbf{b}_1 = 0$. Indeed, since $\mathbb{1}^T\widehat{\mathsf{R}} = 0$ and $\mathsf{N}^T\mathsf{D} = 0$ implies that $\mathbb{1}^T\mathsf{D} = 0$, we multiply the expression of $\mathbf{b}_1$ provided by (55) from the left by $\mathbb{1}^T$ and we find that

$$
\mathbb{1}^T\mathbf{b}_1 = \frac{1}{m_\mathsf{P}}\left(\mathbb{1}^T - \mathbb{1}^T\widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\mathbb{1} + \mathbb{1}^T\mathsf{D}\boldsymbol{\gamma} = \frac{1}{m_\mathsf{P}}\mathbb{1}^T\mathbb{1} = 1. \tag{61}
$$

Similarly, since $\widehat{\mathsf{N}}^T\widehat{\mathsf{R}} = \mathsf{K}_\mathsf{P}\,|\mathsf{P}|$ and $\mathsf{N}^T\mathsf{D} = 0$ implies that $\widehat{\mathsf{N}}^T\mathsf{D} = 0$, we multiply the expression of $\mathbf{b}_1$ provided by (55) from the left by $\widehat{\mathsf{N}}^T$ and we find that

$$
\widehat{\mathsf{N}}^T\mathbf{b}_1 = \frac{1}{m_\mathsf{P}}\left(\widehat{\mathsf{N}}^T - \widehat{\mathsf{N}}^T\widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\mathbb{1} + \widehat{\mathsf{N}}^T\mathsf{D}\boldsymbol{\gamma} = \frac{1}{m_\mathsf{P}}\left(\widehat{\mathsf{N}}^T - \widehat{\mathsf{N}}^T\right)\mathbb{1} = 0. \tag{62}
$$

Now, we split (60) in two independent identities according to the splitting of matrix N, i.e., $\mathsf{N} = [\mathbb{1}, \widehat{\mathsf{N}}]$, cf. (14). Using (61) and $\mathbb{1}^T\widehat{\mathsf{R}} = 0$ in the first row of (60) imply that

$$
\mathbb{1}^T\left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1\mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right) = \mathbb{1}^T - \mathbb{1}^T\mathbf{b}_1\mathbb{1}^T - \mathbb{1}^T\widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T = \mathbb{1}^T - \mathbb{1}^T = 0. \tag{63}
$$

Likewise, using (62) and $\widehat{\mathsf{N}}^T\widehat{\mathsf{R}} = \mathsf{K}_\mathsf{P}\,|\mathsf{P}|$ in the remaining rows of (60) imply that

$$
\widehat{\mathsf{N}}^T\left(\mathsf{I}_{m_\mathsf{P}} - \mathbf{b}_1\mathbb{1}^T - \widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right) = \widehat{\mathsf{N}}^T - \widehat{\mathsf{N}}^T\mathbf{b}_1\mathbb{1}^T - \widehat{\mathsf{N}}^T\widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T = \widehat{\mathsf{N}}^T - \widehat{\mathsf{N}}^T = 0. \tag{64}
$$

Relations (63) and (64) yield (60).

"only if" part.
In this part of the proof, we assume that $\mathbf{b}_1$, $\widehat{\mathsf{B}}$, and $\widehat{\mathsf{A}}$ exist such that $\mathsf{AB} = \mathsf{I}$ and we derive their expressions. As in the proof of Lemma 4.3, we consider the matrix formula for BA in (41). To start with, formula (56) is obtained by the same argument used in the proof of Lemma 4.3. To derive the expression for matrix $\widehat{\mathsf{A}}$, we substitute $\widehat{\mathsf{B}}$ into (41) and we multiply the resulting equation from the left by D^T:

$$
\mathsf{D}^T\mathbf{b}_1\,\mathbb{1}^T + \mathsf{D}^T\widehat{\mathsf{R}}\frac{\mathsf{K}_\mathsf{P}^{-1}}{|\mathsf{P}|}\,\widehat{\mathsf{N}}^T + \mathsf{D}^T\mathsf{D}\widehat{\mathsf{A}}^T = \mathsf{D}^T. \tag{65}
$$

As D is full rank, so is matrix $\mathsf{D}^T\mathsf{D}$, which is, thus, nonsingular. We invert $\mathsf{D}^T\mathsf{D}$ and we obtain the characterization in (57) of the matrix $\widehat{\mathsf{A}}$. A straightforward calculation allows us to verify the orthogonality condition

$\widehat{\mathsf{A}}^T\widehat{\mathsf{R}} = 0$. In fact, we multiply (57) from the right by $\widehat{\mathsf{R}}$, we use the fact that $\mathbb{1}^T\widehat{\mathsf{R}} = 0$ and $\widehat{\mathsf{N}}^T\widehat{\mathsf{R}} = \mathsf{K_P}\,|\mathsf{P}|$ and we obtain that

$$\widehat{\mathsf{A}}^T\widehat{\mathsf{R}} = \left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T\left(\widehat{\mathsf{R}} - \mathbf{b}_1\mathbb{1}^T\widehat{\mathsf{R}} - \widehat{\mathsf{R}}\frac{\mathsf{K_P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\widehat{\mathsf{R}}\right) = \left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T\left(\widehat{\mathsf{R}} - \widehat{\mathsf{R}}\right) = 0. \tag{66}$$

In order to derive (55) for $\mathbf{b}_1$, we first note that $\mathbb{1}$, the two vectors forming the columns of matrix $\widehat{\mathsf{R}}$, and the $m_\mathsf{P} - 3$ columns of matrix D are linearly independent. Indeed, let

$$\mathbb{1}\alpha + \widehat{\mathsf{R}}\boldsymbol{\beta} + \mathsf{D}\boldsymbol{\gamma} = 0 \tag{67}$$

where $\alpha \in \mathbb{R}$, $\boldsymbol{\beta} \in \mathbb{R}^2$, and $\boldsymbol{\gamma} \in \mathbb{R}^{m_\mathsf{P}-3}$. We multiply (67) from the left by $\mathbb{1}^T$ and we obtain:

$$0 = \mathbb{1}^T\mathbb{1}\alpha + \mathbb{1}^T\widehat{\mathsf{R}}\boldsymbol{\beta} + \mathbb{1}^T\mathsf{D}\boldsymbol{\gamma} = m_\mathsf{P}\alpha + 0 + 0; \tag{68}$$

hence $\alpha = 0$. Then, we multiply (67) from the left by $\widehat{\mathsf{N}}^T$ and we obtain

$$0 = \widehat{\mathsf{N}}^T\widehat{\mathsf{R}}\boldsymbol{\beta} + \widehat{\mathsf{N}}^T\mathsf{D}\boldsymbol{\gamma} = \mathsf{K_P}\,|\mathsf{P}|\,\boldsymbol{\beta} + 0; \tag{69}$$

hence $\boldsymbol{\beta} = 0$, and also $\boldsymbol{\gamma} = 0$ because D is a full rank matrix. Therefore, these vectors are a basis for $\mathbb{R}^{m_\mathsf{P}}$ and allow us to decompose $\mathbf{b}_1$ as

$$\mathbf{b}_1 = \alpha\mathbb{1} + \widehat{\mathsf{R}}\boldsymbol{\beta} + \mathsf{D}\boldsymbol{\gamma}, \tag{70}$$

where α, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are a set of (scalar and vector) coefficients to be determined. Since we also require that $\mathsf{AB} = \mathsf{I}_{m_\mathsf{P}}$, see equation (43), vector $\mathbf{b}_1$ must satisfy the conditions $\mathbb{1}^T\mathbf{b}_1 = 1$ and $\widehat{\mathsf{N}}^T\mathbf{b}_1 = 0$. To derive α we multiply (70) from the left by $\mathbb{1}^T$:

$$\mathbb{1}^T\mathbf{b}_1 = \mathbb{1}^T\mathbb{1}\alpha + \mathbb{1}^T\widehat{\mathsf{R}}\boldsymbol{\beta} + \mathbb{1}^T\mathsf{D}\boldsymbol{\gamma} = m_\mathsf{P}\alpha + 0 + 0.$$

As $\mathbb{1}^T\mathbf{b}_1 = 1$ we immediately obtain that $\alpha = 1/m_\mathsf{P}$. To derive $\boldsymbol{\beta}$, we multiply (70) from the left by $\widehat{\mathsf{N}}^T$, we substitute $\alpha = 1/m_\mathsf{P}$, and we obtain

$$\widehat{\mathsf{N}}^T\mathbf{b}_1 = \frac{1}{m_\mathsf{P}}\widehat{\mathsf{N}}^T\mathbb{1} + \widehat{\mathsf{N}}^T\widehat{\mathsf{R}}\boldsymbol{\beta} + \widehat{\mathsf{N}}^T\mathsf{D}\boldsymbol{\gamma} = \frac{1}{m_\mathsf{P}}\widehat{\mathsf{N}}^T\mathbb{1} + \mathsf{K_P}\,|\mathsf{P}|\,\boldsymbol{\beta} + 0,$$

from which we have that

$$\boldsymbol{\beta} = -\frac{\mathsf{K_P}^{-1}}{|\mathsf{P}|}\left(\frac{1}{m_\mathsf{P}}\widehat{\mathsf{N}}^T\mathbb{1}\right)$$

since $\widehat{\mathsf{N}}^T\mathbf{b}_1 = 0$. Substituting these formulas for α and $\boldsymbol{\beta}$ in (70) and leaving $\boldsymbol{\gamma}$ as a free vector we obtain (55).

□

Remark 5.1 *As $\mathbb{1}^T\mathbf{b}_1 = 1$, cf. (61), and $\widehat{\mathsf{N}}^T\mathbf{b}_1 = 0$, cf. (62), we multiply $\mathbf{b}_1$ from the left by the expression of $\widehat{\mathsf{A}}^T$ given in (57) and we find that*

$$\widehat{\mathsf{A}}^T\mathbf{b}_1 = (\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T\left(\mathbf{b}_1 - \mathbf{b}_1\mathbb{1}^T\mathbf{b}_1 - \widehat{\mathsf{B}}\widehat{\mathsf{N}}^T\mathbf{b}_1\right) = (\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T(\mathbf{b}_1 - \mathbf{b}_1) = 0. \tag{71}$$

Relations (61), (62), and (71) together prove that $\mathbf{b}_1$ is the first column of the inverse of A.

Remark 5.2 *By substituting the formula for $\mathbf{b}_1$ in $\widehat{\mathsf{A}}$, we obtain:*

$$\widehat{\mathsf{A}}^T = \left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T\left(\mathsf{I}_{m_\mathsf{P}} - \widehat{\mathsf{R}}\frac{\mathsf{K_P}^{-1}}{|\mathsf{P}|}\widehat{\mathsf{N}}^T\right)\left(\mathsf{I}_{m_\mathsf{P}} - \frac{1}{m_\mathsf{P}}\mathbb{1}\mathbb{1}^T\right) - \boldsymbol{\gamma}\mathbb{1}^T, \quad \boldsymbol{\gamma} \in \mathbb{R}^{m_\mathsf{P}-3}. \tag{72}$$

Thus, the i-th row of $\widehat{\mathsf{A}}$ is defined up to a constant factor (the i-th component of vector $\boldsymbol{\gamma}$) times the row vector $\mathbb{1}^T$.

We now come back to the question posed at the beginning of the present section, namely the *reconstruction problem* of finding the conditions under which a local mimetic matrix is the stiffness matrix associated with a virtual element space. Let D and U be given and defining the local mimetic matrix M^m through (17), (18), and (19). We seek a set of basis functions $\{\chi_i\}_{i=1,\dots,m_{\mathsf{P}}}$ obeying ($\mathsf{P1}^\chi$)-($\mathsf{P3}^\chi$) and such that the associated stiffness matrix is given by M^{m}.

As D is set, let A, B be the transformation matrices given by Lemma 5.1. It easily follows that

$$\mathsf{A}\mathsf{M}^{\mathsf{m}}\mathsf{A}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & |\mathsf{P}|\,\mathsf{K}_{\mathsf{P}} & 0 \\ 0 & 0 & \mathsf{U} \end{bmatrix}. \tag{73}$$

If we can find a basis $\{\phi_i\}_{i=1,\dots,m_{\mathsf{P}}}$ obeying ($\mathsf{P1}^\phi$)-($\mathsf{P2}^\phi$) and such that M^ϕ is given by (73), then transforming back this basis we find the required basis $\{\chi_i\}_{i=1,\dots,m_{\mathsf{P}}}$. Thus, the reconstruction problem stated above is equivalent to find a set of functions $\{\phi_i\}_{i=1,\dots,m_{\mathsf{P}}}$, such that

(i) they are linear on the boundary of P;

(ii) their values $\phi_i(\mathbf{x}_j)$ at the vertices of P coincide with the coefficients a_{ij} of the last $m_{\mathsf{P}}-3$ rows of matrix A defined in Lemma 5.1, i.e., the entries of the matrix $\widehat{\mathsf{A}}$ defined in (57);

(iii) it holds that

$$\big(\mathsf{U}\big)_{i-3,j-3} = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}}\nabla\phi_i\cdot\nabla\phi_j\,dV \quad \text{for } i,j=4,\dots,m_{\mathsf{P}}. \tag{74}$$

Remark 5.3 *As each row of matrix $\widehat{\mathsf{A}}$ contains the degrees of freedom of one of the function ϕ_i, for $i = 4,\dots,m_{\mathsf{P}}$, Remark 5.2 implies that all such functions are defined up to an additive constant. However, this fact is irrelevant here since the quantity $\int_{\mathsf{P}}\mathsf{K}_{\mathsf{P}}\nabla\phi_i\cdot\nabla\phi_j\,dV$ remains unchanged.*

Let us consider the set of harmonic functions $\{\chi_i^H\}_{i=1,\dots,m_{\mathsf{P}}}$ that solve the problems

$$-\mathrm{div}\big(\mathsf{K}_{\mathsf{P}}\nabla\chi_i^H\big) = 0 \quad \text{in } \mathsf{P}, \tag{75}$$

$$\chi_i^H = \delta_i \quad \text{on } \partial\mathsf{P}, \tag{76}$$

where δ_i, $i=1,\dots,m_{\mathsf{P}}$, is the piecewise linear function on $\partial\mathsf{P}$ such that $\delta_i(\mathbf{x}_j)=\delta_{ij}$. This set of harmonic functions satisfy ($\mathsf{P1}^\chi$)-($\mathsf{P3}^\chi$). The transformation rule established by matrix A determines a set of harmonic functions $\{\phi_i^H\}_{i=1,\dots,m_{\mathsf{P}}}$ satisfying ($\mathsf{P1}^\phi$)-($\mathsf{P3}^\phi$), the requirements (i) and (ii) above, and such that

$$\big(\mathsf{U}^H\big)_{i-3,j-3} = \int_{\mathsf{P}}\mathsf{K}_{\mathsf{P}}\nabla\phi_i^H\cdot\nabla\phi_j^H\,dV = \sum_{k,l=1}^{m_{\mathsf{P}}} a_{ik}a_{jl}\int_{\mathsf{P}}\mathsf{K}_{\mathsf{P}}\nabla\chi_k^H\cdot\nabla\chi_l^H\,dV, \quad i,j=4,\dots,m_{\mathsf{P}}. \tag{77}$$

In Theorem 5.1 that follows below we prove that we can suitably modify the functions ϕ_i^H so that the new set of functions satisfy condition (74) if and only if $\mathsf{U}-\mathsf{U}^H$ is a non-negative definite matrix.

The choice of modifying the harmonic functions is a key point in solving the reconstruction problem. Indeed, the solutions of the harmonic problem (75)-(76) provides a set of functions in $H^1(\mathsf{P})$ with the minimum possible energies among those functions that satisfy the assigned boundary conditions on $\partial\mathsf{P}$. In this sense, matrix U^H is a sort of "*minimum energy threshold*" below which the reconstruction problem has no solution. Nonetheless, in the next section we will show that in the special case of a mesh of quadrilateral cells, it is possible to relax this energy threshold at the price of slightly modifying the definition of the space $V_{h,\mathsf{P}}$.

Theorem 5.1 *Let U be a symmetric and positive matrix with size $(m_{\mathsf{P}}-3)\times(m_{\mathsf{P}}-3)$ and D be a matrix with size $m_{\mathsf{P}}\times(m_{\mathsf{P}}-3)$ such that $\mathsf{N}^T\mathsf{D}=0$ and $\mathit{span}(\mathsf{N},\mathsf{D})=\mathbb{R}^{m_{\mathsf{P}}}$, and let M^{m} be the associated mimetic matrix given by (17), (18), and (19). Let U^H be given by (77) using the coefficients of the transformation matrix A defined by Lemma 5.1. Then, a set of functions $\{\chi_i\}_{i=1,\dots,m_{\mathsf{P}}}$ satisfying ($\mathsf{P1}^\chi$)-($\mathsf{P3}^\chi$) and with stiffness matrix given by M^{m} exists if and only if $\mathsf{U}-\mathsf{U}^H$ is a symmetric non-negative definite matrix. Moreover, the functions χ_i, $i=1,\dots,m_{\mathsf{P}}$, are given by*

$$\chi_i = \chi_i^H + v_i, \tag{78}$$

where χ_i^H are the harmonic functions defined by (75), (76) and $v_i\in H_0^1(\mathsf{P})$.

Proof. Let the transformation matrices A and B be fixed in accordance with Lemma 5.1, and let $\{\phi_i^H\}_{i=1,\dots,m_{\mathsf{P}}}$ be the set of functions obtained by applying the transformation matrix A to the set of harmonic functions $\{\chi_i^H\}_{i=1,\dots,m_{\mathsf{P}}}$ given by (75) and (76).

We wish to find a set of functions $w_i \in H_0^1(\mathsf{P})$, $i = 1,\dots,m_{\mathsf{P}}$ such that $\phi_i = \phi_i^H + w_i$, $i = 1,\dots,m_{\mathsf{P}}$ satisfy conditions $(\mathsf{P1}^{\phi})$-$(\mathsf{P2}^{\phi})$ and such that (74) holds.

Condition $(\mathsf{P1}^{\chi})$ requires that $\phi_1 = 1 = \phi_1^H$, $\phi_2 = x = \phi_2^H$, $\phi_3 = y = \phi_3^H$. It follows that $w_1 = w_2 = w_3 = 0$ and we are only allowed to modify the basis functions $\phi_4^H,\dots,\phi_{m_{\mathsf{P}}}^H$ by the functions $w_4,\dots,w_{m_{\mathsf{P}}}$.

A straightforward calculation yields:

$$\begin{aligned}\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i \cdot \nabla\phi_j \, dV &= \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla(\phi_i^H + w_i) \cdot \nabla(\phi_j^H + w_j) \, dV \\ &= \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i^H \cdot \nabla\phi_j^H \, dV + \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla w_i \cdot \nabla w_j \, dV \\ &\quad + \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i^H \cdot \nabla w_j \, dV + \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_j^H \cdot \nabla w_i \, dV, \end{aligned} \tag{79}$$

for $i, j = 4,\dots,m_{\mathsf{P}}$. We integrate by parts the last but one term and we find that

$$\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i^H \cdot \nabla w_j \, dV = -\int_{\mathsf{P}} \mathrm{div}(\mathsf{K}_{\mathsf{P}} \nabla\phi_i^H)\, w_j \, dV + \int_{\partial\mathsf{P}} \mathbf{n}_{\mathsf{P}} \cdot \big(\mathsf{K}_{\mathsf{P}} \nabla\phi_i^H\big)\, w_j \, dS = 0, \tag{80}$$

as ϕ_i^H is a harmonic function on P and $w_{j|\partial\mathsf{P}} = 0$. The last integral of (79) is also zero by the same argument (just interchange the indices i and j). Hence, relation (79) becomes

$$\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i \cdot \nabla\phi_j \, dV = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\phi_i^H \cdot \nabla\phi_j^H \, dV + \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla w_i \cdot \nabla w_j \, dV. \tag{81}$$

Equation (81) can be expressed in the matrix form

$$\mathsf{U} = \mathsf{U}^H + \mathsf{W},$$

where we have introduced the matrix W defined by

$$(\mathsf{W})_{i-3,j-3} = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla w_i \cdot \nabla w_j \, dV, \quad i, j = 4,\dots,m_{\mathsf{P}}.$$

Let Q be the $(m_{\mathsf{P}} - 3) \times (m_{\mathsf{P}} - 3)$ orthogonal matrix that diagonalizes $\mathsf{W} = \mathsf{U} - \mathsf{U}^H$ and $\Lambda = \mathrm{diag}(\lambda_i)$ the diagonal matrix of the eigenvalues of W; it holds that $\mathsf{W} = \mathsf{Q}\Lambda\mathsf{Q}^T$. As we assume that W is nonnegative definite it holds that $\lambda_i \geq 0$ for each $i = 1,\dots,m_{\mathsf{P}} - 3$. Now, let us consider a set of linearly independent $(m_{\mathsf{P}} - 3)$ non zero functions ψ_i in $H_0^1(\mathsf{P})$, such that

$$\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla\psi_i \cdot \nabla\psi_j \, dV = \lambda_i \delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and zero otherwise. The functions ψ_i are easily found by taking $m_{\mathsf{P}} - 3$ functions in $H_0^1(\mathsf{P})$ that are orthogonal in the sense of condition $(\mathsf{P2}^{\phi})$ and rescaling them with the coefficients $\sqrt{\lambda_i}$. Then, for $i = 4,\dots,m_{\mathsf{P}}$, we set

$$w_i = \sum_{j=4}^{m_{\mathsf{P}}} \mathsf{Q}_{i-3,j-3} \psi_{j-3}.$$

We claim that the functions w_i resulting from this construction are a proper choice that allows us to modify the functions ϕ_i^H and obtain the functions ϕ_i and, consequently, the functions χ_i. We first note that $w_{i|\partial\mathsf{P}} = 0$ for $i = 4,\dots,m_{\mathsf{P}}$ because each w_i is a linear combination of the functions ψ_i that have zero trace on $\partial\mathsf{P}$. Furthermore, for $i, j = 4,\dots,m_{\mathsf{P}}$, it holds that

$$\begin{aligned}\int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla w_i \cdot \nabla w_j \, dV &= \sum_{k,l=4}^{m_{\mathsf{P}}} \mathsf{Q}_{i-3,k-3} \mathsf{Q}_{j-3,l-3} \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla \psi_{k-3} \cdot \nabla \psi_{l-3} \, dV \\ &= \sum_{k,l=4}^{m_{\mathsf{P}}} \mathsf{Q}_{i-3,k-3} \mathsf{Q}_{j-3,l-3} \lambda_{k-3} \delta_{k-3,l-3} \\ &= \left(\mathsf{Q} \Lambda \mathsf{Q}^T\right)_{i-3,j-3} \\ &= \mathsf{W}_{i-3,j-3}.\end{aligned}$$

We are left to derive an expression for the functions χ_i. This is quickly done by applying the transformation matrix B to the functions w_i to define

$$\left[v_1 \; v_2 \; \ldots \; v_{m_{\mathsf{P}}}\right]^T = \mathsf{B} \left[0 \; 0 \; 0 \; w_4 \; \ldots \; w_{m_{\mathsf{P}}}\right]^T, \tag{82}$$

and then χ_i can be expressed in the form of equation (78).

It remains to show that the condition in the statement of the theorem is also necessary. This is easily verified by contradiction. Assume that there exists $\mathbf{c} = \left[c_1, \ldots, c_{m_{\mathsf{P}}-3}\right]^T$ such that $\mathbf{c}^T \mathsf{W} \mathbf{c} < 0$, where $\mathsf{W} = \mathsf{U} - \mathsf{U}^H$; also, assume that a set of functions $\{\phi_i\}_{i=1,\ldots,m_{\mathsf{P}}}$ with all the required properties exists achieving U as energy matrix. Then, for $\phi = \sum_{i=4}^{m_{\mathsf{P}}} c_{i-3} \phi_i$ and $\phi^H = \sum_{i=4}^{m_{\mathsf{P}}} c_{i-3} \phi_i^H$, we have

$$0 > \mathbf{c}^T \mathsf{W} \mathbf{c} = \mathbf{c}^T (\mathsf{U} - \mathsf{U}^H) \mathbf{c} = \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla \phi \cdot \nabla \phi \, dV - \int_{\mathsf{P}} \mathsf{K}_{\mathsf{P}} \nabla \phi^H \cdot \nabla \phi^H \, dV. \tag{83}$$

We notice that the two functions ϕ and ϕ^H have the same boundary values as this is true for each couple of functions ϕ_i and ϕ_i^H. Thus, equation (83) yields a contradiction as ϕ^H is harmonic and hence it must achieve the minimal energy among all functions with a given trace. □

6. Matrix invariance

In this section, we address the problem of the relationship between the mimetic scheme and the mimetic matrices D and U that are used in its formulation.

There exist an infinite number of pairs (U, D) that provide the same matrix $\mathsf{M}_1^{\mathsf{m}}$; nonetheless, we will see that the reconstruction problem, which we discussed in the previous subsection, only depends on the scheme and not on its particular matrix representation.

Let (D, U) and $(\mathsf{D}^\dagger, \mathsf{U}^\dagger)$ be two couples of mimetic matrices that specify the same mimetic scheme, i.e., the same matrix M^{m} in (17). In view of the matrix representation (19), it must hold that $\mathsf{M}_1^{\mathsf{m}} = \mathsf{D}^\dagger \mathsf{U}^\dagger \mathsf{D}^{\dagger T} = \mathsf{D} \mathsf{U} \mathsf{D}^T$. Now, D is a matrix of size $(m_{\mathsf{P}} - 3) \times (m_{\mathsf{P}} - 3)$ such that $\mathsf{N}^T \mathsf{D} = 0$ and $\mathsf{span}(\mathsf{N}, \mathsf{D}) = \mathbb{R}^{m_{\mathsf{P}}}$. The key observation is that any other matrix $\mathsf{D}^\dagger$ with the same properties can be written as $\mathsf{D}^\dagger = \mathsf{D}\mathsf{S}$ where S is a non-singular $(m_{\mathsf{P}} - 3) \times (m_{\mathsf{P}} - 3)$ matrix. Indeed, since the columns of D are a set of linearly independent vectors forming a basis of the space orthogonal to the columns of N, any other basis of this space can be expressed as $\mathsf{D}\mathsf{S}$. As a consequence, the couple of matrices (D, U) and the couple of matrices $(\mathsf{D}^\dagger, \mathsf{U}^\dagger)$ leads to the same mimetic method provided that $\mathsf{U}^\dagger = \mathsf{S}^{-1} \mathsf{U} \mathsf{S}^{-T}$. For, the equation $\mathsf{D}^\dagger \mathsf{U}^\dagger \mathsf{D}^{\dagger T} = \mathsf{D} \mathsf{U} \mathsf{D}^T$ implies

$$(\mathsf{D}\mathsf{S}) \mathsf{U}^\dagger (\mathsf{D}\mathsf{S})^T = \mathsf{D} \mathsf{S} \mathsf{U}^\dagger \mathsf{S}^T \mathsf{D}^T = \mathsf{D} \mathsf{U} \mathsf{D}^T \tag{84}$$

and if we multiply on the left by D^T and on the right by D we can simplify at both ends by $\mathsf{D}^T \mathsf{D}$, which is nonsingular, and we obtain $\mathsf{S} \mathsf{U}^\dagger \mathsf{S}^T = \mathsf{U}$, which is equivalent to $\mathsf{U}^\dagger = \mathsf{S}^{-1} \mathsf{U} \mathsf{S}^{-T}$.

Let us also investigate the relation between the stiffness matrices with respect to the "ϕ" functions. The matrix $\widehat{\mathsf{A}}^\dagger$ that corresponds to $\mathsf{D}^\dagger$ is given by equation (72)

$$\widehat{\mathsf{A}}^{\dagger T} = \left(\mathsf{D}^{\dagger T} \mathsf{D}^\dagger\right)^{-1} \mathsf{D}^{\dagger T} \left(\mathsf{I}_{m_{\mathsf{P}}} - \widehat{\mathsf{R}} \frac{\mathsf{K}_{\mathsf{P}}^{-1}}{|\mathsf{P}|} \widehat{\mathsf{N}}^T\right) \left(\mathsf{I}_{m_{\mathsf{P}}} - \frac{1}{m_{\mathsf{P}}} \mathbb{1}\mathbb{1}^T\right), \tag{85}$$

after taking $\boldsymbol{\gamma} = 0$ in view of Remark 5.3.

Since

$$\left(\mathsf{D}^{\dagger T}\mathsf{D}^{\dagger}\right)^{-1}\mathsf{D}^{\dagger T} = \left(\mathsf{S}^T\mathsf{D}^T\mathsf{D}\mathsf{S}\right)^{-1}\mathsf{S}^T\mathsf{D}^T = \mathsf{S}^{-1}\left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{S}^{-T}\mathsf{S}^T\mathsf{D}^T = \mathsf{S}^{-1}\left(\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{D}^T \tag{86}$$

we have

$$\widehat{\mathsf{A}}^{\dagger T} = \mathsf{S}^{-1}\widehat{\mathsf{A}}^T. \tag{87}$$

The rows of $\widehat{\mathsf{A}}^{\dagger T}$ are the degrees of freedom of a set of $(m_{\mathsf{P}}-3)$ basis functions $\widehat{\boldsymbol{\phi}}^{\dagger} = (\phi_4^{\dagger},\dots,\phi_{m_{\mathsf{P}}}^{\dagger})^T$ (recall that $\phi_i^{\dagger}$ for $i=1,2,3$ are determined by $(\mathsf{P1}^{\phi})$), and related to the set of basis functions $\widehat{\boldsymbol{\phi}} = (\phi_4,\dots,\phi_{m_{\mathsf{P}}})^T$ by the relation

$$\widehat{\boldsymbol{\phi}}^{\dagger} = \mathsf{S}^{-1}\widehat{\boldsymbol{\phi}}, \tag{88}$$

and it follows that

$$\mathsf{M}^{\phi^{\dagger}} = \mathsf{S}^{-1}\mathsf{M}^{\phi}\mathsf{S}^{-T}. \tag{89}$$

We conclude that, if the reconstruction problem can be solved for a couple of matrices (D,U), then it can be solved for any other couple of matrices $(\mathsf{D}^{\dagger},\mathsf{U}^{\dagger})$ providing the same mimetic scheme, i.e., such that $\mathsf{D}\mathsf{U}\mathsf{D}^T = \mathsf{D}^{\dagger}\mathsf{U}^{\dagger}\mathsf{D}^{\dagger T}$. This remark will be useful in the next section.

7. The case of a quadrilateral mesh

In this section we exploit the computations of the previous sections in the case of a quadrilateral cell, setting also for simplicity $\mathsf{K} = \mathsf{I}$. We provide a geometrical interpretation of Lemmas 4.3 and 5.1 and we prove a stronger version of Theorem 5.1.

We consider the quadrilateral cell P, not necessarily convex, with vertices $\mathsf{v}_i = (x_i, y_i)$, for $i = 1,\dots,4$, taken counterclockwise. We denote the signed area of the triangle formed by all the vertices of P excluded v_i by $\hat{T}_i$, see, for example, Figure 2 where $\hat{T}_2$ and $\hat{T}_4$ are shown. We can express $\hat{T}_1, \hat{T}_2, \hat{T}_3, \hat{T}_4$ as determinants

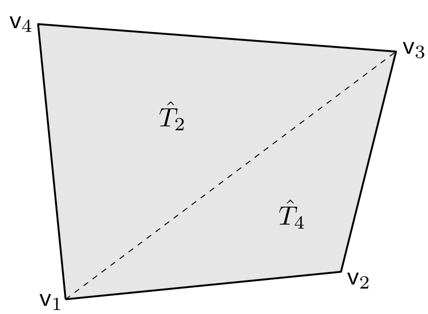

Fig. 2. The quadrilateral P.

$$\hat{T}_1 = \frac{1}{2}\det\begin{bmatrix}1&1&1\\x_2&x_3&x_4\\y_2&y_3&y_4\end{bmatrix},\ \hat{T}_2 = \frac{1}{2}\det\begin{bmatrix}1&1&1\\x_3&x_4&x_1\\y_3&y_4&y_1\end{bmatrix},\ \hat{T}_3 = \frac{1}{2}\det\begin{bmatrix}1&1&1\\x_4&x_1&x_2\\y_4&y_1&y_2\end{bmatrix},\ \hat{T}_4 = \frac{1}{2}\det\begin{bmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{bmatrix}, \tag{90}$$

and note the following identity

$$\hat{T}_1 + \hat{T}_3 = \hat{T}_2 + \hat{T}_4 = |\mathsf{P}|, \tag{91}$$

whose geometrical meaning is obvious.

7.1. *The transformation matrices* A *and* B

We characterize the transformation matrices A and B defined in (35). For a quadrilateral cell, the matrix $\widehat{\mathsf{A}}$ consist of a single column vector in $\mathbb{R}^4$, and thus the matrix A is to be determined in the form

$$
\mathsf{A} = \begin{bmatrix} \mathsf{N}^T \\ \widehat{\mathsf{A}}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{92}
$$

for some $a_1, \ldots, a_4 \in \mathbb{R}$.

We first derive an algebraic identity that will be useful later on. Using (90), the Laplace expansion of the determinant of A by the last row is given by:

$$
\det(\mathsf{A}) = -a_1\, 2\hat{T}_1 + a_2\, 2\hat{T}_2 - a_3\, 2\hat{T}_3 + a_4\, 2\hat{T}_4. \tag{93}
$$

We seek now the conditions on $a_1, \ldots, a_4$ to ensure that A is a basis transformation matrix. In view of Lemma 4.2, we know that the orthogonality condition ($\mathsf{P2}^{\phi}$) is equivalent to the algebraic relation $\widehat{\mathsf{R}}^T \widehat{\mathsf{A}} = 0$. Equation (27) then implies that

$$
a_1\mathbf{g}_1 + a_2\mathbf{g}_2 + a_3\mathbf{g}_3 + a_4\mathbf{g}_4 = 0. \tag{94}
$$

For a quadrilateral cell the vectors $\mathbf{g}_i$, which are defined in (26), are such $\mathbf{g}_1 = -\mathbf{g}_3$ and $\mathbf{g}_2 = -\mathbf{g}_4$ and equation (94) becomes:

$$
(a_1 - a_3)\mathbf{g}_1 + (a_2 - a_4)\mathbf{g}_2 = 0. \tag{95}
$$

Furthermore, since $\mathbf{g}_1$ and $\mathbf{g}_2$ for a nondegenerate quadrilateral must be linearly independent vectors, condition (95) implies that $a_1 = a_3$ and $a_2 = a_4$. In conclusion, the transformation matrix A must be such that

$$
\widehat{\mathsf{A}}^T = \begin{bmatrix} s & t & s & t \end{bmatrix}, \quad \text{with } s, t \in \mathbb{R} \text{ and } \; s \neq t, \tag{96}
$$

where the condition $s \neq t$ is to ensure that A is non singular. Note, indeed, that (93) and (91) imply that

$$
\det(\mathsf{A}) = -s\, 2\hat{T}_1 + t\, 2\hat{T}_2 - s\, 2\hat{T}_3 + t\, 2\hat{T}_4 = 2\,(t - s)\,|\mathsf{P}|\,, \tag{97}
$$

which is different from zero if and only if $s \neq t$.

The matrix D^{ϕ} given by (48) also consists of a single column vector in $\mathbb{R}^4$, and, more precisely, is the fourth column of $\mathsf{B} = \mathsf{A}^{-1}$. A direct inversion of matrix A in (92) yields the formula

$$
\mathsf{D}^{\phi} = \frac{1}{(s - t)} \begin{bmatrix} +\hat{T}'_1 \\ -\hat{T}'_2 \\ +\hat{T}'_3 \\ -\hat{T}'_4 \end{bmatrix}, \tag{98}
$$

with $\hat{T}'_i = \hat{T}_i / |\mathsf{P}|$, providing an interpretation of the result of Lemma 4.3 in terms of the geometric adimensional quantities $\hat{T}'_1$, $\hat{T}'_2$, $\hat{T}'_3$, and $\hat{T}'_4$.

We recall that $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = y$. In view of (35) with the coefficients a_i provided by (96), we have the following formula for the fourth basis function:

$$
\phi_4 = s\,\chi_1 + t\,\chi_2 + s\,\chi_3 + t\,\chi_4 = s\,(\chi_1 + \chi_3) + t\,(\chi_2 + \chi_4). \tag{99}
$$

Since $\chi_1 + \chi_3 = 1 - (\chi_2 + \chi_4)$, we can also express ϕ_4 as

$$
\phi_4 = s - (s - t)(\chi_2 + \chi_4) = (s - t)(\chi_1 + \chi_3) - t. \tag{100}
$$

For the quadrilateral cell of Figure 2, the harmonic functions χ^H that solve problem (75)-(76) are displayed in Figure 3. In Figure 4 we show the function $\phi_4^H = (s - t)(\chi_1^H + \chi_3^H) - t$ corresponding to $s = 1$, $t = -1$. Now, the matrix $\widehat{\mathsf{M}}^{\phi}$ is in this case 1×1 and its value, herein denoted by the same symbol, is given by

$$
\widehat{\mathsf{M}}^{\phi} = \int_{\mathsf{P}} |\nabla \phi_4|^2 \; dV = (s - t)^2 \int_{\mathsf{P}} |\nabla(\chi_2 + \chi_4)|^2 \; dV = (s - t)^2 \int_{\mathsf{P}} |\nabla(\chi_1 + \chi_3)|^2 \; dV. \tag{101}
$$

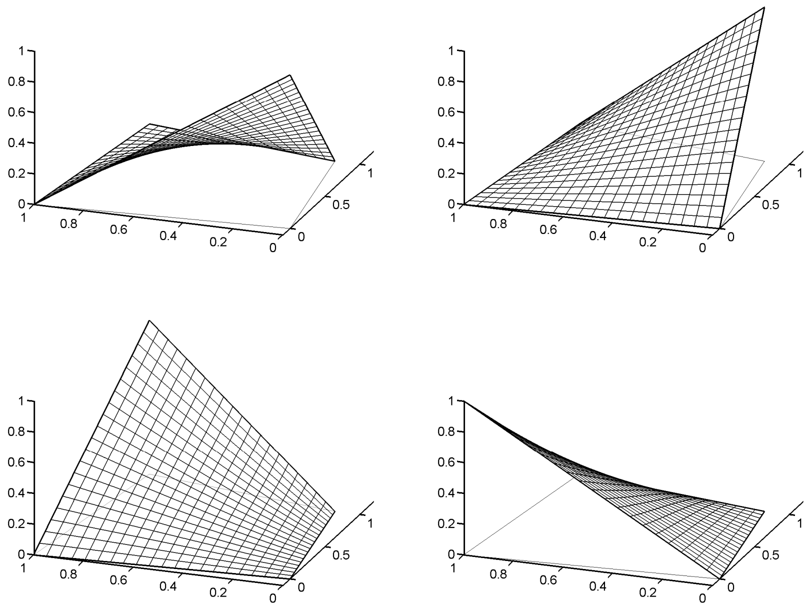

Fig. 3. The functions χ_1^H, χ_2^H, χ_3^H, and χ_4^H (from left to right, top to bottom) that solve the harmonic problem (75)-(76).

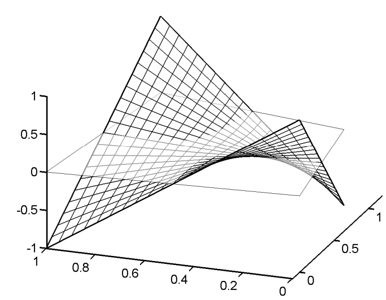

Fig. 4. The function ϕ_4^H for $s = 1$ and $t = -1$.

And, using (101) and (98) we can express the mimetic stabilization term (53) as

$$\mathsf{M}_1^{\chi} = \mathsf{D}^{\phi}\widehat{\mathsf{M}}^{\phi}\big(\mathsf{D}^{\phi}\big)^T = \int_{\mathsf{P}} \left|\nabla(\chi_1 + \chi_3)\right|^2 \, dV \begin{bmatrix} +\hat{T}_1' \\ -\hat{T}_2' \\ +\hat{T}_3' \\ -\hat{T}_4' \end{bmatrix} \big[+\hat{T}_1' \; -\hat{T}_2' \; +\hat{T}_3' \; -\hat{T}_4' \big]. \tag{102}$$

Note that, when P is a parallelogram, $\hat{T}_i' = 1/2$, hence the matrix M_1^{χ} is simply given by

$$(\mathsf{M}_1^{\chi})_{ij} = \frac{(-1)^{i+j}}{4} \int_{\mathsf{P}} \left|\nabla(\chi_1 + \chi_3)\right|^2 \, dV. \tag{103}$$

7.2. *The reconstruction problem for quadrilateral cells*

We consider the reconstruction problem of Section 5 for a quadrilateral cell. We show that, in this particular case, a modification of the virtual finite element space allows the reconstruction problem to be solved unconditionally.

Recall that a mimetic matrix is determined by the choices of the matrices D and U, see equation (19), and the reconstruction problem on a quadrilateral cell consists in finding a local basis $\{\phi_i\}_{i=1,\dots,4}$ that reproduces this mimetic matrix.

In the present case, the matrix D is a single column vector in $\mathbb{R}^4$ and U is a positive scalar. As $\phi_1 = 1$, $\phi_2 = x$, and $\phi_3 = y$, only one basis function, namely ϕ_4, is at our disposal. Moreover, the vector D must be orthogonal to the three columns of N (which are equal to the transpose of the first three rows of A) and this condition is sufficient to determine it up to a constant factor. If we substitute $\begin{bmatrix} a_1\ a_2\ a_3\ a_4 \end{bmatrix}$ with $\begin{bmatrix} 1\ 1\ 1\ 1 \end{bmatrix}$, $\begin{bmatrix} x_1\ x_2\ x_3\ x_4 \end{bmatrix}$, or $\begin{bmatrix} y_1\ y_2\ y_3\ y_4 \end{bmatrix}$ in equation (93), we must have $\det(\mathsf{A}) = 0$, as the matrix A is clearly singular in these cases. Therefore, all vectors orthogonal to the columns of N must be of the form

$$\mathsf{D} = d \begin{bmatrix} +\hat{T}_1 \\ -\hat{T}_2 \\ +\hat{T}_3 \\ -\hat{T}_4 \end{bmatrix}, \quad \text{with } d \in \mathbb{R},\ d \neq 0. \tag{104}$$

A direct application of equation (72) yields

$$\widehat{\mathsf{A}} = \frac{1}{2\,d\,|\mathsf{P}|} \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix} - \gamma \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \end{bmatrix}, \quad \gamma \in \mathbb{R}, \tag{105}$$

which is again of the form $\begin{bmatrix} s & t & s & t \end{bmatrix}^T$ (notice that $s \neq t$) in accordance with (96). The one-column matrix $\widehat{\mathsf{A}}$ determines a function δ defined on $\partial\mathsf{P}$ that is linear on each edge and whose values at the four vertices of P are given by the entries of $\widehat{\mathsf{A}}$.

The parameter matrix U has size 1×1 and is, thus, determined by a single positive scalar factor that we denote by the same symbol U. To reproduce the mimetic scheme, we need to find a function ϕ_4 that is equal to δ on the boundary $\partial\mathsf{P}$ and whose energy $\int_{\mathsf{P}} |\nabla\phi_4|^2\, dV$ is equal to U. Since in view of Remark 5.3 the energy of ϕ_4 is independent of the parameter γ, from now on, we take $\gamma = 0$. With such choice of γ, the integral of δ on each edge is equal to zero.

The harmonic choice ϕ_4^H is the solution with boundary trace δ and with the minimum possible energy U^H. Therefore, when $\mathsf{U} > \mathsf{U}^H$, we can apply the procedure described in the proof of Theorem 5.1 and find a modified function ϕ_4 with energy U. Instead, when $\mathsf{U} < \mathsf{U}^H$ the reconstruction problem has no solution as for the more general polygonal case considered in Section 5. In Figure 5 we show, for $d = 1$, a possible choice for the function w_4 and the modified function $\phi_4 = \phi_4^H + w_4$ defined in Theorem 5.1, while in Figure 6 we show the corresponding modified functions χ_i.

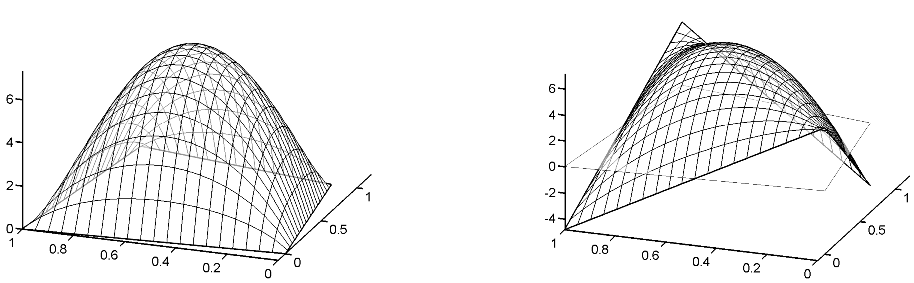

Fig. 5. The function w_4 (left plot) and the modified functions $\phi_4 = \phi_4^H + w_4$ for $d = 1$ (right plot).

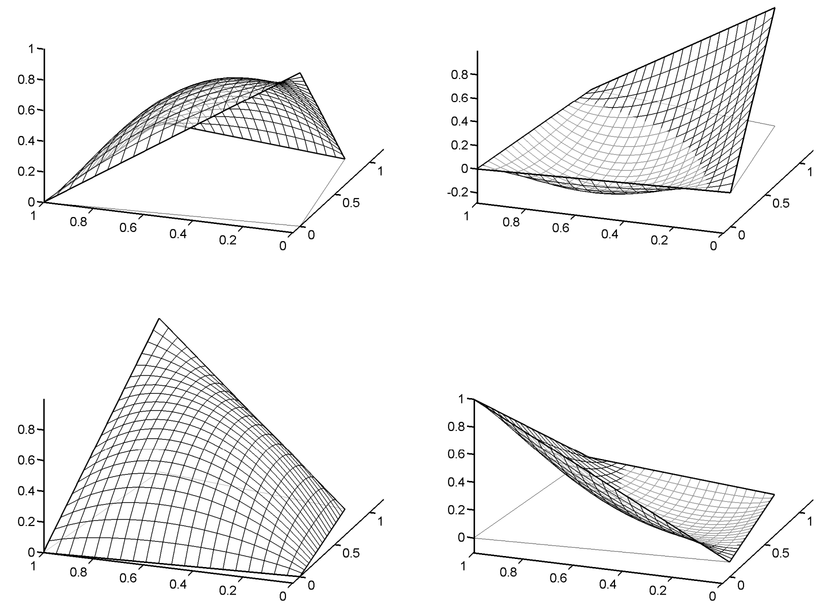

Fig. 6. The functions χ_1, χ_2, χ_3, and χ_4, (from left to right, top to bottom), which are obtained from the functions $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = y$ and the modified function ϕ_4 shown in Figure 5.

We show now that for a quadrilateral cell we can slightly modify the space $V_{h,\mathsf{P}}$ and take into account all positive values of the parameter U. To this end, we replace assumption $(\mathsf{P1}^{\chi})$ with the weaker assumption:

$(\mathbf{P1}^{\boldsymbol{\chi}})^{*}$: $\chi_{i|\mathsf{e}}$, the restriction of χ_i to $\mathsf{e} \in \partial\mathsf{P}$, can be integrated exactly on each edge e by the trapezoidal integration rule.

All the theoretical developments considered so far still hold, but, now, we can use the boundary values of ϕ_4^H to decrease the harmonic energy threshold.

We shall modify the local spaces $V_{h,\mathsf{P}}$ *without* compromising the conformity of the global finite element space V_h. Conformity is ensured if the edge values of all local basis functions are completely determined by the values attained at the edge end points. Hence, as the energy diminishing procedure is dependent on the nodal values, we need to rescale ϕ_4^H first. To do so, we rescale the piecewise linear function δ so that its nodal values are given by alternating $+1$ and -1. And, as the nodal values of δ are given by the entries of $\widehat{\mathsf{A}}$, we need to consider a modified $\widehat{\mathsf{A}}$. This can be done using the matrix invariance property of Section 6. We fix $\mathsf{S}^{-1} = 2d|\mathsf{P}|$ so that from (88) and (105) with $\gamma = 0$ we get $\widehat{\mathsf{A}}^{\dagger} = \begin{bmatrix} +1 & -1 & +1 & -1 \end{bmatrix}^T$. Consequently, $\mathsf{D}^{\dagger} = \mathsf{DS} = \frac{1}{2|\mathsf{P}|}\begin{bmatrix} +\hat{T}_1 & -\hat{T}_2 & +\hat{T}_3 & -\hat{T}_4 \end{bmatrix}^T$ and $\mathsf{U}^{\dagger} = (2d|\mathsf{P}|)^2\mathsf{U}$ and, as noted earlier, the reconstruction procedure will still yield the same mimetic local stiffness matrix as $\mathsf{D}^{\dagger}\mathsf{U}^{\dagger}{\mathsf{D}^{\dagger}}^T = \mathsf{DUD}^T$.

Now, for each edge e we consider an edge bubble function b_{e} belonging to $C_0^{\infty}(\mathsf{e})$ and such that

$$\int_{\mathsf{e}} b_{\mathsf{e}}\, dS = 0.$$

Let $\mathsf{v}_{\mathsf{e}}^1$ and $\mathsf{v}_{\mathsf{e}}^2$ be the end points of e. As $C_0^{\infty}(\mathsf{e})$ is dense in $H^{\frac{1}{2}}(\mathsf{e})$ (see e.g. [16]), the edge bubble b_{e} can be made arbitrary close in $H^{\frac{1}{2}}(\mathsf{e})$ to the linear function that takes the values -1 and $+1$ at $\mathsf{v}_{\mathsf{e}}^1$ and $\mathsf{v}_{\mathsf{e}}^2$, respectively, see Figure 7.

Then, we consider the function ϕ_4^H that is the solution of the harmonic problem on P with boundary value on each edge e given by $g_{|\mathsf{e}} + b_{\mathsf{e}}$ if $g(v_{\mathsf{e}}^1) = 1$ or $g_{|\mathsf{e}} - b_{\mathsf{e}}$ if $g(v_{\mathsf{e}}^1) = -1$. Since the energy of ϕ_4^H satisfies the relation

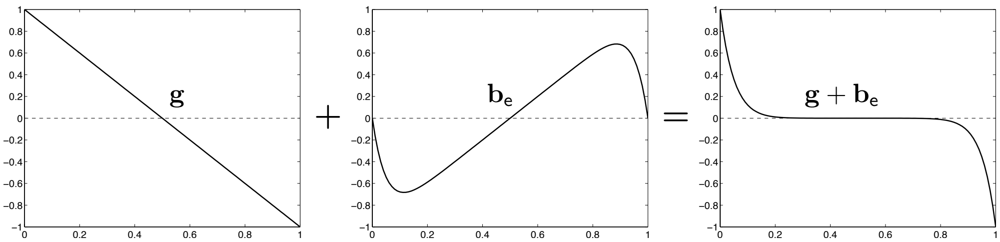

Fig. 7. The modified boundary value $g + b_{\mathsf{e}}$ of ϕ_4^H.

$$\Big(\int_{\mathsf{P}} \big|\nabla \phi_4^H\big|^2 \, dV\Big)^{\frac{1}{2}} = \big|\phi_4^H\big|_{H^1(\mathsf{P})} \leq C \|\phi_4^H\|_{H^{\frac{1}{2}}(\partial \mathsf{P})},$$

we can construct a modified function ϕ_4^H with arbitrarily small energy. Further, the *same* edge bubble function can be chosen to modify the edge value of the basis function relative to both cells sharing the same edge, thus ensuring that conformity is maintained.

Now assume that, for each P, the function ϕ_4^H has been modified in this way in order to achieve $\mathsf{U}^H \leq \mathsf{U}^{\dagger}$. We can now apply the general procedure to fix a bubble function w_4 such that the new basis function $\phi_4 = \mathsf{U}^H + w_4$ has exactly energy $\mathsf{U}^{\dagger}$, thus solving the reconstruction problem.

We end this section by showing how the strategy discussed herein acts on the basis functions χ^H and ϕ^H. In Figure 8 we show the modified function ϕ_4^H. The corresponding modified functions χ_i^H are shown in Figure 9.

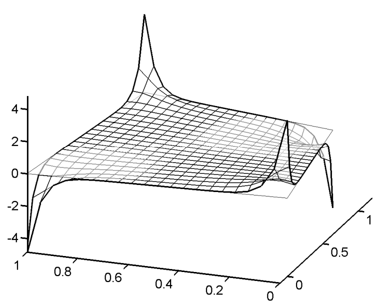

Fig. 8. The harmonic modified functions ϕ_4^H for $d = 1$ in (104).

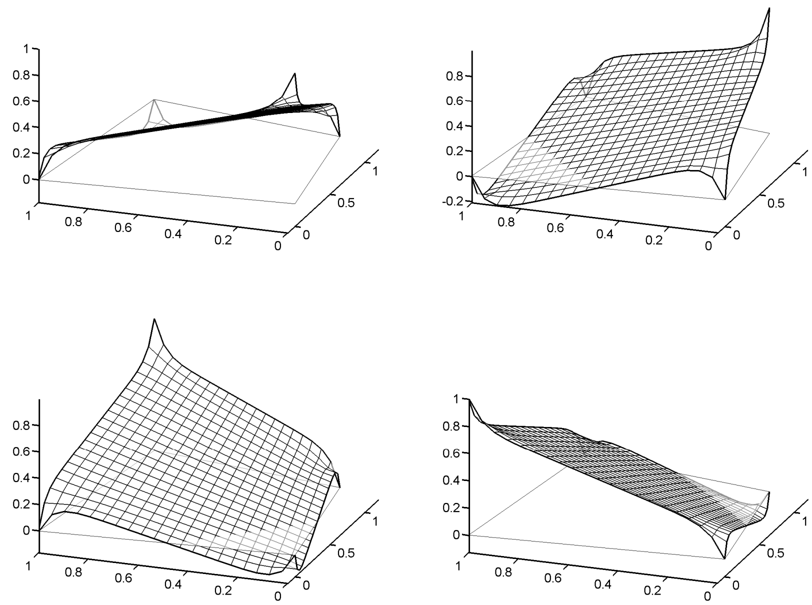

Fig. 9. The final harmonic modified functions χ_1^H, χ_2^H, χ_3^H, and χ_4^H (from left to right, top to bottom) that generate the space $V_{h,\mathsf{P}}$.

8. Numerical Experiments.

We investigate, on a fixed quadrilateral mesh, the effect of the mimetic stabilization term (102) on the accuracy of the MFD solution by varying the parameter

$$\tau_{\mathsf{P}} = \int_{\mathsf{P}} |\nabla(\chi_1 + \chi_3)|^2 \, dV,$$

by which we express the energy of ϕ_4, *cf.* (101).

As discussed in Section 7, on a mesh of quadrilateral cells a relation between a mimetic method and a corresponding virtual element framework can always be established, that is even below the harmonic threshold parameter, herein denoted by τ_{P}^H, corresponding to the special case of harmonic basis. Nonetheless, the choice of τ_{P} has a great impact on the accuracy of the solution on a given mesh due to the stability condition stated in ($\mathsf{S1}$). An accurate numerical solution is provided by such methods only when the parameter τ_{P} is chosen in the range imposed by the stability requirement ($\mathsf{S1}$). For non-quadrilateral meshes, the behavior is similar.

Errors are measured in the mesh-dependent norm (8)-(9), which mimic the energy norm, and the mesh-dependent L^2 norm defined by the relation:

$$|||u|||_{0,h}^2 = \sum_{\mathsf{P}\in\Omega_h} |\mathsf{P}| \sum_{\mathsf{v}\in\partial\mathsf{P}} |u_{\mathsf{v}}|^2, \tag{106}$$

and the associated relative error by

$$\mathcal{E}_{0,h}(u_h) = \frac{|||u - u_h|||_{0,h}}{|||u|||_{0,h}}. \tag{107}$$

We consider the Poisson problem (1)-(2) on $\Omega = (0,1)^2$ with K equal to the identity matrix. We subdivide Ω into the uniform 21×21 square mesh and choose the same value for τ_{P} in all cells in view of showing that the instability phenomenon may appear also on very regular meshes. The forcing term and boundary data are determined in accordance with the exact solution

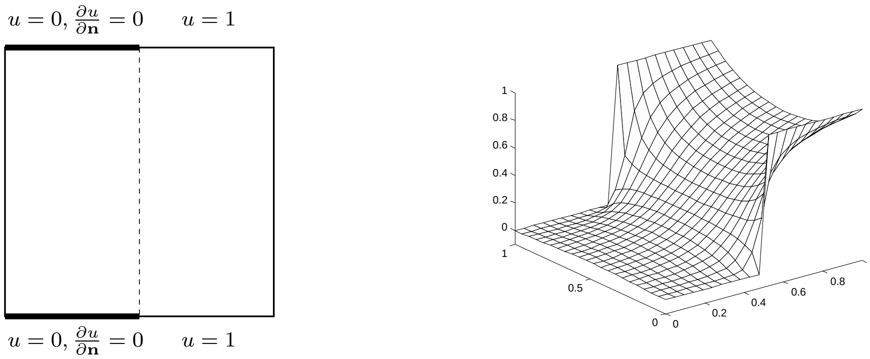

Fig. 10. Sketch of the test case (left) and exact solution on the 21×21 mesh considered for the calculations (right).

$$u(x,y) = \begin{cases} \left(\dfrac{1}{\pi} \mathrm{atan}\Big(\dfrac{x - 1/2}{y(1-y)} \Big) + \dfrac{1}{2} \right)^2 & \text{for } y(1-y) \neq 0 \\ 0 & \text{for } y(1-y) = 0 \text{ and } x < 1/2 \\ 1 & \text{for } y(1-y) = 0 \text{ and } x > 1/2. \end{cases}$$

At the horizontal boundaries $y = 0$ and $y = 1$ the solution is piecewise constant, showing a discontinuity located at $x = 1/2$, and its normal derivative is zero for $0 \leq x < 1/2$, cf. Figure 10.

We run two test cases varying the boundary conditions. In the first, we impose Dirichlet conditions everywhere (labeled by *Dirichlet everywhere* in the figures), while in the second test case we impose the homogeneous Neumann condition for $0 \leq x < 1/2$ and $y = 0$ or $y = 1$, (labeled by *Dirichlet-Neumann* in the figures) and the appropriate Dirichlet condition elsewhere. The reason for these particular choices will be evident from the discussion below. In Figure 11 we show the approximation errors for the two test cases in the range $\tau_{\mathsf{P}} \in \left[10^{-4}, 10^4\right]$, while in Figures 12 and 13 we show the numerical solution at mesh vertices provided by $\tau_{\mathsf{P}} = 10^{-4}$, $\tau_{\mathsf{P}} = \tau_{\mathsf{P}}^H = 2/3$ (harmonic choice), $\tau_{\mathsf{P}} = 2$ (a value close to but bigger than τ_{P}^H) and $\tau_{\mathsf{P}} = 10^4$. Note that the approximate solution is acceptable for a quite large range of values of τ_{P}. However, when the value of τ_{P} is much smaller or much bigger than such acceptable range, the quality of the solution visibly deteriorates. Let us discuss the solution's behaviour in these two extreme cases.

– *Case* τ_{P} *small.* In the current setting, it is easy to show that when $\tau_{\mathsf{P}} = 0$ the MFD scheme reduces to the finite element method related to the so-called *hourglass instability.* Our experiments show that the hourglass instability is present already for small enough τ_{P} and it manifest itself with the presence of spurious oscillations; when $\tau_{\mathsf{P}} = 0$, the resulting matrix can even be singular (see, e.g., [14]), depending on the boundary conditions. Oscillations are not always present; for instance, they appear when the Dirichlet boundary data have a sharp transition between two values and such transition is not resolved by the mesh and in the case of mixed Dirichlet-Neumann boundary conditions. We note that, when P is a parallelogram, the hourglass stabilization term proposed in [14] coincides with M_1^{χ} given by (103) for a particular choice of the parameter τ_{P}.
– *Case* τ_{P} *large.* When τ_{P} becomes large, the approximate solution degrades, regardless of the boundary conditions.

9. Conclusions

We established a finite element characterization of the family of Mimetic Finite Difference methods for diffusion problems presented in [7]. We provided the condition under which the nodal grid functions and the bilinear form used in the MFD discretization can be interpreted as the degrees of freedom and the implementation of a conforming *virtual* finite element method on polygons. The finite element is never constructed in practice, but its existence permits us to reinterpret the approximation and stability properties

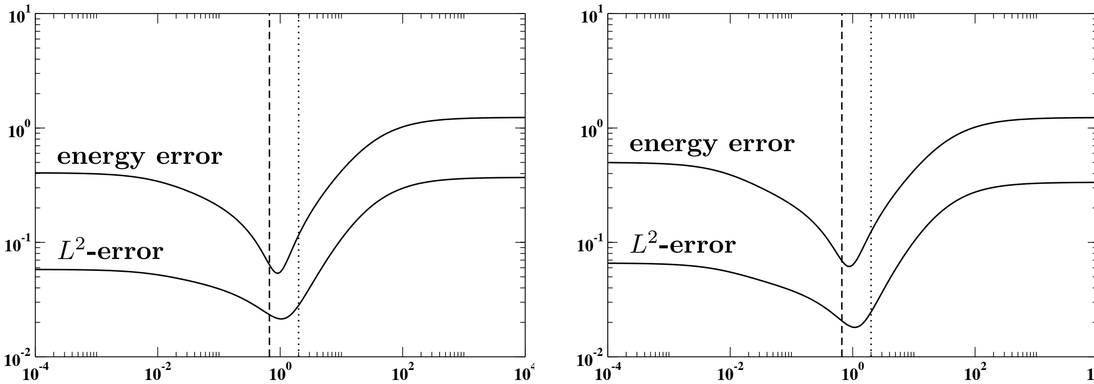

Fig. 11. Relative approximation errors (versus τ) $\mathcal{E}_{0,h}(u_h)$, cf. (107), labeled by L^2-*error* and $\mathcal{E}_{1,h}(u_h)$, cf. (8), labeled by *energy error* for $\tau_{\mathsf{P}} \in [10^{-4}, 10^4]$. We consider Dirichlet condition everywhere (left plot) and Dirichlet-Neumann condition (right plot). The values $\tau_{\mathsf{P}} = 2/3$ (dashed line) and $\tau_{\mathsf{P}} = 2$ (dotted line) are shown in both plots.

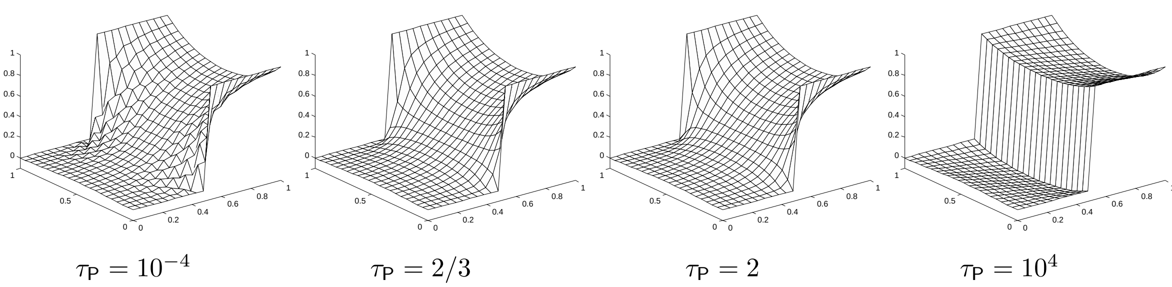

Fig. 12. Profile of the numerical solution at mesh vertices for different values of τ_{P}; boundary conditions: Dirichlet everywhere.

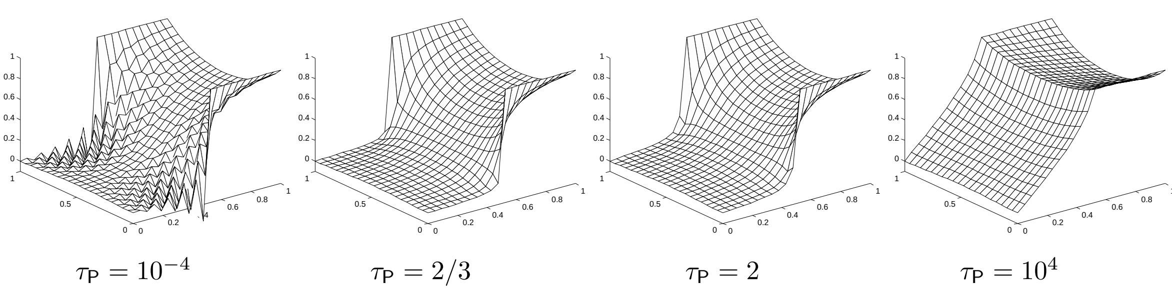

Fig. 13. Profile of the numerical solution at mesh vertices for different values of τ_{P}; boundary conditions: Dirichlet-Neumann.

of the MFD method in a Galerkin framework. In addition, we have shown here that in the particular case of quadrilateral meshes it is possible to drop the condition under which the identification with virtual elements is possible.

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