SYMPLECTIC METHODS FOR OPTIMIZATION AND CONTROL

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§1. Introduction

1. The language of Symplectic geometry is successfully employed in many branches of contemporary mathematics, but it is worth to remind that the original development of Symplectic geometry was greatly influenced by variational problems. In Optimal control crucial role was plaid by the Hamiltonian system of Pontryagin's Maximum principle, which itself is the object of Symplectic geometry. In further development of Optimal control priorities were given to Convex analysis. Though Convex and Functional analysis are very helpful in developing the general theory of Extremal problems, they are not at all effective for investigating essentially nonlinear problems in higher approximations, when the convex approximation fails. Therefore, since the discovery of the Maximum principle, there were always attempts of introducing of geometric methods of investigation, though not as universal as the Convex and Linear methods. These new geometric methods were applied mainly for obtaining optimality conditions of higher orders and constructing the optimal synthesis, and today we already have many ingenious devices and beautiful concrete results.

It seems very probable that there should be a general framework which could unify different directions of these geometric investigations and merge the Maximum Principle with the theory of fields of extremals of the classical Calculus of Variations. We are convinced that the appropriate language for such unification can be provided by Symplectic geometry, and as a justification for such a conviction we consider the statement , according to which "the manifold of Lagrange multipliers in the problem of conditional extremum is a Lagrangian manifold". Thus two "Lagrangian" objects — Lagrange multipliers, the main object of the theory of Extremal problems, and Lagrangian submanifolds, main objects of Symplectic geometry, which existed independently for a long time, can be unified.

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2. The present article has an expository character. Its main aim is to demonstrate the power and productivity of the "symplectic" language in optimization problems and to draw attention of specialists interested in Control theory to this promising direction of investigations. All basic notions and facts from Symplectic geometry, needed in the sequel, are introduced in the article. A big part of our exposition is devoted to reformulation of well–known results into symplectic language, many of which became a folklore. In these cases it is almost impossible to give accurate references, and we do not give them, which should certainly not mean that the authors claim any originality. We also do not claim that we have indicated all essential connections of Optimal control with Symplectic geometry. For example, we omitted matrix Riccati equations, which play essential role in linear–quadratic optimal problems. These equations define flows on Grassmannians, and in case of symmetric matrices on Lagrange Grassmannians. Such flows are studied in detail in many publications. We do not consider in the article such a big and important theme as the variational methods of investigation of Hamiltonian systems — a good example of a feedback influence of Optimization on Symplectic geometry.

In the sequel smoothness always means differentiability of appropriate order, a smooth mapping of manifolds is called *submersion (immersion)* if its differential at every point is surjective (injective). Let M me a smooth manifold. We call a *smooth submanifold in* M an arbitrary immersed submanifold, i.e. every one–to–one immersion of a manifold into M. We call a *Lipschitz submanifold in* M an arbitrary locally–Lipschitz one–to–one mapping $\Phi : W \longrightarrow M$ of a smooth manifold W into M. A *Lipschitz curve* on a Lipschitz manifold is a curve $t \mapsto \Phi(w(t))$, where $w(\cdot)$ is a Lipschitz mapping of a segment into W. To simplify the exposition we shall indicate in the sequel only the image $\Phi(W)$ assuming that the smooth or Lipschitz mapping Φ is given.

We assume that every space of smooth mappings of smooth manifolds is always endowed with the standard topology of uniform convergence of derivatives of all orders on arbitrary compact sets. We say that a generic smooth mapping has a given property if the set of all mappings with this property contains a countable intersection of open everywhere dense subsets in the space of all mappings.

We use in this article standard notations. T_xM denotes the tangent space to M at $x \in M$, T_x^*M is the cotangent space (conjugate to the space T_xM.) The tangent bundle to M is denoted by TM and is represented as a smooth vector bundle with base M, total space $\underset{x\in M}{\cup} T_xM$ and the canonical projection $v \mapsto x,\ v \in T_xM$. Correspondingly, T^*M is the cotangent bundle, which plays a special role in Symplectic geometry. Therefore we introduce a special symbol for the canonical projection of the total space $\underset{x\in M}{\cup} T^*M$ onto the base:

$$\pi_M : T^*M \longrightarrow M,\ \vartheta \mapsto x,\ \vartheta \in T_x^*M\, x \in M.$$

Let $A : E \longrightarrow E'$ be a linear mapping of linear spaces, then $ker\, A$ denotes the kernel of the mapping, $im\, A$ is its image and $rank\, A = dim\,(im\, A)$. A quadratic form $q : E \longrightarrow \mathbb{R}$ is represented as $q(e) = b(e, e)$, where $b(e_1, e_2),\ e_1, e_2 \in E$ is a symmetric bilinear form on E. In this case $ker\, q = \left\{e \in E \,\middle|\, b(e, e') = 0\ \forall e' \in E\right\}$. We write $q > 0$ (< 0) and we say that the form q is positive (negative), if $q(e) >$

0 (< 0) for nonzero $e \in E$. If we substitute symbols $>$ ($<$) by $\geq$ ($\leq$) we obtain nonnegative (nonpositive) forms.

3. The remainder of this section is devoted to some initial facts from Symplectic geometry; for details we recommend [13].

Let Σ be a finite–dimensional real vector space. A *symplectic form* on Σ is an arbitrary real nondegenerate skew–symmetric bilinear form on Σ, i.e. a bilinear mapping $\sigma : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ such that

$$\sigma(z_1, z_2) = -\sigma(z_2, z_1) \ \ \forall z_1, z_2 \in \Sigma,$$

and the relation $\sigma(z, z') = 0 \ \forall z' \in \Sigma$ implies $z = 0$. The space Σ with a given symplectic form σ on it is called *symplectic*. It is easily seen that symplectic forms exist only on even–dimensional spaces and all such forms are transformed into each other by linear substitutions of variables. More precisely, let $dim \Sigma = 2n$, then there exists a basis $e_1, \dots, e_n, f_1, \dots, f_n$ in Σ such that

$$\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0 \ \forall i, j; \ \sigma(e_i, f_j) = 0 \ \forall i \neq j, \ \ \sigma(e_i, f_i) = 1, \ i = 1, \dots, n. \quad (1)$$

Every basis of a symplectic space which satisfies (1) is called a *canonical basis*.

Let $S \subset \Sigma$, the subspace

$$S^{\angle} = \left\{ z \in \Sigma \middle| \sigma(e, z) = 0 \ \forall e \in S \right\} \subset \Sigma.$$

is called the *skew–orthogonal complement to S*. For every subspace $E \subset \Sigma$ the relations

$$dim\, E^{\angle} = 2n - dim\, E, \ \ (E^{\angle})^{\angle} = E$$

hold.

The subspace E is called *isotropic* if $E \subset E^{\angle}$. Every one–dimensional subspace is isotropic, and the dimension of every isotropic subspace does not exceed n. A subspace E is called *Lagrangian* if $E = E^{\angle}$. Thus Lagrangian subspaces in Σ are exactly the n–dimensional isotropic subspaces.

The symplectic group $Sp(\Sigma)$ is the group of all linear transformations of the symplectic space Σ which preserve the symplectic form:

$$Sp(\Sigma) = \left\{ p \in GL(\Sigma) \middle| \sigma(pz_1, pz_2) = \sigma(z_1, z_2) \ \forall z_1, z_2 \in \Sigma \right\}.$$

It is a connected Lie group of dimension $n(2n + 1)$. The elements of this group are called the *symplectic transformations* of the space Σ. The Lie algebra of the Symplectic group is given by the expression

$$sp(\Sigma) = \left\{ A \in gl(\Sigma) \middle| \sigma(Az_1, z_2) = \sigma(Az_2, z_1) \ \forall z_1, z_2 \in \Sigma \right\}.$$

Let h be a real quadratic form on Σ and $d_z h$ be the differential of h at the point $z \in \Sigma$. Then $d_z h$ is a linear form on Σ which depends linearly on z. For every $z \in \Sigma$ there exists a unique vector $\overrightarrow{h}(z) \in \Sigma$ which satisfies the condition

$$\sigma(\overrightarrow{h}(z), \cdot) = d_z h.$$

It is easy to show that the linear operator $\vec{h} : \Sigma \longrightarrow \Sigma$ belongs to $sp(\Sigma)$, and the mapping $h \mapsto \vec{h}$ is an isomorphism of the space of quadratic forms onto $sp(\Sigma)$. We obtain the *linear Hamiltonian system* of differential equations $\dot{z} = \vec{h}(z)$ corresponding to the quadratic *Hamiltonian* h.

Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a canonical basis in Σ and $z = \sum_{i=1}^{n}(x^i e_i + \xi^i f_i)$. The Hamiltonian system takes the standard form in coordinates x^i, ξ^i:

$$\left.\begin{aligned} \dot{x}^i &= \frac{\partial h}{\partial \xi^i}, \\ \dot{\xi}^i &= -\frac{\partial h}{\partial x^i},\ i = 1, \dots, n. \end{aligned}\right\} \tag{2}$$

4. Let $\mathcal{N}$ be a $2n$–dimensional smooth manifold. A smooth nondegenerate closed differential 2–form σ on $\mathcal{N}$ is called a *symplectic structure* on this manifold. The nondegeneracy of the form $\sigma : y \mapsto \sigma_y,\ y \in \mathcal{N}$, means that σ_y is a symplectic form on the tangent space $T_y\mathcal{N}\ \forall y \in \mathcal{N}$; "closed" means that $d\sigma = 0$, where d is the exterior differential. A manifold $\mathcal{N}$ with a given symplectic structure σ on it is called a *symplectic manifold.*

An immersion $\Phi : W \longrightarrow \mathcal{N}$ is called *isotropic* if $\Phi^*\sigma = 0$. An isotropic immersion Φ is called *Lagrangian* if $dim\, W = n$. Correspondingly is defined an isotropic (Lagrangian) smooth submanifold of a symplectic manifold $\mathcal{N}$. It is defined by the condition that its tangent space at every point y must be an isotropic (Lagrangian) subspace of the space $T_y\mathcal{N}$ provided with the symplectic form σ_y.

The most important symplectic manifolds in many situations are cotangent bundels which carry a natural symplectic structure. To define it suppose M is an n–dimensional smooth manifold and let $\pi_M : T^*M \longrightarrow M$ be its cotangent bundle. Let $\vartheta \in T^*M,\quad v \in T_\vartheta(T^*M)$, then $\pi_{M*}v \in T_{\pi_M(\vartheta)}M$. Denote the pairing of the vector $\pi_{M*}v$ with the covector $\vartheta \in T^*_{\pi_M(\vartheta)}M$ by $s_M(v) = \vartheta(\pi_{M*}v)$. The correspondence $v \mapsto s_M(v),\ v \in T(T^*M)$, defines a differential 1–form s_M on T^*M. The closed 2–form

$$\sigma_M \overset{\text{def}}{=} -ds_M$$

defines the standard symplectic structure on the $2n$–dimensional manifold T^*M.

Remark. To avoid ambiguities we emphasize that the covector $\vartheta \in T^*M$ is an element of the $2n$–dimensional manifold T^*M, (the covector defines the fibre to which it belongs.) Often we omit the lower index in the symbols s_M, σ_M, if the manifold M is determined from the context.

A Lipschitz submanifold $\Phi : W \longrightarrow T^*M$ is called *isotropic* if $\int_{\Phi(\gamma)} s_M = 0$ for every closed Lipschitz curve γ which is contractible in W. It is called *Lagrangian* if, additionally, $dim\, W = n$. An immersion $\Phi : W \longrightarrow T^*M$ is isotropic iff Φ^*s_M is a closed form on W. An isotropic (Lagrangian) immersion is called *exact* if the form Φ^*s_M is exact on W. Correspondingly, a Lipschitz isotropic submanifold $\Phi : W \longrightarrow T^*M$ is called *exact* if $\int_{\Phi(\gamma)} s_M = 0$ for every closed Lipschitz curve γ on W.

Among n–dimensional submanifolds in T^*M the smooth sections, i.e. differential 1–forms on M, are singled out. Such a section (1–form) is a Lagrangian submanifold iff the form is closed. It is an exact Lagrangian submanifold iff the 1–form is exact, i.e. is a graph of a differential of a smooth function. The fibres $T^*_x M,\ x \in M$, of the bundle T^*M are exact Lagrangian submanifolds as well.

5. Suppose $\mathcal{N}_i$ is a $2n$–dimensional symplectic manifold with the symplectic form $\sigma_i,\ i = 1, 2$. The diffeomorphism $P : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ is called *symplectomorphism* if $P^*\sigma_2 = \sigma_1$. A well-known theorem of Darboux states that all symplectic manifolds of equal dimension are locally symplectomorphic:

For $\forall z_i \in \mathcal{N}_i$ *there exist neighborhoods* O_{z_i} *of the points* z_i *in* $\mathcal{N}_i,\ i = 1, 2$ *and a symplectomorphism* $P : O_{z_1} \longrightarrow O_{z_2},\ P(z_1) = z_2$.

Suppose $\mathcal{L}$ is an imbedded Lagrangian submanifold of a symplectic manifold $\mathcal{N}$. (An immersed submanifold is imbedded if its topology is induced by the topology of $\mathcal{N}$.) It turns out that in this case a certain neighborhood of $\mathcal{L}$ in $\mathcal{N}$ is symplectomorphic to a neighborhood of the trivial section in $T^*\mathcal{L}$.

Functions on symplectic manifolds are called *Hamiltonians*. Let h be a smooth Hamiltonian on $\mathcal{L}$ and dh be its differential (which is an exact 1–form.) Since σ is a nondegenerate 2–form there exists a uniquely defined vector field $\vec{h}$ on $\mathcal{N}$, satisfying the condition

$$\vec{h} \rfloor \sigma = \sigma(\vec{h}, \cdot) = dh.$$

The field $\vec{h}$ is called a *Hamiltonian* field corresponding to the Hamiltonian h, and the differential equation on $\mathcal{N},\ \dot{z} = \vec{h}(z)$ is the corresponding *Hamiltonian system*.

According to Darboux's theorem symlpectic manifold is locally symplectomorphic to symplectic space: in the neighborhood of an arbitrary point of the manifold we can introduce local coordinates such that in these coordinates the form σ has constant coefficients. If in addition the coordinates correspond to a canonical basis of the symplectic space then the Hamiltonian system in these coordinates with the Hamiltonian function h takes the form (2).

Let $v_\tau,\ \tau \in [0, t]$, be a family of smooth vector fields, measurable in τ and uniformly bounded on every compact in $\mathcal{N}$. Consider the nonstationary differential equation

$$\frac{d}{d\tau} z = v_\tau(z), \tag{3}$$

and suppose that all its solutions can be extended to the whole interval $[0, t]$. Then the equation (3) defines a (nonstationary) flow on $\mathcal{N}$, i.e. a locally Lipschitz, with respect to τ, family of diffeomorphisms $P_\tau : \mathcal{N} \longrightarrow \mathcal{N},\ \tau \in [0, t]$, which satisfies conditions

$$\frac{\partial}{\partial \tau} P_\tau(z) = v_\tau(P_\tau(z)),\ P_0(z) \equiv z.$$

We denote $Symp\,\mathcal{N}$ the group of all symplectomorphisms of $\mathcal{N}$. It is easy to show that $P_\tau \in Symp\,\mathcal{N}\ \forall \tau \in [0, t]$ iff the 1–form $v_\tau \rfloor \sigma$ is closed for almost all $\tau \in [0, t]$. At the same time (3) is a (nonstationary) Hamiltonian system iff the form $v_\tau \rfloor \sigma$ is exact for almost all $\tau \in [0, t]$. Thus the flow generated by a Hamiltonian system consists of symplectomorphisms.

The *Poisson bracket* of smooth Hamiltonians h_1, h_2 is the Hamiltonian

$$\{h_1, h_2\} \stackrel{\text{def}}{=} \sigma(\overrightarrow{h}_2, \overrightarrow{h}_1) = \overrightarrow{h}_1 \rfloor dh_2.$$

The Poisson bracket defines the structure of a Lie algebra on the space of all Hamiltonians: the operation is skew–symmetric and satisfies the Jacobi identity:

$$\{h_1, \{h_2, h_3\}\} - \{h_2, \{h_1, h_3\}\} = \{\{h_1, h_2\}, h_3\}.$$

Furthermore

$$\{h_1, h_2\} \circ P = \{h_1 \circ P, h_2 \circ P\}$$

for $\forall P \in Symp\mathcal{N}$, where $h \circ P(z) = h(P(z)),\ z \in \mathcal{N}$.

Let $P_\tau \in Symp\mathcal{N},\ 0 \le \tau \le t$, be a flow defined by the nonstationary Hamiltonian system

$$\frac{d}{d\tau} z = \overrightarrow{h_\tau}(z), \tag{4}$$

and φ be a smooth Hamiltonian. Then

$$\frac{\partial}{\partial \tau}(\varphi \circ P_\tau) = \{h_\tau, \varphi\} \circ P_\tau. \tag{5}$$

In particular, the function φ is the first integral of the system (4) iff $\{h_\tau, \varphi\} \equiv 0$.

Concluding this introductory section we give the *variation formula* for a pair of nonstationary Hamiltonians h_τ, g_τ.

Let $P_\tau, \widetilde{P}_\tau \in Symp\mathcal{N},\ 0 \le \tau \le t$, be flows defined by the Hamiltonians $h_\tau, h_\tau + g_\tau$:

$$\frac{\partial}{\partial \tau} P_\tau(z) = \overrightarrow{h_\tau}(P_\tau(z)),\ \frac{\partial}{\partial \tau} \widetilde{P}_\tau(z) = (\overrightarrow{h_\tau} + \overrightarrow{g_\tau})(\widetilde{P}_\tau(z)),\ P_0(z) = \widetilde{P}_0(z) = z,\ z \in \mathcal{N}.$$

Then $\widetilde{P}_\tau = P_\tau \circ R_\tau$, where R_τ is the flow generated by the Hamiltonian $g_\tau \circ P_\tau$:

$$\frac{\partial}{\partial \tau} R_\tau(z) = (\overrightarrow{g_\tau \circ P_\tau})(R_\tau(z)).$$

To prove this relation it is sufficient, according to (5), to show that

$$\frac{\partial}{\partial \tau}(\varphi \circ P_\tau \circ R_\tau) = \{h_\tau + g_\tau, \varphi\} \circ P_\tau \circ R_\tau$$

for an arbitrary smooth function φ on $\mathcal{N}$. Using again (5) we obtain

$$\frac{\partial}{\partial \tau}(\varphi \circ P_\tau \circ R_\tau) = (\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \theta})(\varphi \circ P_\tau \circ R_\theta)\Big|_{\theta = \tau} =$$
$$\{h_\tau, \varphi\} \circ P_\tau \circ R_\tau + \{g_\tau \circ P_\tau, \varphi \circ P_\tau\} \circ R_\tau = \{h_\tau + g_\tau, \varphi\} \circ P_\tau \circ R_\tau.$$

Propositions, theorems, formulas and figures are numbered independently in each section. References to other sections use double numbering.

§2. Lagrange multipliers and Lagrangian submanifolds

1. Consider a standard problem of conditional extremum. Let U, M be smooth manifolds, $f : U \longrightarrow M$ be a smooth mapping, and let $\varphi : U \longrightarrow \mathbb{R}$ be a smooth real–valued function. We have to minimize φ on a level set of the mapping f.

Put $N = \mathbb{R} \times M$ and define the mapping $F : U \longrightarrow N$ by the relation $F(u) = (\varphi(u), f(u)),\ u \in U$. If at a given point u the function φ attains its minimum on the level set of f then $im\, F$ and the ray $\left\{(\varphi(u) - t, f(u)) \in N \middle| t > 0\right\}$ have an empty intersection. In particular, $F(u)$ is a boundary point of $im\, F$.

Let $F'_u : T_u U \longrightarrow T_{F(u)} N$ be the differential of the mapping F at u (which is a linear mapping of the corresponding tangent spaces). We remind that a point of the manifold U is called a *regular point* of the mapping F if the differential of F at this point is surjective. If the point is not regular it is called a *critical point* of the mapping. The image of a critical point of F is called a *critical value* of F.

The implicit function theorem implies that the image of a regular point of the mapping F belongs to the interior of the set $im\, F$. Thus if $F(u)$ is a boundary point of $im\, F$ then $im\, F'_u \neq T_{F(u)} N$. Hence there exists a nonzero covector $\omega \in T^*_{F(u)} N$ which is orthogonal to $im\, F'_u$.

Let V be an arbitrary linear space. We shall denote the pairing of a covector $\vartheta \in V^*$ with a vector $v \in V$ simply by ϑv considering the expression as a product of a row by a column. In particular, the relation $\omega \perp im\, F'_u$ can be written as $\omega F'_u v = 0\ \forall v \in T_u U$, or simply as $\omega F'_u = 0$.

Thus we obtained the simplest form of the Lagrange multiplier rule: *if the function φ attains at u an extremal value on the level set of the mapping f then there exists a nonzero $\omega \in T^*_{F(u)} N$ such that $\omega F'_u = 0$.*

2. Definition. Let U, N be arbitrary smooth manifods and $F : U \longrightarrow N$ be a smooth mapping. We call a *Lagrangian point of the mapping F* an arbitrary pair (ω, u), where $u \in U$ and $\omega \in T^*_{F(u)} N \setminus 0$ satisfies the equality $\omega F'_u = 0$. The covector ω is called a *Lagrange multiplier* and u is called a *critical point* corresponding to the Lagrangian point (ω, u). The set of all Lagrangian points of the mapping F is denoted by C_F.

Let $F^*(T^*N)$ be a vector bundle over U, induced by the bundle T^*N under the mapping $F : U \longrightarrow N$. As usually, we shall denote the total space of a bundle and the bundle itself with the same letter. According to the definition of the induced bundle we have

$$F^*(T^*N) = \left\{(\vartheta, u) \middle| u \in U,\ \vartheta \in T^*_{F(u)} N\right\} \subset T^*N \times U.$$

We identify the manifold U with the trivial section in the bundle $F^*(T^*N)$:

$$U = \left\{(0, u) \middle| u \in U\right\} \subset F^*(T^*N).$$

Then, $C_F \subset F^*(T^*N) \setminus U$.

Let $(\vartheta, u) \in F^*(T^*N)$, hence $\vartheta F'_u \in T^*U$.

Definition. The mapping F is called a *Morse mapping* if the mapping $(\vartheta, u) \mapsto \vartheta F'_u$, $(\vartheta, u) \in F^*(T^*N) \setminus U$ is transversal to the trivial section in T^*U. In other words, the mapping F is said to be a Morse mapping if the system of equations $\vartheta F'_u = 0$ is regular at $\vartheta \neq 0$.

For $N = \mathbb{R}$, i.e. in case F is a real–valued function, the condition that F is a Morse mapping is equivalent to the assertion that the Hessian of the function F is a nondegenerate quadratic form at every critical point.

If the mapping F is a Morse mapping then C_F is a smooth submanifold in $F^*(T^*N)$ for every N. Since the dimension of the total space of the bundle $F^*(T^*N)$ is equal to $dim\, U + dim\, N$ and the codimension of trivial section in T^*U is equal to $dim\, U$ we obtain

$$dim\, C_F = dim\, N.$$

From Thom's transversality theorem now easily follows

Proposition 1. . *For arbitrary manifolds U, N a generic mapping $F : U \longrightarrow N$ is a Morse mapping.*

Let $dim\, N = n$ and suppose the mapping $F : U \longrightarrow N$ is a Morse mapping. If $(\omega, u) \in C_F$ then $(t\omega, u) \in C_F$ $\forall t \in \mathbb{R} \setminus 0$. Hence the n–dimensional submanifold $C_F \subset F^*(T^*N)$ defines an $(n-1)$–dimensional submanifold $\mathbb{P}C_F$ in the projectivization

$$\mathbb{P}F^*(T^*N) = F^*(T^*N)/\left\{(\vartheta, u) \sim (t\vartheta, u), t \in \mathbb{R} \setminus 0\right\}$$

of the bundle $F^*(T^*N)$.

The projection $(\omega, u) \mapsto u$ maps the manifold C_F onto the set of critical points of F. The latter set is the main object of investigation in the singularity theory of smooth mappings. According to Thom–Bordman for every typical F the set of its critical points is a finite union of submanifolds in U of dimensions $\leq (n-1)$. At the same time, the set of critical points is not necessarily a smooth submanifold and has, in general, highly complicated singularities even for generic F. Proposition 1 implies that all these singularities are resolved by simply adjoining the Lagrange multipliers.

3. Let $F : U \longrightarrow N$ be a smooth mapping, $u \in U$. We shall define the notion of the second derivative of F, which is not quite trivial if we attempt to obtain an invariant notion, independent of the choice of local coordinates in U and N, and which should reflect the local structure of F "in the second order". For example, if u is regular for F then, according to the implicit function theorem, F is represented in some local coordinates as a linear mapping, and there is no sense to speak about the second derivative. To define the second derivative in a critical point it is appropriate to complement it, as above, by Lagrange multipliers.

Thus let $(\omega, u) \in C_F$, $v \in ker\, F'_u$, and suppose $\underline{\omega}$ and $\underline{v}$ are smooth sections of vector bundles $F^*(T^*N)$, T_*U respectively, such that $\underline{\omega}(u) = \omega$, $\underline{v}(u) = v$. Consider a smooth function $\underline{\omega}F'\underline{v} : \widehat{u} \mapsto \underline{\omega}(\widehat{u})F'_u\underline{v}(\widehat{u})$ on U. It is easily seen that the differential $(\underline{\omega}F'\underline{v})'_u$ of this function at u depends only on ω, v and does not depend on the sections $\underline{\omega}, \underline{v}$.

Definition. *The second derivative* of a smooth mapping F at a Lagrangian point (ω, u) is the linear mapping

$$\omega F''_u : ker\, F'_u \longrightarrow T^*_u U,$$

defined by the relation $\omega F''_u v = (\underline{\omega} F' \underline{v})'_u,\ v \in ker\, F'_u$. The *Hessian* of F at a Lagrangian point (ω, u) is defined to be the real–valued quadratic form ωF^h_u on $ker\, F'_u$, which is defined by the relation $\omega F^h_u(v) = (\omega F''_u v)v,\ v \in ker\, F'_u$.

It is easily seen that the bilinear form $(v_1, v_2) \mapsto (\omega F''_u v_1)v_2$ is symmetric, hence it is restored by the quadratic form ωF^h_u.

The following test for being a Morse mapping is established by a straightforward computation.

Proposition 2. *A smooth mapping F is a Morse mapping iff the linear mapping $\omega F''_u$ is injective at every Lagrangian point (ω, u).*

Suppose $(\omega, u) \in C_F$. If the Hessian ωF^h_u is a nondegenerate quadratic form then $\omega F''_u$ is injective. The opposite is not always true. For example, for the following Morse mapping

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \mapsto \begin{pmatrix} u^1 \\ u^1 u^2 + (u^2)^3 \end{pmatrix},\ u^i \in \mathbb{R},\ i = 1, 2,$$

for which the origin is a cusp point, the Hessian is equal to zero for $\omega = (0, 1),\ u^1 = u^2 = 0$.

4. Denote by

$$F_C : C_F \longrightarrow T^*N$$

the mapping $(\omega, u) \mapsto \omega,\ (\omega, u) \in C_F$. We remind that T^*N is a symplectic manifold.

Proposition 3. *If $F : U \longrightarrow N$ is a Morse mapping then F_C is an exact Lagrangian immersion.*

Proof. We use local coordinates for proof, and to simplify the exposition we shall suppose that U, N are vector spaces. Hence we can assume that $T^*U = U^* \times U,\ T^*N = N^* \times N,\ C_F = \left\{(\xi, u) \in N^* \times U \,\middle|\, \xi \frac{dF}{du} = 0\right\}$,

$$T_{(\xi,u)} C_F = \left\{(\eta, v) \in N^* \times U \,\middle|\, \eta \frac{dF}{du} + \frac{d}{du}(\xi \frac{dF}{du} v) = 0\right\}, \qquad (1)$$

$F_C : (\xi, u) \mapsto (\xi, F(u))$.

According to Proposition 2 the property of F to be a Morse mapping is equivalent to the relation $\frac{d}{du}(\xi \frac{dF}{du} v) \neq 0$ for $(\xi, u) \in C_F,\ v \in ker\, \frac{dF}{du}$.

Suppose that F_C is not an immersion at (ξ, u). Then $\exists\, (\eta, v) \in T_{(\xi,u)} C_F$ such that $\eta = 0,\ v \in ker\, \frac{dF}{du}$. The definition of $T_{(\xi,u)} C_F$ implies that $\frac{d}{du}(\xi \frac{dF}{du} v) = 0$. We come to the contradiction with the assumption that F is a Morse mapping.

It is left to show that the immersion F_C is Lagrangian. We have $s_N = \xi dy,\ (\xi, y) \in N^* \times N = T^*N$, hence $F^*_C s_N = \xi dF = 0$. Thus for a Morse mapping F the mapping $F_C : C_F \longrightarrow T^*N$ is an immersion. At the same time, the composition of F_C with the projection $\pi_N : T^* \longrightarrow N$ can not be an immersion since

$$\pi_N \circ F_C(\alpha\omega, u) = \pi_N(\alpha\omega) = \pi_N(\omega) \ \ \forall(\omega, u) \in C_F, \ \alpha \in \mathbb{R} \setminus 0.$$

We have proved that $\pi_N \circ F_C$ defines a smooth mapping of a $(dim\, N-1)$–dimensional manifold $\mathbb{P}F_C$ into N.

The following assertion formulates conditions under which the latter mapping is an immersion.

Proposition 4. *Let F be a Morse mapping and $(\omega, u) \in C_F$. The differential of the mapping $\pi_N \circ F_C$ has rank $dim\, N - 1$ at (ω, u) iff $rank\, F'_u = dim\, N - 1$ and the Hessian ωF^h_u of the mapping F at (ω, u) is a nondegenerate quadratic form.*

Proof. We use the notations from the proof of Proposition 3. We have

$$ker\, (\pi_N \circ F_C)'_{(\omega,u)} = (N^* \times ker\, F'_u) \cap T_{(\xi,u)}C_F, \tag{2}$$

where $\omega = (\xi, F(u))$, cf. (1). Let $v \in ker\, F'_u$. According to (1) the existence of a pair (η, v) contained in the space (2) is equivalent to the relation $v \in ker\, \omega F^h_u$. Moreover the pair $(\eta, 0)$ belongs to the space (2) iff $\eta \perp im\, F'_u$.

5. We return to the situation considered in n^o1, when $N = \mathbb{R} \times M$, $F(u) = (\varphi(u), f(u))$. Let (ω, u) be a Lagrangian point, $\omega \in T^*_{F(u)}N$, $u \in U$. Then $\omega = (\alpha, \lambda)$, $\alpha \in \mathbb{R}$, $\lambda \in T^*_{f(u)}M$, $0 = \omega F'_u = \lambda f'_u + \alpha\varphi'_u$. According to the usual terminology in the theory of extremal problems the Lagrangian point $((\alpha, \lambda), u)$ will be called *normal* if $\alpha \neq 0$, and will be called *abnormal* if $\alpha = 0$. Consider the set of normal Lagrangian points. Since the Lagrange multipliers $\omega = (\alpha, \lambda)$ are defined up to a nonzero multiplier we can normalize them by fixing the value of α arbitrarily. This procedure reduces the dimension and permits to consider the normal Lagrangian points as elements of the space $f^*(T^*M)$ rather then of $F^*(T^*N)$. We shall use the normalization $\alpha = -1$. Put

$$C_{f,\varphi} = \left\{ (\lambda, u) \in f^*(T^*M) \middle| \, u \in U, \ \lambda \in T^*_{f(u)}M, \ \lambda f'_u - \varphi'_u = 0 \right\},$$
$$f_C : (\lambda, u) \mapsto \lambda, \ (\lambda, u) \in C_{f,\varphi}.$$

The following assertions are easy modifications of the results of n^o4.

Proposition 5. *If $F = (\varphi, f)$ is a Morse mapping then $C_{f,\varphi}$ is a smooth submanifold in $f^*(T^*M)$, and $f_C : C_{f,\varphi} \longrightarrow T^*M$ is a Lagrangian immersion, where $f^*_C s_M = d\varphi$.*

Proposition 6. *Let $F = (\varphi, f)$ be a Morse mapping and $(\lambda, u) \in C_{f,\varphi}$. The differential of the mapping $\pi_M \circ f_C$ at (λ, u) is invertible iff f'_u is surjective and the Hessian of the mapping F at $((-1, \lambda), u)$ is a nondegenerate quadratic form.*

Let $(\lambda, u) \in C_{f,\varphi}$, $\omega = (-1, \lambda)$, and ωF^h_u be a Hessian of the mapping F at the Lagrangian point (ω, u). If the quadratic form ωF^h_u is negative definite (positive definite) then it is easily seen that u is a point of strong local minimum (maximum) of the function φ on the level set of the mapping f. On the other hand, if f'_u is surjective and the quadratic form ωF^h_u is indefinite then u is not a point of a local extremum of the function φ on the level set of the mapping F. Combining these assertions with Propositions 5,6 we come to the

Corollary. *Let $F = (\varphi, f)$ be a Morse mapping and W be a connected open subset in $C_{f,\varphi}$ such that $\pi_M \circ f_C\Big|_W$ is a diffeomorphism of W onto $f(W) \subset M$. If there exists a point $(\lambda_0, u_0) \in W$ such that u_0 is a point of a local minimum (maximum) of the function φ on the level set of f then $\forall (\lambda, u) \in W$ u is a point of strong local minimum (maximum) of φ on the level set of f. Furthermore,*

$$\varphi(u_1) - \varphi(u_0) = \int_\gamma s_M$$

for $\forall (\lambda_i, u_i) \in W,\ i = 0, 1$ and for every smooth curve $\gamma : [0,1] \longrightarrow f_C(W)$, satisfying the condition $\pi_M(\gamma(i)) = f(u_i),\ i = 0, 1$.

§3. The problem of optimal control

1. Among many similar formulations of the optimal control problem we choose the problem with the optimized integral functional, free time, and fixed end–points.

Let M be a smooth manifold and V a subset in $\mathbb{R}^r$. We call *admissible controls* locally bounded measurable mappings $v(\cdot) : \mathbb{R}_+ \longrightarrow V$, where $\mathbb{R}_+ = \left\{ t \in \mathbb{R} \middle| t \geq 0 \right\}$. Let $g^0 : M \times V \longrightarrow \mathbb{R}$ be a continuous function, smooth in the first variable, and let $g : M \times V \longrightarrow T_*M$ be a continuous mapping, smooth in the first variable , and subject to the condition $g(x, v) \in T_xM\ \forall x \in M,\ v \in V$. In particular, for every fixed v the mapping $x \mapsto g(x, v)$ is a smooth vector field on M. We consider an initial point $x_0 \in M$ which will not change in the sequel. For every admissible control $v(\cdot)$ there exists a unique absolutely continuous curve $\tau \mapsto x(\tau; v(\cdot))$ in M, defined on a half–interval $\tau \in [0, t_v)$, which satisfies the condition $x(0; v(\cdot)) = x_0$ and is a solution of the differential equation

$$\frac{dx}{d\tau} = g(x, v(\tau)). \tag{1}$$

Denote by $\mathcal{U}$ the set of pairs $(t, v(\cdot))$, such that $x(\tau; v(\cdot))$ is defined for $\tau \in [0, t]$. Define a function $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$ and a mapping $f : \mathcal{U} \longrightarrow M$ by relations

$$\varphi(t, v(\cdot)) = \int_0^t g^0(x(\tau; , v(\cdot)), v(\tau))d\tau, \quad f(t, v(\cdot)) = x(t; v(\cdot)).$$

Since the infinite–dimensional space $\mathcal{U}$ is not necessarily a smooth manifold we can not apply directly to φ and f the results of previous section. Nevertheless we have in the given situation a reasonable substitute of a smooth structure for $\mathcal{U}$, but to define it we have to consider generalized controls, cf. [9]. We shall not give here the appropriate definitions which would lead us too far aside. We only indicate that the role of the differential at $(t, v(\cdot))$ for f is taken by the mapping

$$f'_{(t,v)} : (\theta, w(\cdot)) \mapsto \int_0^t P^t_{\tau *}(g(x(\tau; v), v(\tau) + w(\tau)) - g(x(\tau; v), v(\tau)))d\tau + g(x(t; v))\theta,$$

where $(t+\theta, v+w) \in U$, (and assuming that t is a density point for $v(\cdot)$). Here P_τ^t is a diffeomorphism of a neighborhood of the point $x(\tau; v)$ on a neighborhood of $x(t; v)$ which is uniquely defined by the conditions that $P_\tau^\tau \equiv x$ and that the curves $\tau \mapsto P_{\tau'}^\tau(x)$, $\tau \in [\tau', t]$, are absolutely continuous solutions of the differential equation (1).

Let $F = (\varphi, f) : \mathcal{U} \longrightarrow \mathbb{R} \times M$. Substituting M by $\mathbb{R} \times M$, the point x_0 by $(0, x_0) \in \mathbb{R} \times M$, and the differential equation (1) by the equation

$$\frac{d}{dt}(x^0, x) = \left(g^0(x, v(\tau)), g(x, v(\tau))\right),$$

we obtain an explicit expression for $F'_{(t,v)} = (\varphi'_{(t,v)}, f'_{(t,v)})$. We call the point $(t, v) \in \mathcal{U}$ a *critical point* of F if the image of the mapping $F'_{(t,v)}$, which belongs to $\mathbb{R} \times T_{x(t,v)}M$, does not contain a neighborhood of the origin. It is easily seen that the image of $F'_{(t,v)}$ is convex, hence criticality of (t, v) is equivalent to the existence of a nonzero $\omega = (\alpha, \lambda) \in \mathbb{R} \times T^*_{x(t,v)}M$ such that $\omega F'_{(t,v)}(\theta, w) \le 0$ for $\forall(\theta, w)$ satisfying the condition $(t+\theta, v+w) \in \mathcal{U}$. We call ω the*Lagrange multiplier* and the triple $\omega, (t, v)$ a*Lagrange point for F*. If (t, v) is a critical point for F then (τ, v) is also a critical point for F for $\forall \tau \le t$. Put $\lambda_\tau = P_\tau^{t*}\lambda \in T^*_{x(\tau,v)}M$, then (α, λ_τ) is a Lagrange multiplier corresponding to the critical point (τ, v).

It is easily seen that the curve $\tau \mapsto \lambda_\tau$ in T^*M is a trajectory of a nonstationary Hamiltonian system on T^*M corresponding to a nonstationary Hamiltonian

$$h_\tau(\xi) = \xi g(\pi_M \xi, v(\tau)) + \alpha g^0(\pi_M \xi, v(\tau)), \ \xi \in T^*M, \ 0 \le \tau \le t. \tag{2}$$

An elementary calculation leads to the following

Proposition 1. *The system $(\alpha, \lambda), (t, v)$ is a Lagrangian point of the mapping $F = (\varphi, f)$ iff for this system the Pontryagin Maximum Principle holds:*

$$0 = h_\tau(\lambda_\tau) = \max_{u \in V}\left(\lambda_\tau g(x(\tau; v), u) + \alpha g^0(x(\tau; v), u)\right), \ 0 \le \tau \le t,$$

where h_τ is given by (2) and the curve $\tau \mapsto \lambda_\tau$ is a trajectory of the Hamiltonian system in T^*M defined by the nonstationary Hamiltonian h_τ, $\tau \in [0, t]$, with the boundary condition $\lambda_t = \lambda$.

2. A Lagrangian point $(\alpha, \lambda), (t, v)$ is called *normal* if $\alpha \ne 0$ and *abnormal* in the opposite case. As in §2 we normalize the Lagrangian multipliers for normal points by putting $\alpha = -1$. Let

$$C_{f,\varphi} = \left\{(\lambda, t, v) \middle| \, (-1, \lambda), (t, v) \text{ is a Lagrangian point of the mapping } (\varphi, f)\right\},$$
$$f_C : (\lambda, t, v) \mapsto \lambda, \ (\lambda, t, v) \in C_{f,\varphi}.$$

Put

$$H(\lambda) = \max_{u \in V}(\lambda g(\pi_M \lambda, u) - g^0(\pi_M \lambda, u)), \ \lambda \in T^*M.$$

Proposition 1 implies $im\, f_C \subset H^{-1}(0)$.

Let $(\lambda, t, v) \in C_{f,\varphi}$, then there exists an absolutely continuous curve $\lambda_\tau \in T^*M,\ 0 \le \tau \le t$, such that $(\lambda_\tau, \tau, v) \in C_{f,\varphi}\ \forall\tau,\ \lambda_t = \lambda$.

Without any regularity conditions it is difficult to expect that $C_{f,\varphi}$ could be provided with the structure of a Lagrangian manifold, and that f_C is a Lagrangian immersion. Still it turns out that if a Lagrangian manifold is contained in $H^{-1}(0)$ and contains the curve $\lambda_\tau,\ 0 \le \tau \le t$, we can make some essential conclusions about the optimality of the control $v(\cdot)$ independently from any assumptions about the analytic nature of $C_{f,\varphi}$ and f_C.

Theorem 1. *Let $\mathcal{L} \subset H^{-1}(0)$ be an exact Lagrangian Lipschitz submanifold in T^*M such that the preimage of an arbitrary point in $\pi_M(\mathcal{L})$ under the mapping $\pi_M\Big|_{\mathcal{L}} : \mathcal{L} \longrightarrow M$ is a connected Lipschitz complex in $\mathcal{L}$, and the preimage of an arbitrary Lipschitz curve in M under the mapping $\pi_M\Big|_{\mathcal{L}}$ is a Lipschitz complex in $\mathcal{L}$. Let $\lambda\tau \in \mathcal{L},\ 0 \le \tau \le t$, be an absolutely continuous curve such that $(\lambda_\tau, \tau, v) \in C_{f,\varphi}$ for some admissible control $v(\cdot)$. Then for $\forall(\widehat{t}, \widehat{v}(\cdot)) \in \mathcal{U}$ such that $x(\tau; \widehat{v}) \in \pi_M(\mathcal{L}),\ g(x(\tau; v), v(\tau)) \ne 0$ for $0 \le \tau \le \widehat{t},\ \widehat{x}(\widehat{t}; \widehat{v}) = x(t; v)$, the inequality*

$$\int_0^t g^0(x(\tau; v), v(\tau))d\tau \le \int_0^{\hat{t}} g^0(x(\tau; \widehat{v}), \widehat{v}(t))d\tau$$

holds.

Proof. We denote $x(\tau) = x(\tau; v(\tau)),\ \widehat{x}(\tau) = x(\tau; \widehat{v}(\tau))$. Under the assumptions of Theorem 1 one can show that there exists a Lipschitz curve $\widehat{\lambda}_\tau \in \mathcal{L}$ and a nondecreasing Lipschitz function $\theta(\tau),\ o \le \tau \le 1$, such that the following conditions are satisfied

(1) $\widehat{\lambda}_0 = \lambda_0,\ \widehat{\lambda}_1 = \lambda_1$;
(2) $\theta(0) = 0,\ \theta(1) = \widehat{t}$;
(3) $\pi_M(\widehat{\lambda}_\tau) = \widehat{x}(\theta(\tau)),\ 0 \le \tau \le 1$.

We obtain

$$\int_0^{\hat{t}} g^0(\widehat{x}(\theta), \widehat{v}(\theta))d\theta = \int_0^1 \dot{\theta} g^0(\widehat{x}(\theta(\tau)), \widehat{v}(\theta(\tau))d\tau \ge$$

$$\ge \int_0^1 \dot{\theta}\widehat{\lambda}_\tau g(\widehat{x}(\theta(\tau)),\ \widehat{v}(\theta(\tau))d\tau = \int_0^1 \widehat{\lambda}_\tau \frac{d}{d\tau}\widehat{x}(\theta(\tau))d\tau = \int_\gamma s_M,$$

where $\widehat{\gamma}$ denotes the curve $\tau \mapsto \widehat{\lambda}_\tau$. (We used here the relations $0 = H(\widehat{\lambda}_\tau) = \max_{u \in V}(\widehat{\lambda}_\tau g(x(\theta(\tau)), u) - g^0(x(\theta(\tau)), u)))$. Denote by γ the curve $\tau \mapsto \lambda_\tau$. Since the form s_M is exact on $\mathcal{L}$ we have

$$\int_\gamma s_M = \int_\gamma s_M = \int_0^t \lambda_\tau \dot{x}(\tau)d\tau = \int_0^t \lambda_\tau g(x(\tau), v(\tau))d\tau = \int_0^t g^0(x(\tau), v(\tau))d\tau.$$

Remark. The exact Lagrangian submanifold $\mathcal{L}$ evidently satisfies conditions of Proposition 2 if the projection $\pi_M\Big|_{\mathcal{L}}$ is Lipschitz invertible, fig.1; "vertical pieces" are also permitted, fig.2, but "folds" are not allowed, fig.3.

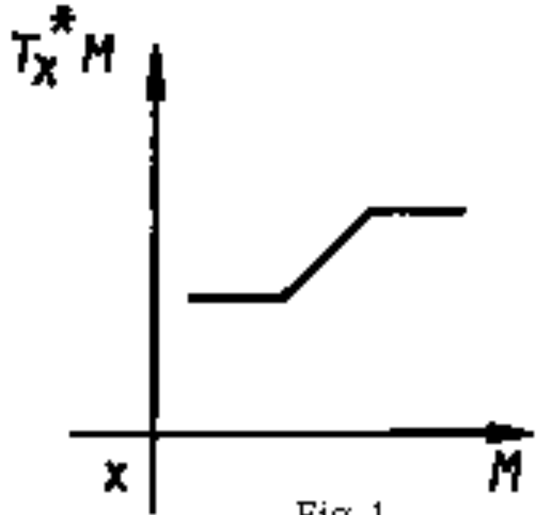

Fig. 1

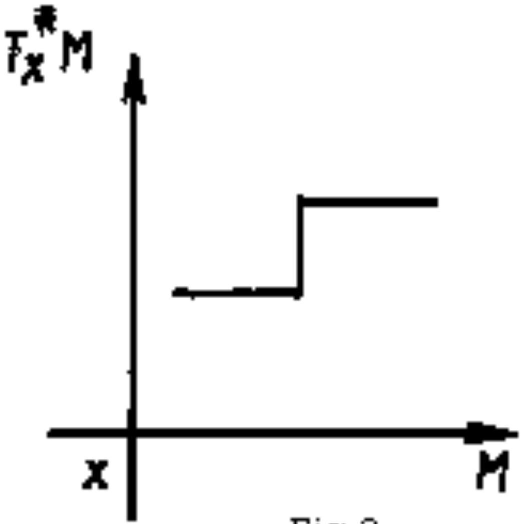

Fig.2

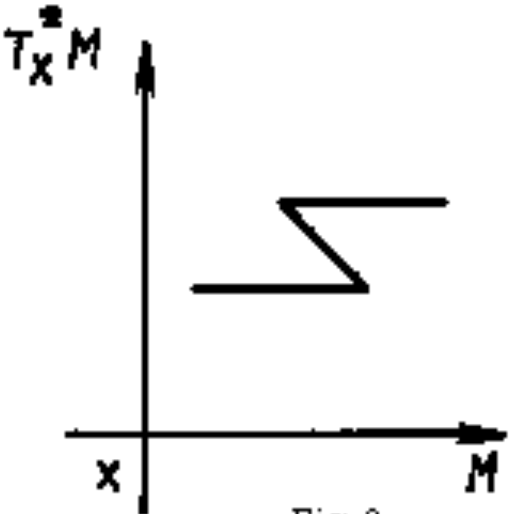

Fig.3

3. Let $(\lambda, t, v) \in C_{f,\varphi}$ and $\lambda_\tau,\ 0 \le \tau \le t$, be a trajectory of the Hamiltonian system defined by the nonstationary Hamiltonian

$$h_\tau(\xi) = \xi g(\pi(\xi), v(\tau)) - g^0(\pi(\xi), v(\tau)),\ \xi \in T^*M,$$

with the boundary condition $\lambda_t = \lambda$. We call the curve λ_τ a *normal Pontryagin extremal* corresponding to the control $v(\cdot)$. According to Proposition 1 $(\lambda_\tau, \tau, v) \in C_{f,\varphi}$ for $0 \le \tau \le t$.

The simplest case when a Lagrangian manifold can be constructed, which contains a given Pontryagin extremal λ_τ, is given when the Hamiltonian

$$H(\xi) = \underset{u \in V}{max}(\xi g(\pi(\xi), u) - g^0(\pi(\xi), u)),\ \xi \in T^*M \tag{3}$$

is smooth. It is easy to show that in this case λ_tau is a trajectory of the Hamiltonian system defined by the Hamiltonian H.

The following assertion is a geometric formulation of the classical method of characteristics for solution of differential equations in partial derivatives of first order.

Proposition 2. *Let $H : T^*M \longrightarrow \mathbb{R}$ be a smooth function and $\mathcal{L}_0 \subset T^*M$ be a smooth Lagrangian submanifold. Suppose that $H'_\lambda\Big| T_\lambda\mathcal{L}_0 \neq 0\ \forall\lambda \in \mathcal{L}_0 \cap H^{-1}(0)$. Let $t \mapsto p(t,\lambda)$ be a trajectory of the Hamiltonian system with Hamiltonian H and the initial condition $p(0,\lambda) = \lambda \in \mathcal{L}_0 \cap H^{-1}(0)$. Then the mapping p, which is defined on an open set in $\mathbb{R} \times (\mathcal{L}_0 \cap H^{-1}(0))$ and with values in $H^{-1}(0)$, is a Lagrangian immersion.*

Proof. Condition $H'_\lambda\Big| T_\lambda\mathcal{L}_0$ is equivalent to the statement that $\overrightarrow{H}(\lambda)$ is not skeworthogonal to $T_\lambda\mathcal{L}_0$ in the symplectic space $T_\lambda(T^*M)$. Since $\mathcal{L}_0$ is a Lagrangian manifold the last statement is equivalent to $\overrightarrow{H}(\lambda)$ not being tangent to $\mathcal{L}_0$. Hence p is indeed an immersion. Since the Hamiltonian flow preserves the symplectic structure it is sufficient to check that the immersion $(t,\lambda) \mapsto p(t,\lambda)$ is Lagrangian only for $t = 0$. We have

$$\frac{\partial}{\partial t}p(t,\lambda)\Big|_{t=0} = \overrightarrow{H}(\lambda),\ \sigma(\overrightarrow{H}(\lambda),\vartheta_1) = H'_\lambda\vartheta_1 = 0,$$
$$\sigma(\vartheta_1,\vartheta_2) = 0\ \forall\vartheta_1,\vartheta_2 \in T_\lambda(\mathcal{L}_0 \cap H^{-1}(0)).$$

Proposition 2 can be used to construct a Lagrangian manifold from Theorem 1 if we take for $\mathcal{L}_0$ a Lagrangian submanifold in $T^*_{x_0}M$, or more generally, if we take a Lagrangian submanifold

$$\left\{\lambda \in T^*_xM \,\Big|\, x \in X,\ \lambda \perp T_xX\right\}, \tag{4}$$

where X is a smooth submanifold in M. The submanifold (4) is often used in problems of optimal control with a variable left end–point $x(0) \in X$, and an obvious generalization of Theorem 1 is valid for this case. For nonsmooth Hamiltonians (3) there might not exist even a nonsmooth flow consisting of Pontryagin extremals — the extremals are inevitably intersecting and branching near any singular extremal. Therefore it is important to emphasize that to apply Theorem 1 it is not at all necessary to have a Hamiltonian flow of extremals: the corresponding Lagrangian manifold can be constructed in a different way. It is often possible to construct an auxiliary Morse mapping, cf. §2, for which the Lagrange multipliers coincide with the Lagrange multipliers of the optimal problem in consideration and hence constitute a Lagrangian manifold in the level set of the Hamiltonian (3) corresponding to the value 0. We shall return to this problem later.

After the appropriate Lagrangian submanifold $\mathcal{L} \subset T^*M$ is constructed we have to examine its projection on M. The local properties of the projection $\pi\Big|_{\mathcal{L}}$ are defined by a mutual disposition of the Lagrangian subspaces $T_\lambda\mathcal{L}$ and $T_\lambda(T^*_{\pi(\lambda)}M)$ in $T_\lambda(T^*M)$, $\lambda \in \mathcal{L}$.

The next section contains some basic information about the manifold of Lagrangian subspaces of a symplectic space. This information turns out to be useful not only for investigation of the projections $\pi\Big|_{\mathcal{L}}$, but in many other cases as well.

§4. Geometry of Lagrange Grassmanians

1. Let Σ be a symplectic vector space with a symplectic form σ with $dim\,\Sigma = 2n$. We denote by $L(\Sigma)$ the set of all Lagrangian subspaces in Σ and call it the Lagrange Grassmannian. The subspaces $\Lambda_0, \Lambda_1 \in L(\Sigma)$ are transversal iff $\Lambda_0 \cap \Lambda_1 = 0$. The set of all Lagrangian subspaces transversal to a given $\Lambda_0 \in L(\Sigma)$ will be denoted by $\Lambda_0^{\pitchfork}$.

If $p : \Sigma \longrightarrow \Sigma$ is a linear symplectic transformation and $\Lambda \subset \Sigma$ is a Lagrangian subspace then $p\Lambda$ is also Lagrangian. Thus the symplectic group of all symplectic transformations acts naturally on $L(\Sigma)$. This action automatically defines an action of $L(\Sigma)$ on the sequences of k points from $L(\Sigma)$:

$$p : (\Lambda_1, \dots, \Lambda_k) \mapsto (p\Lambda_1, \dots, p\Lambda_k),\ p \in Sp(\Sigma),\ \Lambda_i \in L(\Sigma),\ i = 1, \dots, k,\ k \geq 1.$$

We have $dim\,(p\Lambda_i \cap p\Lambda_j) = dim\,(\Lambda_i \cap \Lambda_j)$.

Proposition 1. *The group $Sp(\Sigma)$ acts transitively on the set*

$$\left\{(\Lambda_0, \Lambda_1)\Big|\, \Lambda_0, \Lambda_1 \in Sp(\Sigma),\ \Lambda_o \cap \Lambda_1 = 0\right\}$$

of pairs of transversal Lagrangian subspaces. The restriction $p \mapsto p\Big|_{\Lambda_0}$ defines an isomorphism of stable subgroups $\left\{p \in Sp(\Sigma)\Big|\, p\Lambda_i = \Lambda_i,\ i = 0, 1\right\}$ on $GL(\Lambda_0)$.

Proof. The bilinear form

$$(\lambda_1, \lambda_0) \mapsto \sigma(\lambda_1, \lambda_0), \lambda_i \in \Lambda_i,\ i = 0, 1, \tag{1}$$

defines a nondegenerate pairing of spaces Λ_0 and Λ_1. Let $e_1, \dots, e_n$ be a basis in Λ_1 and $f_1, \dots, f_n$ the dual basis in Λ_0. Then $e_1, \dots, e_n, f_1, \dots, f_n$ is a canonical basis in Σ. Proposition 1 follows from the assertion that a linear transformation in Σ is symplectic iff it maps a canonical basis into a canonical basis.

Proposition 1 implies that $L(\Sigma)$ is a homogeneous space of the group $Sp(\Sigma)$; in particular, $L(\Sigma)$ has the structure of a C^ω–manifold. We shall indicate a standard family of coordinate neighborhoods on $L(\Sigma)$ defined by pairs of transversal Lagrangian subspaces.

Let $\Lambda_0, \Lambda_1 \in L(\Sigma)$, $\Lambda_0 \cap \Lambda_1 = 0$. The pairing (1) defines an isomorphism $\Lambda_0 \approx \Lambda_1^*$. Correspondingly $\Sigma = \Lambda_o \oplus \Lambda_1 \approx \Lambda_1^* \oplus \Lambda_1$. The indicated isomorphism identifies σ with the standard symplectic form

$$\begin{aligned} ((\xi_1, x_1), (\xi_2, x_2)) \mapsto \xi_2 x_1 - \xi_1 x_2,& \\ x_i \in \Lambda_1,\ \xi_i \in \Lambda_1^*,\ i = 1, 2.& \end{aligned} \tag{2}$$

The form (2) will be also denoted by σ.

Every n–dimensional subspace $H \subset \Lambda_1^* \oplus \Lambda_1 \equiv \Sigma$ transversal to Λ_1^*, (in particular, every subspace sufficiently close to Λ_1), is a graph of a linear mapping $Q_H : \Lambda_1 \longrightarrow \Lambda_1^*$:

$$H = \left\{(\xi, x) \in \Lambda_1^* \oplus \Lambda_1 \Big|\, \xi = Q_H x\right\}.$$

It is easy to show that H is a Lagrangian subspace in $\Lambda_1^* \oplus \Lambda_1$ iff $Q_H^* = Q_H$.

Let $I^0 : \Sigma \longrightarrow \Lambda_1^* \oplus \Lambda_1$ be the isomorphism of symplectic spaces defined above, which sends Λ_0 into Λ_1^*. Then

$$\Lambda \mapsto Q_{I^0\Lambda}, \quad \Lambda \in \Lambda_0^{\pitchfork} \tag{3}$$

is a diffeomorphism of $\Lambda_0^{\pitchfork}$ onto the space of selfadjoint mappings of Λ_1 into Λ_1^*¿ Thus $\Lambda_0^{\pitchfork}$ is a coordinate neighborhood in $L(\Sigma)$, and the mapping (3) defines local coordinates. Selfadjoint transformations from Λ_1 into Λ_1^* are actually equivalent to quadratic forms on Λ_1: to a mapping $Q : \Lambda_1 \longrightarrow \Lambda_1^*$ there corresponds the form $q : x \mapsto (Qx)x,\ x \in \Lambda_1$. The space of all real–valued quadratic forms on Λ_1 is denoted $\mathcal{P}(\Lambda_1)$. We have

$$\Lambda_0^{\pitchfork} \equiv \mathcal{P}(\Lambda_1),\ dim\, L(\Sigma) = dim\, \mathcal{P}(\Lambda_1) = \frac{n(n+1).}{2}$$

It is easily seen that $L(\Sigma) = \overline{\Lambda_0^{\pitchfork}}$. The manifold $L(\Sigma)$ is compact. Taking into account the isomorphism $\Lambda_0^{\pitchfork} \equiv \mathcal{P}(\Lambda_1)$ we can say that $L(\Sigma)$ is a compactification of the space of quadratic forms of n real variables. We shall now show that the geometry of the Lagrange Grassmannian considered as a homogeneous space of the group $Sp(\Sigma)$ is intimately connected with the geometry of the space of quadratic forms, which explains the effectiveness of symplectic methods in many problems where we have to deal with families of quadratic forms.

Consider a subgroup in $Sp(\Sigma)$ which preserves $\Lambda_0^{\pitchfork}$. A linear transformation of the space Σ preserves $\Lambda_0^{\pitchfork}$ iff it preserves Λ_0. The isomorphism I^0 permits to consider correspondingly $L(\Lambda_1^* \oplus \Lambda_1)$ and Λ_1^* instead of $L(\Sigma)$, and Λ_0. A linear transformation of $\Lambda_1^* \oplus \Lambda_1$ is symplectic and preserving Λ_1^* iff it is represented as

$$\begin{gathered}(\xi, x) \mapsto (A^*\xi + BA^{-1}x,\ A^{-1}x) \\ \xi \in \Lambda_1^*,\, x \in \Lambda_1,\ A \in GL(\Lambda_1),\ B : \Lambda_1 \longrightarrow \Lambda_1^*,\ B^* = B.\end{gathered} \tag{4}$$

The mapping (4) transforms the subspace $\left\{(\xi, x) \middle| \xi = Qx\right\} \subset \Lambda_1^* \oplus \Lambda_1$ into subspace $\left\{(\xi, x) \middle| \xi = (A^*QA + B)x\right\}$. In other words, the Lagrangian subspace corresponding to the quadratic form $q(x) = (Qx)x,\ x \in \Lambda_1$ is transformed into the subspace corresponding to the quadratic form $q(Ax) + (Bx)x$, obtained from q by coordinate transformation and translation. Thus the symplectic transformations which preserve $\Lambda_0^{\pitchfork} \equiv \mathcal{P}(\Lambda_1)$ are "variable substitutions" and translations in the space of quadratic forms of n variables. It turns out that the group $Sp(\Sigma)$ is generated already by translations and a single transformation which interchanges the subspaces Λ_0 and Λ_1.

Suppose that a scalar product $(\cdot|\cdot)$ is defined on Λ_1 which identifies Λ_1^* with Λ_1 and $\Lambda_1^* \oplus \Lambda_1$ with $\Lambda_1 \oplus \Lambda_1$, so that

$$\sigma((y_1, x_1)|(y_2, x_2)) = (x_1|y_2) - (x_2|y_1),\ x_i, y_i\Lambda_1. \tag{5}$$

Proposition 2. *The symplectic group of the space $\Lambda_1 \oplus \Lambda_1 \equiv \Sigma$ with the symplectic form (5) is generated by transformations*

$$(y,x) \mapsto (y + Bx, x), \ b \in gl(\Lambda_1), \ B = B^*,$$

and the transformation

$$J : (y,x) \mapsto (-x, y).$$

Suppose $Q : \Lambda_1 \longrightarrow \Lambda_1, \ Q = Q^*$. The intersection of the Lagrangian subspace

$$\left\{(y,x) \middle| \, y = Qx\right\} \subset \Lambda_1 \oplus \Lambda_1 \tag{6}$$

with $0 \oplus \Lambda_1$ coincides with the subspace $0 \oplus ker\, Q$. In particular, the subspace (6) is not transversal to $0 \oplus \Lambda_1$ iff Q is degenerate. If it is nondegenerate then the symplectic mapping J transforms the subspace (6) into the subspace $\left\{(y,x) \middle| \, y = -Q^{-1}x\right\}$, corresponding to the operator $-Q^{-1}$. But, contrary to the operation of matrix inversion, J is defined for degenerate Q as well. Speaking not quite formally, we can say that the Lagrange Grassmannian of a $2n$–dimensional symplectic space is a compactification of the space of symmetric $n \times n$–matrices (the space of quadratic forms of n variables)for which the inversion of all symmetric matrices, including the degenerate matrices, is possible: the cone of the "ideal matrices", which are inverses to degenerate matrices, is "attached at the infinity" to the space of symmetric matrices.

Repeating in the opposite order the identifications of the symplectic spaces we have made, we obtain

$$\Lambda_0^{\pitchfork} \setminus \Lambda_1^{\pitchfork} \approx \left\{q \in \mathcal{P}(\Lambda_1) \middle| \, ker\, q \neq 0\right\}. \tag{7}$$

2. Let $\wedge^k(\Sigma)$ be the subspace of k–linear skew–symmetric forms on $\Sigma, \ k = 0, 1, \dots,$ $2n$; $\sigma \in \wedge^2(\Sigma)$. If $\omega \in \wedge^k(\Sigma)$, then $ker\, \omega \stackrel{\text{def}}{=} \left\{x \in \Sigma \middle| \, x \rfloor \omega = 0\right\}$. It is easy to show that $dim\, ker\, \omega \leq 2n - k$ for every nonzero form $\omega \in \wedge^k(\Sigma)$. The form ω is called *decomposable* if $dim\, ker\, \omega = 2n - k$. A decomposable form is restored by its kernel up to a nonzero scalar multiplier.

Let $\mathbb{P}\wedge^k(\Sigma) = (\wedge^k(\Sigma) \setminus 0)/\{\omega \equiv \alpha\omega, \omega \in \Sigma, \alpha \in \mathbb{R}, \alpha\omega \neq 0\}$ be a projectivization of the space $\wedge^k(\Sigma)$, and denote by $\overline{\omega}$ the image of a nonzero form ω under the canonical factorization $\wedge^k(\Sigma) \setminus 0 \longrightarrow \mathbb{P} \wedge^k(\Sigma)$. The mapping

$$ker\, \omega \mapsto \overline{\omega}, \ \omega \text{ is decomposable in } \wedge^k(\Sigma), \tag{8}$$

is a standard projective Plücker imbedding of the Grassmannian of $(2n - k)$–dimensional subspaces in Σ. It turns out that for $k = n$ the image of the Lagrange Grassmannian under the imbedding (8) is an intersection of the image of the standard Grassmannian with a projective subspace in $\mathbb{P} \wedge^k(\Sigma)$. Namely, the following assertion is valid.

Proposition 3. *Let $\omega \in \wedge^n(\Sigma)$ be a decomposable form. Then*

$$ker\, \omega \in L(\Sigma) \iff \sigma \wedge \omega = 0.$$

We consider in more detail the Lagrange Grassmannians for $n = 1, 2$. Every one–dimensional subspace in $\mathbb{R}^2$ is Lagrangian, hence $L(\mathbb{R}^2) = \mathbb{RP}^1$. Furthermore $Sp(\mathbb{R}^2 = SL(\mathbb{R}^2)$, therefore symplectic transformations coincide with the orientation preserving projective transformations. Thus $L(\mathbb{R}^2)$ is an oriented projective line, topologically — an oriented circle.

To describe $L(\mathbb{R}^4)$ we use the Plücker imbedding. The form $\omega \in \wedge^2(\mathbb{R}^4)$ is decomposable iff

$$\omega \wedge \omega = 0. \tag{9}$$

The equation (8) defines a quadric if signature $(+++---)$ in $\mathbb{P} \wedge^2 (\mathbb{R}^4) = \mathbb{RP}^5$. To obtain the image of $L(\mathbb{R}^4)$ under the Plücker imbedding we must intersect the quadric with the hyperplane, defined by the equation $\sigma \wedge \omega = 0$. As a result we obtain a quadric of signature $(+++--)$ in $\mathbb{RP}^4$. Topologically this quadric, and hence also $L(\mathbb{R}^4)$, are represented as a quotient of $S^1 \times S^2$ relative to the equivalence relation $(z_1, z_2) \approx (-z_1, -z_2)$, $z_i \in S^i$, $i = 1, 2$. This is a three–dimensional nonorientable manifold, cf. [10], where two "proves" are given of the relation $L(\mathbb{R}^4) \approx S^1 \times S^2$.

3. Proposition 1 asserts that the symplectic group acts transitively on pairs of transversal Lagrangian subspaces. Now we shall consider the action of this group on triples of Lagrangian subspaces.

Let $\Lambda_i \in L(\Sigma)$, $i = 0, 1$, where $\Lambda_0 \cap \Lambda_1 = 0$. The isomorphism introduced in $n^o 1$, $I_0 : \Sigma \longrightarrow \Lambda_1^* \oplus \Lambda_1$, transforms Λ_0 into Λ_1^*, hence for $\forall \Lambda \in \Lambda_0^{\pitchfork}$ we have a selfadjoint linear mapping of Λ_1 into Λ_1^*:

$$I^0 \Lambda = \left\{ (\xi, x) \in \Lambda_1^* \oplus \Lambda_1 \middle| \, \xi = Q_{I^0 \Lambda} x \right\}.$$

The subgroup in $Sp(\Lambda_1^* \oplus \Lambda)$ which preserves Λ_1 and Λ_1^* consists of transformations of the form

$$(\xi, x) \mapsto (A^* \xi, A^{-1} x), \ A \in GL(\Lambda_1).$$

The Lagrangian subspace defined by the equation $\xi = Qx$ is transformed by this mapping into the subspace defined by the equation $\xi = A^* Q A x$. We correspond to each $\Lambda \in \Lambda_0^{\pitchfork}$ the quadratic form

$$q_{I^0 \Lambda}(x) = (Q_{I^0 \Lambda} x) x, \ x \in \Lambda_1.$$

The given considerations imply that the existence of a symplectic transformation which carries the triple of Lagrangian subspaces $(\Lambda_0, \Lambda_1, \Lambda)$ into the triple $(\Lambda_0, \Lambda_1, \Lambda')$ is equivalent to the existence of a linear change of coordinates transforming the form $q_{I^0 \Lambda}$ into the form $q_{I^0 \Lambda'}$.

If q is a real–valued quadratic form on a vector space E we denote by $sgn\, q$ the difference between the number of positive and negative squares in the diagonal form of q:

$$sgn\, q = max \left\{ dim\, H \middle| \, q|_H > 0, \ H \subset E \right\} - max \left\{ dim\, H \middle| q|_H < 0, \ H \subset E \right\}.$$

Hence the forms q and q' are transformed into each other by a linear change of variables iff $sgn\, q = sgn\, q'$, $dim\, ker\, q = dim\, ker\, q'$. Note that $ker\, q_{I^0 \Lambda} = \Lambda \cap \Lambda_1$.

Denote

$$\mu(\Lambda_0, \Lambda_1, \Lambda) = sgn\, q_{I_0 \Lambda}.$$

The number $\mu(\Lambda_0, \Lambda_1, \Lambda)$ is called the *Maslov index* of the triple of the Lagrangian subspaces $\Lambda_0, \Lambda_1, \Lambda$. We shall give now another, more invariant definition of the Maslov index, which does not presupposes the assumption about the transversality of the subspaces Λ and Λ_1 to Λ_0.

Let $\Lambda_i \in L(\Sigma)$, $i = 0, 1, 2$. Define a quadratic form q on $\Lambda_1 \cap (\Lambda_0 + \Lambda_2)$ by relations

$$q(\lambda_1) = \sigma(\lambda_0, \lambda_2),\ \lambda_1 = \lambda_0 + \lambda_2,\ \lambda_i \in \Lambda_i,\ i = 0, 1, 2.$$

The vector $\lambda_1 \in \Lambda_1 \cap (\Lambda_0 + \Lambda_2)$ is in general represented as a sum of vectors from $\Lambda_0 + \Lambda_2$ not uniquely, but the skew–scalar product of these vectors depends only of λ_1, which is a direct consequence of the isotropy of Λ_0, Λ_2. The Maslov index of the triple $\Lambda_0, \Lambda_1, \Lambda_2$ is called the number

$$\mu(\Lambda_0, \Lambda_1, \Lambda_2) = sgn\, q.$$

It is easily seen that $q = q_{I^0 \Lambda_1}$ if $\Lambda_2 \cap \Lambda_0 = \Lambda_1 \cap \Lambda_0 = 0$. The given definition of the Maslov index uses only the symplectic structure of the space, therefore the index is a symplectic invariant of the triple of Lagrangian subspaces. We can give still another definition of the Maslov index, in which the subspaces Λ_i, $i = 0, 1, 2$, enter symmetrically.

Define on the $3n$–dimensional space $\Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$ a quadratic form $\widehat{q}$ by the relation

$$\widehat{q}(\lambda_0, \lambda_1, \lambda_2) = \sigma(\lambda_0, \lambda_1) - \sigma(\lambda_0, \lambda_2) + \sigma(\lambda_1, \lambda_2).$$

One can show, cf. [11], that $\mu(\Lambda_0, \Lambda_1, \Lambda_2) = sgn\, \widehat{q}$.

The last representation implies the skew–symmetry of the Maslov index in all three variables:

$$\mu(\Lambda_0, \Lambda_1, \Lambda_2) = -\mu(\Lambda_1, \Lambda_0, \Lambda_2) = -\mu(\Lambda_0, \Lambda_2, \Lambda_1).$$

Somewhat more difficult is to prove the following identity — the chain rule, cf.[11]:

$$\begin{gathered} \mu(\Lambda_0, \Lambda_1, \Lambda_2) - \mu(\Lambda_0,\Lambda_1, \Lambda_3) + \mu(\Lambda_0, \Lambda_2, \Lambda_3) - \mu(\Lambda_1, \Lambda_2\, \Lambda_3) = 0, \\ \forall \Lambda_i \in L(\Sigma),\ i = 0, 1, 2, 3. \end{gathered} \tag{10}$$

Except the Maslov index there are the following trivial invariants of the triple $\Lambda_0, \Lambda_1, \Lambda_2$:

$$dim(\Lambda_i \cap \Lambda_j),\ 0 \le i < j \le 2,\ dim(\bigcap_{i=0}^{2} \Lambda_i).$$

We have proved that there are no other invariants if $\Lambda_0 \cap \Lambda_1 = \Lambda_0 \cap \Lambda_2 = 0$. It turns out that this is true without any assumptions.

Proposition 4. *A triple of Lagrangian subspaces $(\Lambda_0, \Lambda_1, \Lambda_2)$ can be transformed into the triple $(\Lambda'_0, \Lambda'_1, \Lambda'_2)$ by a symplectic mapping iff*

$$\mu(\Lambda_0, \Lambda_1, \Lambda_2) = \mu(\Lambda'_0, \Lambda'_1, \Lambda'_2),\ dim(\Lambda_i \cap \Lambda_j) = dim(\Lambda'_i \cap \Lambda'_j),$$
$$0 \le i < j \le 2,\ dim(\overset{2}{\underset{i=0}{\cap}} \Lambda_i) = dim(\overset{2}{\underset{i=0}{\cap}} \Lambda'_i).$$

Consider the case $n = 1$ in more detail. Since $L(\mathbb{R}^2)$ is an oriented circle the Maslov index is an invariant of a triple of points on the oriented circle. If two of the three points coincide the index is zero. Let s_0, s_1, s_2 be three different points located on S^1 in such an order that if we move along S^1 in the positive direction we pass consecutively through the points $s_{i_1}, s_{i_2}, s_{i_3}$. The pairity of the substitution (i_1, i_2, i_3) does not depend on the choice of the initial state of the movement. The Maslov index $\mu(s_1, s_2, s_3)$ is equal to 1 if this substitution is even and is equal to -1 if it is odd.

4. We consider in more detail the tangent spaces $T_\Lambda L(\Sigma)$, $\Lambda \in L(\Sigma)$. To every quadratic form $h \in \mathcal{P}(\Sigma)$ there corresponds a linear Hamiltonian field $\vec{h}$ and a one–parameter subgroup $t \mapsto e^{t\vec{h}}$ in $Sp(\Sigma)$. Consider the linear mapping

$$h \mapsto \frac{d}{dt} e^{t\vec{h}} \Lambda \Big|_{t=0},\ h \in \mathcal{P}(\Sigma) \tag{11}$$

of the space of quadratic forms to $T_\Lambda L(\Sigma)$. The set of all linear Hamiltonian fields is a Lie algebra of the group $Sp(\Sigma)$. At the same time the action of the group $Sp(\Sigma)$ on $L(\Sigma)$ is transitive. Hence the mapping (11) is surjective. This mapping is certainly not invertible since $dim\, \mathcal{P}(\Sigma) = n(2n + 1)$ and $dim\, L(\Sigma) = \frac{n(n+1)}{2}$. It is easy to show that the kernel of the mapping (10) consists of all quadratic forms which vanish on Λ. Thus to two different forms from $\mathcal{P}(\Sigma)$ correspond equal vectors from $T_\Lambda L(\Sigma)$ iff the restrictions of these forms on Λ coincide. We obtained a natural identification of the space $T_\Lambda L(\Sigma)$ with the space $\mathcal{P}(\Lambda)$ of the quadratic forms on Λ.

The correspondence $T_\Lambda L(\Sigma) \longrightarrow \mathcal{P}(\Lambda)$ could be described more explicitly, without considering quadratic forms on Σ. Suppose $\Lambda(t)$ is a smooth curve in $L(\Sigma)$, $\Lambda(0) = \Lambda$. We correspond to every smooth curve $\lambda_t \in \Lambda(t)$ the number $\frac{1}{2}\sigma(\frac{d\lambda_t}{dt}\Big|_{t=0}, \lambda_0)$. The isotropy of the spaces $\Lambda(t)$ imply that this number depends only on λ_0, $\frac{d\Lambda(t)}{dt}\Big|_{t=0}$. In other words, to the tangent vector $\frac{d\Lambda(t)}{dt}\Big|_{t=0} \in T_\Lambda L(\Sigma)$ there corresponds the quadratic form

$$\lambda_0 \mapsto \frac{1}{2}\sigma\left(\frac{d\lambda_t}{dt}\Big|_{t=0}, \lambda_0\right),\ \lambda \in \Lambda.$$

It is not difficult to show that that this correspondence coincides with the isomorphism $T_\Lambda L(\Sigma) \approx \mathcal{P}(\Lambda)$ defined above. We shall use this identification in the sequel without any further mentioning.

The cone of non–negative quadratic forms defines a partial ordering in $\mathcal{P}(\Lambda)$. We call a curve $\Lambda(t) \in L(\Sigma)$ *monotone non–decreasing, (monotone non–increasing)* if $\frac{d\Lambda(t)}{dt} \ge 0$ ($\frac{d\Lambda(t)}{dt} \le 0$) $\forall t$.

Suppose that the curve $\Lambda(t)$ is contained in a coordinate neighborhood of the manifold $L(\Sigma) \approx L(\Lambda_1^* \oplus \Lambda_1)$ considered in $n^o 1$. Thus

$$\Lambda_t = \left\{ (\xi, x) \in \Lambda_1^* \oplus \Lambda_1 \middle| \, \xi = Q_t x \right\},$$

where $Q_t : \Lambda_1 \longrightarrow \Lambda_1^*$ is a selfadjoint mapping smoothly depending on T. To the tangent vector $\frac{d\Lambda(t)}{dt}$ corresponds a quadratic form on Λ_1. It is easily seen that this is the form

$$x \mapsto -(\frac{dQ_t}{dt} x)x, \ x \in \Lambda_1.$$

Thus the curve Λ_t is non–decreasing (non–increasing) iff $\frac{dQ_t}{dt} \leq 0$ ($\frac{dQ_t}{dt} \geq 0$).

We know that $\Lambda_1^{\pitchfork}$ is isomorphic to the space of all quadratic forms on $\mathbb{R}^n$, though not canonically isomorphic. Consider the remaining part of the Lagrange Grassmannian

$$\mathcal{M}_{\Lambda_1} = L(\Sigma) \setminus \Lambda_1^{\pitchfork} = \left\{ \Lambda \in L(\Sigma) \middle| \, \Lambda \cap \Lambda_1 \neq 0 \right\},$$

which according to V.Arnold is called the *train* of the Lagrangian subspace Λ_1. The relation (7) implies that the intersection of $\mathcal{M}_{\Lambda_1}$ with an arbitrary coordinate neighbirhood $\Lambda_0^{\pitchfork}$, where Λ_0, is transversal to Λ_1, coincides with the set of all degenerate quadratic forms on $\mathbb{R}^n$. To a subspace Λ, which has a k–dimensional intersection with Λ_1, there corresponds a form with a k–dimensional kernel.

Degenerate forms constitute an algebraic hypersurface in the space of all forms $\mathcal{P}(\mathbb{R}^n)$. This hypersurface has singularirties: its singular points are all the forms which have at least two–dimensional kernel. At the same time the forms with at least two–dimensional kernel constitute an algebraic subset of codimension 3 in $\mathcal{P}(\mathbb{R}^n)$, cf. fig.1, which represents the hypersurface of the degenerate forms in the three–dimensional space $\mathcal{P}(\mathbb{R}^2)$.

Fig. 1

From this we obtain that $\mathcal{M}_{\Lambda_1}$ is an algebraic hypersurface in $L(\Sigma)$, and its singular points constitute an algebraic subset of codimension 3 in $\mathcal{M}_{\Lambda_1}$. Hence $\mathcal{M}_{\Lambda_1}$ is a pseudo–manifold.

Let Λ be a nonsingular point of the hypersurface $\mathcal{M}_\Lambda$. It is not difficult to show that vectors from $T_\Lambda L(\Sigma)$ corresponding to the positive definite and negative definite quadratic forms on Λ are not tangent to the hyperplane $\mathcal{M}_{\Lambda_1}$, cf. fig.5, where the dispositions of the positive and negative cones in $T_\Lambda L(\Sigma)$ relative to $\mathcal{M}_{\Lambda_1}$ are given for $n = 2$.

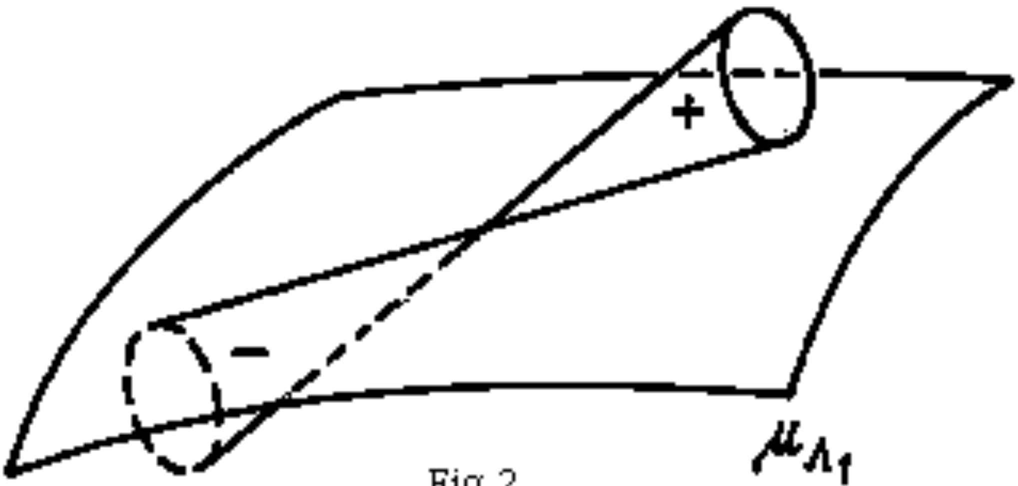

Fig.2

We define a *canonical coorientation* of the hyperplane $\mathcal{M}_{\Lambda_1}$ in $L(\Sigma)$ at a nonsingular point Λ, considering as positive that side of the hyperplane towards which the positive definite elements of $T_\Lambda L(\Sigma)$ are directed and as negative — towards which the negative definite elements are directed.

The defined coorientation of the train permits to define correctly the intersection index $\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}$ of an arbitrary continuous curve $\Lambda(t)$ in $L(\Sigma)$, with endpoints outside of $\mathcal{M}_{\Lambda_1}$, with the hypersurface $\mathcal{M}_{\Lambda_1}$. If $\Lambda(t)$ is smooth and transversally intersecting $\mathcal{M}_{\Lambda_1}$ in nonsingular points the index is defined in the usual way: every intersection point of $\Lambda(\hat{t})$ adds $+1$ or -1 into the value of the intersection index according to the direction of the vector $\frac{d\Lambda}{dt}\Big|_{t=\hat{t}}$ respectively to the positive or negative side of $\mathcal{M}_{\Lambda_1}$. At the same time, since the singularities of $\mathcal{M}_{\Lambda_1}$ are of codimension 3 in $L(\Sigma)$, not only the generic curves, but as well as generic homotopies of such curves, do not intersect the singularities of $\mathcal{M}_{\Lambda_1}$. Thus with a small change of an arbitrary continuous curve with endpoints outside of $\mathcal{M}_{\Lambda_1}$ we can bring it into a transversal position with $\mathcal{M}_{\Lambda_1}$, with the intersection index not depending on the perturbation. Furthermore, the intersection index is constant under an arbitrary homotopy which leaves the endpoints outside $\mathcal{M}_{\Lambda_1}$.

Let $\Lambda(t),\ t \in S^1$, be a closed curve. Then the intersection index $\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}$ does not depend on Λ_1. Indeed, for $\forall \Lambda' \in L(\Sigma)$ there exists a $P \in Sp(\Sigma)$ such that $\Lambda' = P\Lambda_1$. Since the definitions of a train and of the intersection index are invariant under symplectic transformations we have $P\mathcal{M}_{\Lambda_1} = \mathcal{M}_{P\Lambda_1}$, $(P\Lambda(\cdot))\cdot\mu_{P\Lambda_1} = \Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}$. The group $Sp(\Sigma)$ is arcwise connected, hence there exists a continuous curve $P_t \in Sp(\Sigma)$ such that $P_0 = id,\ P_1 = P$. Taking into account the homotopic invariance of the intersection index we obtain

$$\Lambda(\cdot)\cdot\mathcal{M}_{P_t\Lambda_1} = (P_t^{-1}\Lambda(\cdot))\cdot\mathcal{M}_{\Lambda_1} = \Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}.$$

Definition. The intersection index $\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}$ of a closed curve $\Lambda(\cdot)$ with $\mathcal{M}_{\Lambda_1}$, which does not depend on Λ_1, is called the *Maslov index* of the closed curve and is denoted $Ind\,\Lambda(\cdot)$.

There is a close connection of the described intersection index with the Maslov index of the triples of Lagrangian subspaces. The latter definition permits to express explicitly the intersection index without bringing the curve into general position or solving nonlinear equations.

If a section of the curve $\Lambda(t)$ for $t_0 \le t \le t_1$ belongs to the coordinate neighborhood $\Lambda_0^{\pitchfork} \approx \mathcal{P}(\Lambda_1)$, q_t is the quadratic form corresponding to $\Lambda(t)$, and to the subspace Λ_1 corresponds the vanishing form, then it is easily shown that

$$\left(\Lambda\Big|_{[t_0,t_1]}\right)\cdot\mathcal{M}_{\Lambda_1} = \frac{1}{2}(sgn\,q_{t_0} - sgn\,q_{t_1}).$$

Since $sgn\,q_t = \mu(\Lambda_0,\Lambda_1,\Lambda(t))$ then

$$\left(\Lambda\Big|_{[t_0,t_1]}\right)\cdot\mathcal{M}_{\Lambda_1} = \frac{1}{2}(\mu(\Lambda_0,\Lambda_1,\Lambda(t_0)) - \mu(\Lambda_0,\Lambda_1,\Lambda(t_1))).$$

We can subdivide an arbitrary curve $\Lambda(t)$ into sections $\Lambda\Big|_{[t_i,t_{i+1}]}$, $t_0 < t_1 < \ldots < t_{l+1}$ for which $\Lambda(t)$ is transversal to some $\Delta_i \in L(\Sigma)$ for $t_i \le t \le t_{i+1}$. Then

$$\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1} = \frac{1}{2}\sum_{i=0}^{l}(\mu(\Delta_i,\Lambda_1,\Lambda(t_i)) - \mu(\Delta_i,\Lambda_1,\Lambda(t_{i+1})), \tag{12}$$

where the relation (12) is valid also in case $\Lambda(t_i)\cap\Lambda_1 \ne 0,\ 1 \le i \le l$.

5. For a global study of $\Sigma \approx \mathbb{R}^{n*}\oplus\mathbb{R}^n$ it is convenient to use a special complex structure $\mathbb{C}^n$ on $\mathbb{R}^{n*}\oplus\mathbb{R}^n$. To do this we put $z = \xi + ix,\ \xi \in \mathbb{R}^{n*}, x \in \mathbb{R}^n$. The structure $\mathbb{C}^n$ has a standard Hermitian form $h(z,w) = \sum\limits_{j=1}^{n} z^j\overline{w}^j$. The real part of h is a usual scalar product

$$(z_1|z_2) = \sum_{j=1}^{n}(\xi_1^j\xi_2^j + x_1^jx_2^j), \tag{13}$$

and the imaginary part coincides with the symplectic form σ. Thus

$$h(z,w) = (z|w) + i\sigma(z,w).$$

The unitary group $\mathbb{U}(\mathbb{C}^n)$ preserves h, hence it preserves σ, and $\mathbb{U}(\mathbb{C}^n) \subset Sp(\mathbb{R}^{n*}\oplus\mathbb{R}^n) = Sp(\Sigma)$. We shall show that $\mathbb{U}(\mathbb{C}^n)$ acts transitively on $L(\Sigma)$.

It is enough to show that an arbitrary Lagrangian subspace Λ can be obtained by a unitary transformation from the real subspace $\Lambda_0 = \left\{\xi + i0\,\Big|\,\xi \in \mathbb{R}^{n*}\right\}$. Indeed, suppose $e_1,\ldots,e_n$ is an orthonormal basis relative to the real scalar product. Since Λ is a Lagrangian subspace we have $\sigma(e_j,e_k) = 0$, hence the basis is orthonormal also relative to the Hermitian form h. Therefore the transformation which carries over the standard basis of the arithmetic space $\mathbb{C}^n$ into $e_1, \dots, e_n$ is unitary. Furthermore, the unitary transformation $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is carrying over the real subspace $\Lambda_0 \subset \mathbb{C}^n$ into itself iff its matrix is real, in other words, when U belongs to the orthogonal group: $\mathbb{O}(\mathbb{R}^n) \subset \mathbb{U}(\mathbb{C}^n)$. Thus the Lagrange Grassmannian is the homogeneous space

$$Ł(\Sigma) \approx \mathbb{U}(\mathbb{C}^n)/\mathbb{O}(\mathbb{R}^n).$$

Using this representation we can obtain an imbedding of $L(\Sigma)$ into the space of complex symmetric $n \times n$–matrices as a Lagrangian submanifold. We emphasize that the space of complex symmetric, not selfadjoint matrices is considered. A symplectic structure on this space is given by the imaginary part of the Hermitian form

$$(B_1, B_2) \mapsto tr\,(B_1, \overline{B}_2),\ B_j^{\intercal} = B_j,\ j = 1, 2,$$

where $\intercal$ denotes the transposition of a matrix. The imbedding

$$L(\Sigma) \approx \mathbb{U}(\mathbb{C}^n)/\mathbb{O}(\mathbb{C}^n)$$

into the indicated space of matrices is given by the relation

$$U\mathbb{O}(\mathbb{R}^n) \mapsto UU^{\intercal},\ U \in \mathbb{U}(\mathbb{C}^n).$$

We compute now the fundamental group $\pi_1(L(\Sigma))$. Since

$$\pi_1(\mathbb{U}(\mathbb{C}^n)) = \mathbb{Z},\ \pi_1(\mathbb{O}(\mathbb{R}^n)) = \mathbb{Z}_2$$

, we have $\pi_1(\mathbb{U}(\mathbb{C}^n)/\mathbb{O}(\mathbb{R}^n)) = \mathbb{Z}$. The mapping $U\mathbb{O}(\mathbb{R}^n) \mapsto detU^2$ from $\mathbb{U}(\mathbb{C}^n)/\mathbb{O}(\mathbb{R}^n)$ into $S^1 \subset \mathbb{C}$ induces an isomorphism of fundamental groups. Thus if with every closed curve $\Lambda(t) = U(t)\mathbb{O}(\mathbb{R}^n),\ t \in S^1$ from $L(\Sigma) \approx \mathbb{U}(\mathbb{C}^n)/\mathbb{O}(\mathbb{C}^n)$ we associate the degree of the mapping $t \mapsto detU^2(t)$ from S^1 into S^1 we obtain an isomorphism of $\pi_1(L(\Sigma))$ onto $\mathbb{Z}$. There is a simple explicit formula for this isomorphism:

$$deg(det\, U(\cdot)^2) = \frac{1}{\pi i} \int\limits_{S^1} tr\left(\frac{dU}{dt} U(t)^{-1}\right) dt \tag{14}$$

We have introduced in the previous n^o the Maslov index of a closed curve, $Ind\,\Lambda(\cdot)$, which is a homotopy invariant and induces a homomorphism of the fundamental group $\pi_1(L(\Sigma))$ into $\mathbb{Z}$. It turns out that this homomorphism coincides with the isomorphism (14). To prove this we can compute the Maslov index and the right–hand side of (14) for an arbitrary nontrivial curve. We omit this simple calculation. Thus we have

$$Ind\,\Lambda(\cdot) = \frac{1}{\pi i} \int\limits_{S^1} tr\,(\dot{U}(t) U(t)^{-1}) dt,\ \Lambda(t) = U(t)\mathbb{O}(\mathbb{R}^n).$$

We give now another integral formula for Maslov index which does not use the representation of $L(\Sigma)$ as a homogeneous space. The expression $\dot{\Lambda}(t) \in T_{\Lambda(t)} L(\Sigma)$ is interpreted as the quadratic form $\dot{\Lambda}(t) : \lambda_t \mapsto \frac{1}{2}\sigma(\dot{\lambda}_t, \lambda_t)\ \lambda_t \in \Lambda(t)$ on $\Lambda(t)$. The restriction of the scalar product (13) on the subspace $\Lambda(t)$ determines a representation of this form as

$$\dot{\Lambda}(t)(\lambda) = (Q_{\dot{\Lambda}(t)}\lambda|\, \lambda),\ \ \lambda \in \Lambda(t),$$

where $Q_{\dot{\Lambda}(t)} : \Lambda(t) \longrightarrow \Lambda(t)$ is a symmetric operator. We have

$$Ind\,\Lambda(\cdot) = \frac{2}{\pi}\int\limits_{S^1} tr\, Q_{\dot{\Lambda}(t)}dt.$$

Since the tangent space $T_\Lambda L(\Sigma)$ is identified with the space of linear symmetric operators $Q : \Lambda \longrightarrow \Lambda$ a Riemannian structure is defined

$$(Q_1|Q_2) \mapsto tr(Q_1Q_2),\ (Q|Q) = tr\, Q^2 = \sum_{j=1}^{n} \alpha_j(Q)^2,$$

where $\alpha_1(q), \dots, \alpha_n(Q)$ are the eigenvalues of the linear operator $Q : \Lambda \longrightarrow \Lambda$.

Let $l(\Lambda(\cdot))$ be the length of the curve $\Lambda(\cdot)$. Then

$$l(\Lambda(\cdot)) = \int\limits_{S^1}\sqrt{\sum_{j=1}^{n}\alpha_j(Q_{\dot{\Lambda}(t)})^2dt},\ \ Ind\,\Lambda(\cdot) = \frac{2}{\pi}\int\limits_{S^1}\sum_{j=1}^{n}\alpha_j(Q_{\dot{\Lambda}(t)})dt.$$

Monotone nondecreasing curves in $L(\Sigma)$ are characterized by the condition $\alpha_j(Q_{\dot{\Lambda}(t)}) \geq 0,\ j = 1, \dots, n$. For nonnegative α_j the inequality

$$n^{-\frac{1}{2}}\sum_{j=1}^{n}\alpha_j \leq \sqrt{\sum_{j=1}^{n}\alpha_j^2} \leq \sum_{j=1}^{n}\alpha_j$$

holds. Hence for a nondecreasing curve $\Lambda(\cdot)$ the inequalities

$$\frac{\pi}{2\sqrt{n}}Ind\,\Lambda(\cdot) \leq l(\Lambda(\cdot)) \leq \frac{\pi}{2}Ind\,\Lambda(\cdot)$$

are obtained.

6. Concluding this survey of the geometry of Lagrange Grassmannians we shall prove a multidimensional generalization of Sturm's classical theorems about the zeros of the solutions of differential equations of second order.

Proposition 5. *Let* $\Lambda(t),\ t_0 \leq \tau \leq t_1$, *be a continuous curve in* $L(\Sigma)$, *(not necessarily closed), and suppose that* $\Lambda_1, \Lambda_2 \in L(\Sigma)$ *satisfy the relations* $\Lambda_i \cap \Lambda(t_0) = \Lambda_i \cap \Lambda(t_1) = 0,\ i = 1, 2$. *Then*

$$|\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1} - \Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_2}| \leq n.$$

Proof. Since $\Lambda_1^{\pitchfork}$ is arcwise connected we can complement the curve $\Lambda(\cdot)$ to a closed curve $\overline{\Lambda}(\cdot)$, joining $\Lambda(t_1)$ and $\Lambda(t_0)$ by a curve $\Delta \subset \Lambda_1^{\pitchfork}$. Then

$$\Lambda(\cdot) \cdot \mathcal{M}_{\Lambda_1} = \overline{\Lambda}(\cdot) \cdot \mathcal{M}_{\Lambda_1} = Ind\,\overline{\Lambda}(\cdot) = \overline{\Lambda}(\cdot) \cdot \mathcal{M}_{\Lambda_2} = \Lambda(\cdot) \cdot \mathcal{M}_{\Lambda_2} + \Delta \cdot \mathcal{M}_{\Lambda_2}.$$

At the same time

$$\Delta \cdot \mathcal{M}_{\Lambda_2} = \frac{1}{2}(\mu(\Lambda_1, \Lambda_2, \Lambda(t_1)) - \mu(\Lambda_1, \Lambda_0, \Lambda(t_0))).$$

Hence $|\Delta \cdot \mathcal{M}_{\Lambda_2}| \leq n$.

Note that we not only estimated the difference of intersection indices, but expressed this difference explicitly as a function of the endpoints of $\Lambda(\cdot)$.

Corollary. *If $\Lambda(\cdot)$ is a monotone curve, nondecreasing or nonincreasing, then the difference between the numbers of its intersection points with $\mathcal{M}_{\Lambda_1}$ and $\mathcal{M}_{\Lambda_2}$ is not greater then n.*

Indeed, every intersection point of a monotone curve with $\mathcal{M}_{\Lambda_i}$ gives an increment into the index of the same sign.

Proposition 6. *Let $P_\tau \in Sp(\Sigma)$, $0 \leq \tau \leq t$, be a continuous curve in $Sp(\Sigma)$, $P_0 = id$, and suppose $\Lambda_0, \Lambda_0' \in L(\Sigma)$. Put $\Lambda(\tau) = P_\tau \Lambda_0$, $\Lambda'(\tau) = P_\tau \Lambda_0'$. Then for $\forall\, \Lambda_1 \in L(\Sigma)$ the inequality*

$$|\Lambda(\cdot) \cdot \mathcal{M}_{\Lambda_1} - \Lambda'(\cdot) \cdot \mathcal{M}_{\Lambda_1}| \leq n \tag{14}$$

holds.

Proof. We join Λ_0 and Λ_0' with a continuous curve Δ and construct a curvelinear quadrangle

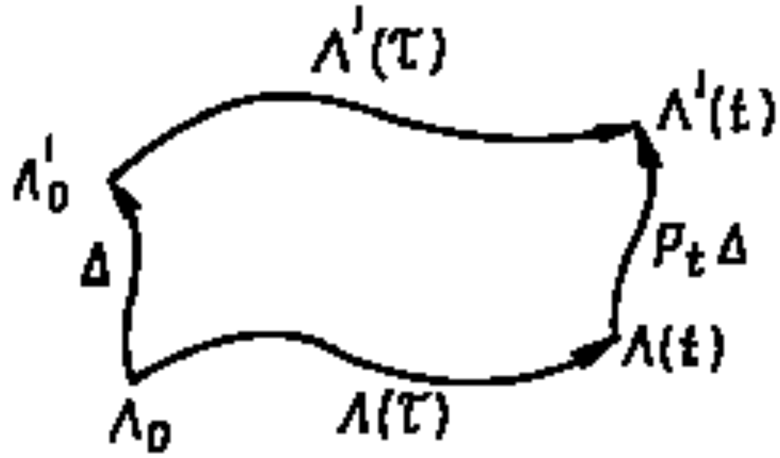

Fig.3

The closed curve $\Lambda(\cdot)\circ P_t\Delta\circ\Lambda'(\cdot)^{-1}\circ\Delta^{-1}$, which is obtained by a successive passage of $\Lambda(\cdot)$, $P_t\Delta$, $\tau\mapsto\Lambda'(t-\tau)$, and the reversed curve of Δ, is contractible. Indeed, the homotopy $(\theta,\Lambda)\mapsto P_{(1-\theta)t}\Lambda$, $\theta\in[0.1]$, $\Lambda\in L(\Sigma)$, contracts the considered closed curve onto the curve $\Delta\circ\Delta^{-1}$. Hence

$$Ind(\Lambda(\cdot)\circ P_t\Delta\circ\Lambda'(\cdot)^{-1}\circ\Delta^{-1})=0,\ \Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}+P_t\Delta\cdot\mathcal{M}_{\Lambda_1}=\Lambda'(\cdot)\mathcal{M}_{\Lambda_1}+\Delta\cdot\mathcal{M}_{\Lambda_1}.$$

At the same time $P_t\Delta\cdot\mathcal{M}_{\Lambda_1}=\Delta\cdot\mathcal{M}_{P_t^{-1}\Lambda_1}$ and hence, according to Proposition 5, $|\Delta\cdot\mathcal{M}_{P_t^{-1}\Lambda_1}-\Delta\cdot\mathcal{M}_{\Lambda_1}|\le n$.

Remark. Suppose P_τ is an absolutely continuous curve. Then P_τ can be represented as a flow generated by a nonstationary linear Hamiltonian system in Σ:

$$\frac{\partial}{\partial\tau}P_\tau z=\overrightarrow{q}_\tau(P_\tau z),\ z\in\Sigma,$$

where $\overrightarrow{q}$ is a linear Hamiltonian field on Σ generated by the quadratic Hamiltonian q_τ. If q_τ is nonnegative (nonpositive) then for $\forall\,\Lambda\in L(\Sigma)$ the curve $\tau\mapsto P_\tau\Lambda$ is monotone nondecreasing (nonincreasing), and the intersection index in (14) could be substituted by the number of the intersection points.

Theorem 1. *Let q_τ,h_τ be quadratic nonstationary Hamiltonians on Σ, where $h_\tau\ge 0$, $0\le\tau\le t$, and let $P_\tau,\widetilde{P}_\tau\in Sp(\Sigma)$ be linear Hamiltonian flows on Σ, generated by the Hamiltonian fields $\overrightarrow{q}_\tau$, $\overrightarrow{q}_\tau+\overrightarrow{h}_\tau$:*

$$\frac{\partial}{\partial\tau}P_\tau=\overrightarrow{q}_\tau P_\tau,\ \frac{\partial}{\partial\tau}\widetilde{P}_\tau=(\overrightarrow{q}_\tau+\overrightarrow{h}_\tau)\widetilde{P}_\tau,\ P_0=\widetilde{P}_0=id.$$

Finally, let $\Lambda(\tau),\widetilde{\Lambda}(\tau)$ be trajectories of the corresponding flows on $L(\Sigma)$:

$$\Lambda(\tau)=P_\tau\Lambda(0),\ \widetilde{\Lambda}(\tau)=\widetilde{P}_\tau\widetilde{\Lambda}(0),\ 0\le\tau\le t.$$

Then for $\forall\,\Lambda_1\in L$, which is transversal to the endpoints of the curves $\Lambda(\cdot),\widetilde{\Lambda}(\cdot)$, the inequality

$$\Lambda(\cdot)\cdot\mathcal{M}_{\Lambda_1}-n\le\widetilde{\Lambda}(\cdot)\cdot\mathcal{M}_{\Lambda_1}$$

is valid.

Proof.

The variation formula, given in Introduction, implies that $\widetilde{P}_\tau=P_\tau R_\tau$, where R_τ is a flow corresponding to the nonnegative nonstationary Hamiltonian $r_\tau:z\mapsto h_\tau(P_\tau z)$, $z\in\Sigma$. In other words,

$$\frac{\partial}{\partial\tau}R_\tau=\overrightarrow{r}_\tau R_\tau,\ 0\le\tau\le t,\ R_0=id.$$

Consider the monotone nondecreasing curve $\Delta(\tau)=R_\tau\widetilde{\Lambda}(0)$ and the curve $\Lambda'(\tau)=P_\tau\Delta(t)$, $0\le\tau\le t$, in $L(\Sigma)$. The identity $\widetilde{P}_\tau\equiv P_\tau R_\tau$ implies that the closed curve $\Delta(\cdot)\circ\Lambda'(\cdot)\circ\widetilde{\Lambda}(\cdot)^{-1}$ is contractible. Hence

$$Ind(\Delta\circ\Lambda'\circ\widetilde{\Lambda}^{-1})=0,\ \widetilde{\Lambda}\cdot\mathcal{M}_{\Lambda_1}=\Delta\cdot\mathcal{M}_{\Lambda_1}+\Lambda'\cdot\mathcal{M}_{\Lambda_1}.$$

At the same time, $\Delta\cdot\mathcal{M}_{\Lambda_1}\ge 0$, and, according to Proposition 6, $\Lambda'\cdot\mathcal{M}_{\Lambda_1}\ge\Lambda\cdot\mathcal{M}_{\Lambda_1}-n$.

§5. The Index of the Second Variation and the Maslov Index

1. We shall consider 1–cocycles on manifolds with half–integer values, and start with definitions. We call a *real–valued 1–cochain* on the manifold M an arbitrary real function c defined on the set of all continuous curves on M subject to the condition

$$c(\gamma|_{[t_0,t_1]}) = c(\gamma|_{[t_0,t]}) + c(\gamma|_{[t,t_1]})\ \forall t \in (t_0, t_1),$$

where $\gamma : [t_0, t_1] \longrightarrow M$ is an arbitrary continuous curve on M. A 1–cochain is a 1–*cocycle* if for every $x \in M$ there exists a function $d_x : O_x \longrightarrow \mathbb{R}$, defined on a neighborhood O_x of x, such that $c(\gamma) = d_x(\gamma(t_1)) - d_x(\gamma(t_0))$ for $\forall \gamma : [t_0, t_1] \longrightarrow O_x$. Every 1–cocycle vanishis on all singular cycles, homologous to zero, (in particular, on all contractible closed curves.) A 1–cocycle is called *exact* or *cohomologous to zero* if there exists a funcion $d : M \longrightarrow \mathbb{R}$ such that

$$c(\gamma) = d(\gamma(t_1)) - d(\gamma(t_0))$$

for every continuous curve $\gamma : [t_0, t_1] \longrightarrow M$. Evidently, exact cocycles vanish on every closed curve. Two cocycles are *cohomologous* if their difference is exact.

Suppose $\Pi \in L(\Sigma)$. We define a cocycle Ind_Π on $L(\Sigma)$ such that $Ind_\Pi\Lambda(\cdot) = \Lambda(\cdot) \cdot \mathcal{M}_\Pi$ for every curve in $L(\Sigma)$ with endpoints outside of $\mathcal{M}_\Pi$, in particular, $Ind_\Pi\Lambda(\cdot) = Ind\Lambda(\cdot)$ for every closed curve $\Lambda(\cdot)$. It is sufficient to define Ind_Π on parts of the curve, which belong to some coordinate neighborhoods.

Let $\Lambda(\cdot)$ be an arbitrary continuous curve in $L(\Sigma)$ and suppose that $\Lambda(t) \in \Delta^\pitchfork$ for $t_0 \le t \le t_1$, for some $\Delta \in L(\Sigma)$. Put

$$Ind_\Pi(\Lambda|_{[t_0,t_1]}) = \frac{1}{2}(\mu(\Delta, \Pi, \Lambda(t_0)) - \mu(\Delta, \Pi, \Lambda(t_1)). \tag{1}$$

We have to prove that the right–hand side of (1) does not depend on the choice of Δ. Using the chain rule (4.10) we obtain

$$Ind_\Pi(\Lambda|_{[t_0,t_1]}) = \frac{1}{2}(\mu(\Pi, \Lambda(t_0), \Lambda(t)) - \mu(\Delta, \Lambda(t_0), \Lambda(t_1)). \tag{2}$$

Hence it is remained only to prove that $\mu(\Delta, \Lambda(t_0), \Lambda(t_1))$ does not depend on the choice of Δ. Let $\Lambda(t) \in \Delta^\pitchfork \cap \Delta'^\pitchfork, t_0 \le t \le t_1$. The chain rule (4.10) and the equality (4.12) imply

$$\begin{gathered}\mu(\Delta, \Lambda(t_0), \Lambda(t_1)) - \mu(\Delta', \Lambda(t_0), \Lambda(t_1)) = \\ \mu(\Delta', \Delta, \Lambda(t_0)) - \mu(\Delta', \Delta, \Lambda(t_1)) = (\Lambda|_{[t_0,t_1]}) \cdot \mathcal{M}_\Delta = 0.\end{gathered}$$

Thus formulas (1) — (2) define a cocycle Ind_Π on $L(\Sigma)$ correctly. Formula (4.12) implies that if the ends of the curve $\Lambda(\cdot)$ are outside of $\mathcal{M}_\Pi$ then $Ind_\Pi\Lambda(\cdot)$ coincides with the intersection index $\Lambda(\cdot) \cdot \mathcal{M}_\Pi$. The cocycles Ind_Π are cohomologous for different $\Pi \in L(\Sigma)$. Indeed, the equality (2) implies that

$$\begin{gathered}Ind_{\Pi_1}(\Lambda|_{[t_0,t_1]}) - Ind_{\Pi_2}(\Lambda|_{[t_0,t_1]}) = \\ \frac{1}{2}(\mu(\Pi_1, \Lambda(t_0), \Lambda(t_1)) - \mu(\Pi_2, \Lambda(t_0), \Lambda(t_1))).\end{gathered} \tag{3}$$

Let $(s, t) \mapsto \Lambda_s(t),\ s \in [0, 1],\ t \in [\alpha, \beta]$ be a homotopy of continuous curves in $L(\Sigma)$, where $dim\,(\Lambda_s(\alpha) \cap \Pi)$ and $dim\,(\Lambda_s(\beta) \cap \Pi)$ do not depend on $s \in [0, 1]$. It is easy to show that in this case $Ind_\Pi\Lambda_s(\cdot)$ is also independent on s.

2. Vector bundle is called *symplectic* if every fibre carries a symplectic structure continuously depending on the fibre. A subbundle of a vector bundle is called *Lagrangian* if its fibres are Lagrangian subspaces. Cocycles on a Lagrange Grassmannian, considered in the previous section, permit to assign to every pair of Lagrangian subbundles a "characteristic 1–cocycle" in the base. We shall consider here the most important particular case of this procedure: we describe the Maslov cocycle of a Lagrangian immersion.

Let $\mathcal{L}$ be a smooth manifold, $dim\,\mathcal{L} = dim\,M$, and $\Phi : \mathcal{L} \longrightarrow T^*M$ be a Lagrangian immersion. For $x \in \mathcal{L}$ we denote

$$\Sigma_x = T_{\Phi(x)}(T^*M),\ \Pi_x = T_{\Phi(x)}(T^*_{\pi_M(\Phi(x))}M),\ \Lambda_x = \Phi_*(T_x\mathcal{L}).$$

Then Σ_x is a symplectic space, $\Pi_x,\ \Lambda_x \in L(\Sigma_x)$, where Π_x is the tangent space at $\Phi(x)$ to the fibre in T^*M, and Λ_x is the tangent space to $\Phi(\mathcal{L})$ at the same point.

Let $x(\cdot)$ be a continuous curve in $\mathcal{L}$, and $[t_0, t_1]$ is a segment in the domain of definition of the curve such that there exists a continuous family of Lagrangian subspaces $\Delta_t \in \mathcal{L}(\Sigma_{x(t)}),\ t_0 \le t \le t_1$, satisfying conditions

$$\Delta_t \cap \Pi_{x(t)} = \Delta_t \cap \Lambda_{x(t)} = 0,\ \forall t \in [t_0, t_1]. \tag{4}$$

It is clear that for sufficiently small segments such families exist. Put

$$m_\Phi(x|_{[t_0,t_1]}) = \frac{1}{2}\left(\mu(\Delta_{t_0}, \Pi_{x(t_0)}, \Lambda_{x_{t_0}}) - \mu(\Delta_{t_1}, \Pi_{x(t_1)}, \Lambda_{x(t)})\right). \tag{5}$$

The expression (4) does not depend on the choice of the family Δ_t. Indeed, the symplectic group acts transitively on pairs of transversal Lagrangian subspaces, hence there exists a continuous family of symplectic mappings $A_t : \Sigma_{x(t)} \longrightarrow \Sigma_{x(t_0)}$, such that $A_t\Delta_{x(t)} = \Delta_{x(0)},\ A_t\Pi_{x(t)} = \Pi_{x(0)}$. Therefore, $\mu(\Delta_t, \Pi_{x(t)}, \Lambda_{x(t)}) = \mu(\Delta_{t_0}, \Pi_{x(t_0)}, A_t\Lambda_{x(t)})$. Hence the right–hand side of (5) coincides with $Ind_{\Pi_{x(t_0)}}(A.\ \Lambda_x|_{[t_0,t_1]})$.

Suppose that $\Delta'_t \in L(\Sigma_{x(t)})$ is another family of Lagrangian subspaces satisfying conditions (4), and suppose $A'_t : \Sigma_{x(t)} \longrightarrow \Sigma_{x(t_o)}$ is a family of symplectic mappings such that $A'_t\Delta'_{x(t)} = \Delta'_{x(0)},\ A'_t\Pi_{x(t)} = \Pi_{x(0)}$. Then $A'_t = P_tA_t$, where $P(t) \in Sp(\Sigma_{x(0)}),\ P(0) = id,\ P(t)\Pi_{x(0)} = \Pi_{x(0)},\ t_0 \le t \le t_1$. Furthermore, $\mu(\Delta'_t, \Pi_{x(t)}, \Delta_{x(t)}) = \mu(\Delta'_0, \Pi_{x(0)}, P(t)A_t\Lambda_{x(t)})$, therefore, if we substitute in (5) the family Δ_t by the family Δ'_t, we obtain for the corresponding index the expression $Ind_{\Pi_{x(t_0)}}(P(\cdot)A.\Lambda_x|_{[t_0,t_1]})$.

Put $\Lambda^s(t) = P(st)A_t\Lambda_{x(t)},\ s \in [0,1],\ t \in [t_0, t_1]$. Since the transformation $P(\tau)$ preserves $\Pi_{x(0)}\ \forall\tau$ the value of the expression $dim(\Lambda^s(t) \cap \Pi_{x_0})$ does not depend on s. Hence $Ind_{\Pi_{x(t)}}\Lambda^s(\cdot)$ also does not depend on s. In particular,

$$\begin{aligned} Ind_{\Pi_{x(t_0)}}(A.\,\Lambda_x|_{[t_0,t_1]}) = Ind_{\Pi_{x(t_0)}}\Lambda^0(\cdot) = Ind_{\Pi_{x(t_0)}}\Lambda^1(\cdot) = \\ Ind_{\Pi_{x(t_0)}}(P(\cdot)A.\,\Lambda_x|_{[t_0,t_1]}). \end{aligned}$$

Thus the right–hand side of (4) does not depend on the choice of Δ_t and hence defines a 1–cocycle m_Φ on $\mathcal{L}$. m_Φ is called the *Maslov cocycle* of the Lagrangian immersion $\Phi : \mathcal{L} \longrightarrow T^*M$.

Consider the composition $\pi_M \circ \Phi$ of the immersion Φ with the canonical projection of T^*M on M. The critical points of the mapping $\pi_M \circ \Phi$ are those $x \in \mathcal{L}$ for which $\Lambda_x \cap \Pi_x \neq 0$. For a generic immersion the set

$$\left\{ x \in \mathcal{L} \middle| \Lambda_x \cap \Pi_x \neq 0 \right\} \tag{6}$$

is a pseudomanifold of codimension 1 in $\mathcal{L}$ with a natural coorientation, cf. [5]. In this case the intersection index of the hypersurface (6) with an arbitrary curve in $\mathcal{L}$, with endpoints outside of (6), is correctly defined. This index is called the *Maslov–Arnold cocycle*. The results of n^o1 imply that it coincides with m_Φ. At the same time , the cocycle m_Φ is always defined — for every immersion and every continuous curve, and is computed according to simple explicit formulas, which do not require either the general position considerations nor the location of intersection points.

3. We shall pass now from arbitrary Lagrangian immersions to manifolds of Lagrangian points described in §2. Let U, M be smooth manifolds, $f : U \longrightarrow M$ be a smooth mapping and $\varphi : U \longrightarrow \mathbb{R}$ be a smooth real–valued function. Suppose that $F = (\varphi, f) : U \longrightarrow \mathbb{R} \times M$ is a Morse mapping. Let $C_{f,\varphi}$ be the manifold of (normed) normal Lagrangian points and $f_c : C_{f,\varphi} \longrightarrow T^*M$ be the corresponding Lagrangian immersion, cf. §2, n^o5. Furthermore, let $(\lambda, u) \in C_{f,\varphi},\ \omega = (-1, \lambda) \in \mathbb{R} \oplus T^*_{f(u)}M$. We remind that ωF^h_u denotes the Hessian of the mapping F at the Lagrangian point (ω, u). Thus ωF^h_u is a real–valued quadratic form on the space $ker\, F'_u = ker\, f'_u$.

Theorem 1. *Let $\gamma(t) = (\lambda_t, u_t) \in C_{f,\varphi},\ t \in [0, 1]$ be an arbitrary continuous curve in $C_{f,\varphi}$ and $\omega_t = (-1, \lambda_t)$. Then*

$$sgn(\omega_1 F^h_{u_1}) - sgn(\omega_0 F^h_{u_0}) = 2m_{f_c}(\gamma(\cdot)). \tag{7}$$

In other words, the Maslov cocycle of the Lagrangian immersion $f_c : C_{f,\varphi} \longrightarrow T^*M$ is the coboundary of the function $(\lambda, u) \mapsto \frac{1}{2} sgn((-1, \lambda)F^h_u)$ on $C_{f,\varphi}$ with half–integer values.

Note that the right–hand side of (7) depends only on the $dim\, M$dimensional manifold of normal Lagrangian points, at the same time the left–hand side contains the signatures of quadratic forms, defined on spaces of dimensions not less than $dim\, U - dim\, M$.

In the extremal problems $dim\, M$ represents the "number of relations", $dim\, U$ is the number of “variables”, and the quadratic form ωF^h_u is the “second variation of the functional”. Usually the number of variables is much greater than the number of relations, therefore Theorem 1 gives an effective method to compute the signature of the second variation.

We considered the finite–dimensional case as a preparation to the study of the optimal control problem, cf. §3, in which U is substituted by the infinite–dimensional space of admissible controls. Therefore it is reasonable to rewrite the equation (7) in a form which is meaningful, at least formally, in case of an infinite–dimensional U as well. We have to rewrite only the left–hand side of (7) only since right side contains the Maslov cocycle of the Lagrangian immersion into T^*M, which arise in optimal control problems naturally, as it was shown in §3.

Let q be a quadratic form on a linear space E. *Positive (negative) index of inertia* of q is given by the expression

$$ind^+ q = sup\left\{dim\, E' \middle| E' \subset E,\ q|_{E'} > 0\right\},$$
$$\left(ind^- q = sup\left\{dim\, E' \middle| E' \subset E,\ q|_{E'} < 0\right\}\right),$$

hence $sgn\, q = ind^+ q - ind^- q$. If E is infinite–dimensional, $ker\, q$ is finite–dimensional, then $ind^+ q$ or $ind^- q$ is infinite and the expression for $sgn\, q$ has no sense. At the same time, the left–hand side of (7) is meaningful if at least one of the indices $ind^+ q, ind^- q$ is finite.

Corollary. *Under the conditions of Theorem 1 we have*

$$\begin{aligned}&\left(ind^+(\omega_1 F^h_{u_1}) + \frac{1}{2} dim\, ker\, \omega_1 F^h_{u_1} - \frac{1}{2} corank\, f'_{u_1}\right) - \\ &\left(ind^+(\omega_0 F^h_0) + \frac{1}{2} dim\, ker\, \omega_0 F^h_{u_0} - \frac{1}{2} corank\, f'_{u_0}\right) = m_{f_c}(\gamma(\cdot)).\end{aligned} \tag{8}$$

Returning to the optimal control problem of §3 we suppose that the Hamiltonian (3.3) is smooth and satisfies to the conditions of Proposition 3.2 for $\mathcal{L}_0 = T^*_{x_0} M$. The Lagrangian immersion in Proposition 3.2

$$(t, \lambda) \mapsto p(t, \lambda),\ t > 0,\ \lambda \in \mathcal{L}_0 \cap H^{-1}(0)$$

substitutes in this case the immersion f_c. Let $\lambda_0 \in \mathcal{L}_0 \cap H^{-1}(0)$. If we put $\gamma(\tau) = (\tau, \lambda_0),\ \tau > 0$, then the equality (8) turns into a direct generalization of the classical Morse formula stating that the increment of the inertia index of the second variation of a regular integral functional along an extremal equals to the sum of multiplicities of conjugate points of the Jacobi equation. Indeed, the curve $\lambda_\tau = p(\tau, \lambda_0)$ is a Pontryagin extremal. The solutions of the Jacobi equation along this extremal are the curves $\xi_\tau = \left(\frac{\partial}{\partial \lambda} p(\tau, \lambda)\big|_{\lambda=\lambda_0}\right) \xi_0,\ \xi_\tau \in T_{\lambda_\tau}(T^*M)$. The time–instant $t > 0$ is conjugate to zero for the Jacobi equation, if there exists a nonzero solution ξ_τ of the equation satisfying conditions $H'_{\lambda_0} \xi_0 = 0,\ \pi_{M*} \xi_0 = \pi_{M*} \xi_t = 0$. (Condition $H'_{\lambda_0} \xi_0 = 0$ arises since we consider the problem with free time.) The multiplicity of a conjugate point is the dimension of the space of such solutions. Employing notations of $n^o 2$ for the considered Lagrangian immersion we obtain that the multiplicity of a conjugate point t is equal to $dim\,(\Lambda_{\gamma(t)} \cap \Pi_{\gamma(t)})$. This is equal to the contribution of the point $\gamma(t)$ into the intersection index of the curve $\gamma(\cdot)$ with the hypersurface

$$\left\{(\tau, \lambda) \in \mathbb{R}_+ \times \mathcal{L}_0 \middle| H(p(\tau, \lambda)) = 0,\ \Lambda_{(\tau,\lambda)} \cap \Pi_{(\tau,\lambda)} \neq 0\right\},$$

in case when the intersection point is isolated and the intersection has the positive sign. In the regular case of a variational problem the intersection points, i.e. the conjugate points of the Jacobi equation, are indeed isolated, but intersections take negative sign if we adopt the orientations of this paper. Hence the sum of multiplicities of the conjugate points is equal $-m_p \gamma(\cdot)$. In more general cases the conjugate points are not necessarily isolated, but the explicit formulas for the Maslov cocycle given in this section permit to ignore this phenomenon. In addition, the Maslov index can be computed along any curve, and not necessarily along extremals.

Concluding this section we note that if the index of the second variation is infinite, formula (8) read from right to left, defines the "increment of the index along a curve", which turns out to be quite useful for developing of the analogs of the Morse theory for the indefinite functionals.

§6 Bang–bang extremals

1. We return to the Optimal Control problem of §3 and consider the case when the Maximum principle generates a nonsmooth Hamiltonian. The problem of description of (possibly nonsmooth) Lagrangian submanifolds contained in the level sets of such Hamiltonians should play a central role for the optimal synthesis. Here will be given some preliminary results in this direction, more detailed results will be published elsewhere.

We start with bang–bang controls. Let $\widetilde{v}(\tau) = v_i,\ \tau_{i-1} < \tau < \tau_i,\ i = 1, \dots, l$, where $0 = \tau_0 < \tau_1 < \dots < \tau_l,\ \widetilde{v}(\tau) = v_{l+1}$ for $\tau > \tau_l$. Suppose $g(x_0, v_0) \neq 0$ and let $\widetilde{x}(t) = x(\tau; \widetilde{v}(\cdot))$ be a trajectory of (3.1) corresponding to the control $\widetilde{v}(\cdot)$. Suppose $\widetilde{\lambda}_\tau \in T^*_{x(\tau)}M,\ 0 \leq \tau \leq t_1,\ \widetilde{\lambda}_\tau \neq \widetilde{\lambda}_{\tau'}$ for $\tau \neq \tau'$, is a normal Pontryagin extremal, corresponding to the control $\widetilde{v}(\cdot)$. Thus $\widetilde{\lambda}_\tau,\ 0 \leq \tau \leq t_1$, is a trajectory of the Hamiltonian system in T^*M defined by the Hamiltonian

$$h_\tau(\lambda) = \lambda g(\pi(\lambda), \widetilde{v}(\tau)) - g^0(\pi(\lambda), \widetilde{v}(\tau)),\ \lambda \in T^*M.$$

Furthermore,

$$0 = h_\tau(\widetilde{\lambda}_\tau) = H(\widetilde{\lambda}_\tau),\ 0 \leq \tau \leq t_1,$$

where, according to (3.3), we have

$$H(\widetilde{\lambda}_\tau) = \max_{u \in V}(\widetilde{\lambda}_\tau g(\widetilde{x}(\tau), u) - g^0(\widetilde{x}(\tau), u)).$$

. Since the control $\widetilde{v}(\cdot)$ is piece–wise constant the Hamiltonian $h_\tau(\lambda)$ depends on τ also piece–wise constantly: $h_\tau(\lambda) = h_i(\lambda),\ \tau_{i-1} < \tau < \tau_i$. Denote $\theta = (\theta^1, \dots, \theta^l)$, where θ^i are real numbers, $\theta^0 = 0$. If the vector θ is sufficiently close to $\overline{\tau} = (\tau_1, \dots, \tau_l)$ then the control

$$v^\theta(\tau) = v_i \text{ for } \theta_{i-1} < \tau < \theta_i,\ i = 1, \dots, l,\ v^\theta(\tau) = v_{l+1} \text{ for } \tau > \theta_l$$

is well defined. The control $v^\theta(\cdot)$ defines a Hamiltonian

$$h_t^\theta(\lambda) = h_i(\lambda) \text{ for } \theta_{i-1} < t < \theta_i,\ i = 1, \dots, l.$$

In particular, $h_t(\lambda) = h_t^{\overline{\tau}}(\lambda)$.

Put

$$f(t, \theta) = x(t; v^\theta(\cdot)),\ \varphi(t, \theta) = \int_0^t g^0(x(\tau; v^\theta(\tau)), v^\theta(\tau)), d\tau,\ F(t, \theta) = (\varphi(t, \theta), f(t, \theta)).$$

The mapping F is a Lipschitz mapping. Furthermore, it is smooth in the region

$$\mathcal{D} = \left\{ (t, \theta) \middle| t \neq \theta_i,\ \theta_{i-1} < \theta_i,\ i = 1, \dots, l \right\}.$$

Proposition 1. *Let* $(t,\theta) \in \mathcal{D}$, $t \in (\theta_{i-1}, \theta_i)$, $\lambda_t^\theta \in T^*_{f(t,\theta)}M$. *Then* $(\lambda_t^\theta, t, \theta) \in C_{f,\varphi}$ *iff* $h^j(\lambda^\theta_{\theta_j}) = 0$, $j = 1, \dots, i$, *where* $\tau \mapsto \lambda_\tau^\theta$ *is a trajectory of the Hamiltonian system defined by the Hamiltonian* h_τ^θ.

Proposition 2. *Let* $t \in (\tau_i, \tau_{i+1})$. *The restriction of the mapping* F *on some neighborhood of the point* $(t, \overline{\tau})$ *is a Morse mapping if* $\{h_{j+1}, h_j\}(\widetilde{\lambda}_{\tau_j}) \neq 0$, $j = 1, \dots, i$.

Propositions 1,2 together with the Proposition 2.5 imply the following

Theorem 1. *Suppose that*

$$H(\lambda) = max\left\{h_i(\lambda) \middle| \, i =, \dots, l\right\}$$

for all $\lambda \in T^*M$ *sufficiently close to the points of the curve* $\widetilde{\lambda}_\tau$, $0 \leq \tau \leq t_1$. *Let, furthermore,* $h_j(\widetilde{\lambda}_\tau) < 0$ *for* $\tau_{i-1} < \tau < \tau_i$, $j \neq i$; $h_j(\widetilde{\lambda}_{\tau_i}) < 0$ *for* $j \neq i, i+1$, *and* $\{h_{i+1}, h_i\}(\widetilde{\lambda}_{\tau_i}) \neq 0$, $i =, 1 \dots, l$. *Then there exists a neighborhood of* $\left\{\widetilde{\lambda}_\tau \middle| 0 \leq \tau \leq t_1\right\}$ *in* T^*M *such that the set*

$$\left\{\lambda_t^\theta \in \mathcal{O} \middle| H(\lambda_\tau^\theta) = 0, \ 0 \leq \tau \leq t\right\} \tag{1}$$

is a Lagrangian submanifold in T^*M. *(We suppose here, as well as in Proposition 1, that* $\frac{\partial}{\partial \tau}\lambda_\tau^\theta = \overrightarrow{h}_\tau^\theta(\lambda_\tau^\theta)$.*)*

According to Theorem 3.1 the trajectory $\widetilde{x}(\tau)$, $\tau \in [0, t]$ is locally optimal if the submanifold (1) is "well–projected" onto M. To construct the manifold (1) is much harder than to proof its existence. Neither is it simple to investigate how it is projected onto M. In case of a smooth Lagrangian submanifold one can avoid these difficulties by the following assertion:

If the Maslov cocycle vanishes on every part of a given curve then some neighborhood of this curve is well projected onto M.

At the same time, to compute the value of the Maslov cocycle it is sufficient to know the tangent spaces to the Lagrangian submanifold only in several points of the curve. Since the Lagrangian submanifold (1) is a Lipschitz manifold, (not everywhere smooth,) hence the Maslov cocycle is not defined on an arbitrary curve as the tangent space does not exist at every point of the curve. Nevertheless, it turns out that the corresponding function with half–integer values can be defined in a natural way, at least for the extremals λ_τ^θ. In accordance with the results of §5 the doubled value of this function on the extremal λ_τ^θ, $\tau \in [t_0, t_1]$, coincides with the difference of the signatures of the Hessians of the mapping F in the Lagrangian points $\lambda_{t_1}^\theta, (t_1, \theta)$ and $\lambda_{t_0}^\theta, (t_0, \theta)$.

2. We shall need a modification of the Maslov index of a triple of Lagrangian subspaces. Suppose $\Lambda_i \in L(\Sigma)$, $i = 0, 1, 2$. Put

$$ind_{\Lambda_0}(\Lambda_1, \Lambda_2) = \frac{1}{2}(\mu(\Lambda_0, \Lambda_1, \Lambda_2) - dim(\Lambda_1 \cap \Lambda_2) + n), \tag{2}$$

where $dim\Sigma = 2n$.

According to the definition $ind_{\Lambda_0}(\Lambda_1, \Lambda_2)$ is a symplectic invariant of the triple of Lagrangian subspaces with half–integer values. One can show, cf. [2], §3, that the invariant (2) is nonnegative and satisfies the triangle inequality:

$$ind_{\Lambda_0}(\Lambda_1, \Lambda_3) \leq ind_{\Lambda_0}(\Lambda_1, \Lambda_2) + ind_{\Lambda_0}(\Lambda_2, \Lambda_3) \ \forall \Lambda_i \in L(\Sigma), \ i = 0, 1, 2, 3.$$

A continuous curve $\Lambda(\tau) \in L(\Sigma)$, $t_0 \leq \tau \leq t_1$, is called *simple* if $\exists \Delta \in L(\Sigma)$ such that $\Lambda(\tau) \cap \Delta = 0 \ \forall \tau \in [t_0, t_1]$.

Proposition 3. *([2], §3). If $\Lambda(\tau)$, $\tau \in [t_0, t_1]$, is a simple nondecreasing curve in $L(\Sigma)$ and $\Pi \in L(\Sigma)$, then*

$$Ind_{\Pi}(\Lambda(\cdot)) = ind_{\Pi}(\Lambda(t_0), \Lambda(t_1)).$$

In particular, the value of the cocycle $Iind_{\Pi}$ on the simple nondecreasing curve depends only on Π and the endpoints of the curve , cf. the expression (5.1) valid for an arbitrary simple curve.

Now under the assumptions of Theorem 1 suppose that λ_t^θ belongs to the manifold (1). If, in addition, $(t, \theta) \in \mathcal{D}$ then the manifold (1) is smooth near the point λ_τ^θ. Let $\Lambda_t^\theta \subset T_{\lambda_t^\theta}(T^*M)$ be the tangent space to the manifold (1) at λ_t^θ. These spaces can be described more explicitly.

Let $P_\tau^\theta \in Symp(T^*M)$, $\tau \geq 0$ be a piecewise smooth curve in $Symp(T^*M)$, which is smooth on the intervals (θ_{i-1}, θ_i), $i = 1, \dots, l$, $(\theta_l, +\infty)$ and defined by conditions

$$\begin{aligned} &\frac{\partial}{\partial \tau} P_\tau^\theta(\lambda) = \vec{h}_i(P_\tau^\theta(\lambda)), \ \theta_{i-1} < \tau < \theta_i, \ P_{\theta_i + 0}(\lambda) \equiv \lambda, \\ &1, \dots, l; \ \frac{\partial}{\partial \tau} P_\tau^\theta(\lambda) = \vec{h}_{l+1}(P_\tau^\theta(\lambda)), \ \theta_l < \tau. \end{aligned} \tag{3}$$

The subspaces Λ_t^θ are uniquely determined by the conditions

$$\begin{aligned} &\Lambda_{-0}^\theta = \Pi_{\lambda_0^\theta}, \ \Lambda_{\theta_i+0}^\theta = span(\Lambda_{\theta_i-0}^\theta + \vec{h}_{i+1}(\lambda_{\theta_i}^\theta)) \cap \vec{h}_{i+1}(\lambda_{\theta_i}^\theta)^\angle, \\ &\Lambda_\tau^\theta = P_{\tau *}^\theta \Lambda_{\theta_i+0}^\theta, \ \theta_i < \tau < \theta_{i+1}, \ i \geq 0, \end{aligned} \tag{4}$$

where $\Pi_\lambda = T_\lambda(T^*_{\pi(\lambda)}M)$. The procedure which generates Lagrange subspaces given in (4) can be considered as a linearization of the method of characteristics described in Proposition 3.2. Note that $P_{\tau^*}\Pi_\lambda = \Pi_{P_\tau(\lambda)} \ \forall \tau, \lambda$. Hence the Maslov class vanishes on the curves $\Lambda_\cdot^\theta\Big|_{(\theta_i, \theta_{i+1})}$, i.e. $m(\Lambda_\cdot^\theta\Big|_{(\theta_i, \theta_{i+1})}) = 0$, $i = 0, 1, \dots$. Define

$$m\left(\Lambda_\cdot^\theta\Big|_{[t_0, t_1)}\right) = \sum_{t_0 \leq \theta_i < t_1} ind_{\Pi_{\lambda_{\theta_i}^\theta}}(\Lambda_{\theta_i-0}^\theta, \Lambda_{\theta_i+0}^\theta), \ \forall t_0, t_1. \tag{5}$$

In other words, to find the value of m on the curve $\tau \mapsto \lambda_\tau^\theta$ we must first paste the discontinuities of the curve of tangent spaces $\tau \mapsto \Lambda_\tau^\theta$ with simple nondecreasing curves in Lagrange Grassmannians and then compute the Maslov cocycle in the usual way.

Proposition 4. *Let λ_t^θ be contained in the manifold (1), and suppose $\Pi \cap \Lambda_{t-0}^\theta = 0$. (We use here and in the sequel the abbreviated notation $\Pi = \Pi_{\lambda_t^\theta}$.) Then Hessian of F at the Lagrangian point $\lambda_t^\theta, (t,\theta)$ is nondegenerate and its positive index of inertia is computed according to the formula*

$$ind^*(\omega F_{(t,\theta)}^h) = m\left(\Lambda_\cdot^\theta\Big|_{[0,t)}\right) - \frac{n}{2}, \tag{6}$$

where $\omega = (-1, \lambda_t^\theta)$.

Corollary. *Under the assumptions of Proposition 4 and provided $m(\Lambda_\cdot^\theta\Big|_{[0,t)}) > \frac{n}{2}$ the control $v^\theta\Big|_{[0,t]}$ is not locally optimal.*

Since $dim(\Lambda_{\theta_i-0}^\theta \cap \Lambda_{\theta_i+0}^\theta) = n-1$ every summand in the right side of (2) is 0, or $\frac{1}{2}$, or 1. At the same time $ind_\Pi(\Lambda_{\theta_i-0}^\theta, \Lambda_{\theta_i+0}^\theta) = \frac{1}{2}$ iff $dim(\Pi \cap \Lambda_{\theta_i-0}^\theta) \neq dim(\Pi \cap \Lambda_{\theta_i+0}^\theta)$. Note also that

$$dim(\Lambda_\tau^\theta \cap \Pi) \geq n - i \text{ for } \theta_{i-1} < \tau < \theta_i,$$

and for a local injectivity of the mapping $\lambda \mapsto \pi(\lambda)$ of the manifold (1) into M at the point λ_t^θ it is necessary that $\Lambda^\theta \cap \Pi = 0$. Thus local injectivity can hold only in case $t > \theta_{n-1}$. For $t \leq \theta_{n-1}$ the mapping $\lambda \mapsto \pi(\lambda)$, though not locally injective, is still "good" (in the sense of Theorem 3.1) near the point λ_t^θ, if the intersection $\Lambda_{t+0}^\theta \cap \Pi$ has the minimal possible dimension.

Theorem 2. *Suppose that under the assumptions of Theorem 1 and for every $\tau_i \leq t$ the relations*

$$ind_\Pi(\Lambda_{\tau_i-0}^{\overline{\tau}}, \Lambda_{\tau_i+0}^{\overline{\tau}}) = \begin{cases} \frac{1}{2}, 0 \leq i \leq n-1 \\ 0, i \geq n, \end{cases}$$

hold, then there exists a neighborhood $\mathcal{L}$ of the curve $\widehat{\lambda}\Big|_{[0,t]}$ in the manifold (1) satisfying the conditions of Theorem 3.1.

Remark. It is easy to show that under the condition $\Lambda_{\tau-0}^{\overline{\tau}} \cap \Pi = \Lambda_{\tau+0}^{\overline{\tau}} \cap \Pi = 0$ the local injectivity of the mapping $\lambda \mapsto \pi(\lambda)$ of the Lagrangian manifold (1) into M near the point $\widehat{\lambda}_\tau$ is equivalent to the equality $ind_\Pi(\Lambda_{\tau-0}^{\overline{\tau}}, \Lambda_{\tau+0}^{\overline{\tau}}) = 0$. Typical locations of the trajectories $x(\cdot, v^\theta) = \pi(\lambda_\cdot^\theta)$ for different values of $ind_\Pi(\Lambda_{\tau-0}^{\overline{\tau}}, \Lambda_{\tau+0}^{\overline{\tau}})$ are represented on the figure.

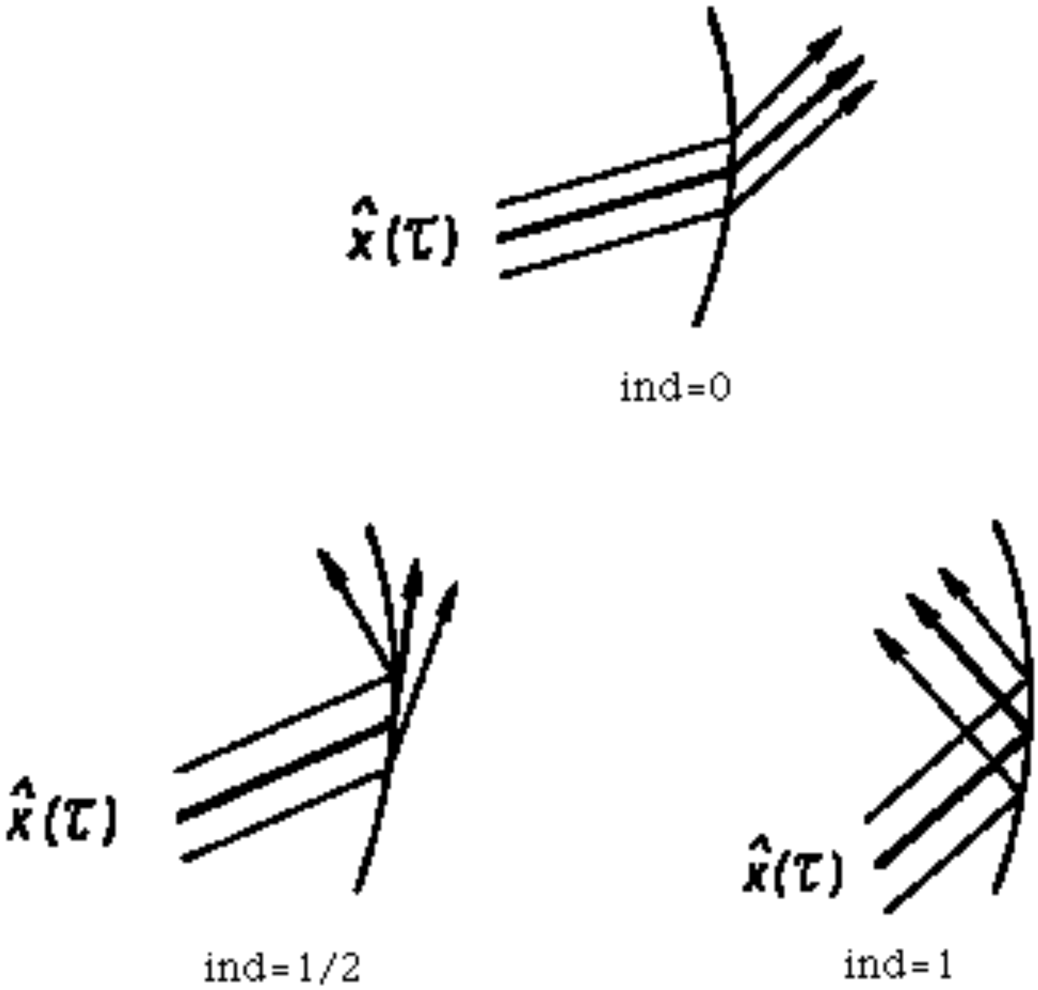

Fig. 1

Standard topological considerations lead to the following global version of Theorem 2. Suppose W is a closed subset in

$$\left\{(t,\theta) \in \mathbb{R}_+ \times \mathbb{R}_+^l \middle| \theta_{i-1} < \theta_i,\ i = 1, \dots, l, \theta_0 = 0\right\}$$

, where $(t,\theta) \in W \Longrightarrow (\tau,\theta) \in W\ \forall \tau \in [0,t]$, and put

$$\mathcal{L}_W = \left\{ \lambda_t^\theta \in T^*M \middle| (t,\theta) \in W, \frac{\partial}{\partial \tau}\lambda_\tau^\theta = \vec{h}_\tau^\theta(\lambda_\tau^\theta), h(\lambda_\tau^\theta) = H(\lambda_\tau^\theta) = 0,\ 0 \le \tau \le t \right\}.$$

To every $\lambda_t^\theta \in \mathcal{L}_W$ we assign a Lagrangian subspace Λ_t^θ in $T_{\lambda_t^\theta}(T^*M)$ defined by the relations (3), (4).

Theorem 3. *Suppose for every $\lambda_t^\theta \in \mathcal{L}_W,\ \theta_i \le t$ the following relations hold:*

(1) $\{h_i, h_{i-1}\}(\lambda_{\theta_i}^\theta) \ne 0$ *for* $i > 0$;

(2) $ind_\Pi(\Lambda_{\theta_i - 0}^\theta, \Lambda_{\theta_i+0}^\theta) = \begin{cases} \frac{1}{2} \ \textit{if}\ 0 \le i \le n-1 \\ 0 \ \textit{if}\ i \ge n. \end{cases}$

If $\mathcal{L}_W$ is arcwise connected and $\pi(\mathcal{L}_W)$ is simply connected then there exists a Lagrangian submanifold $\mathcal{L} \supset \mathcal{L}_W$ in T^*M satisfying the conditions of Theorem 3.1

§7 Jacobi curves

1. The results of §6 can be applied, first of all, to the case when the set V of control parameters is finite. In this case the sets of admissible velocities $g(x, V)$ are also finite, and hence the optimal control problem is badly formulated: the set of admissible trajectories is not closed in the uniform metric, (except the trivial case when $g(x, V)$ is a single point,) though V is compact. As is well known the natural completion (relaxation) of the initial system is the system with the sets of admissible controls $conv\, g(x, V),\ x \in M$. All Pontryagin extremals are preserved under the relaxation, but new ones appear, the so–called singular extremals.

After evident changes in notations the relaxed problem takes the form

$$\dot{x} = \sum_{j=0}^{r} v^j g_j(x),\ v^j \geq 0,\ \sum_{j=0}^{r} v^j = 1,\ x(0) = x_0. \tag{1}$$

The functional

$$\varphi(t, v(\cdot)) = \int_0^t \sum_{j=0}^{r} v^j(\tau) g_j^0(x(\tau), v(\cdot)) d\tau \tag{2}$$

is minimized on the level set of the mapping $f : (t, v(\cdot)) \mapsto x(t, v(\cdot))$. Here $v(\cdot) = (v^0(\cdot), v^1(\cdot), \dots, v^r(\cdot))$ is an admissible control. The space of all admissible controls, i.e. of measurable vector–functions on $\mathbb{R}_+$ with values in

$$\Delta^r = \left\{ (v^0, \dots, v^r) \Big| \, v^j \geq 0,\ \sum_{j=0}^{r} v^j = 1 \right\},$$

denote by $\mathcal{V}$. Every problem with right–hand side affine in controls and with a convex polyhedron as the set of control parameters can be represented as (1), (2).

Let $\overline{\lambda}_\tau,\ 0 \leq \tau \leq t_1$, be a normal Pontryagin extremal for the problem (1), (2), corresponding to an admissible control $\overline{v}(\cdot) = (\overline{v}^0(\cdot) \dots, \overline{v}^r(\cdot))$. Then $\overline{\lambda}_\tau$ is a trajectory of the nonstationary Hamiltonian system generated by the Hamiltonian

$$\overline{h}_\tau(\lambda) = \sum_{j=0}^{r} \overline{v}^j(\tau)(\lambda g_j(\pi(\lambda)) - g_j^0(\pi(\lambda))),\ \lambda \in T^*M.$$

Additionally,we have

$$0 = \overline{h}_\tau(\overline{\lambda}_\tau) = H(\overline{\lambda}_\tau),\ o \leq \tau \leq t,$$

where $H(\lambda) = \underset{0 \leq j \leq r}{max} (\lambda g_j(\pi(\lambda)) - g_j^0(\pi(\lambda)))$.

The mapping $\Phi : \mathbb{R}^k \longrightarrow \mathcal{V}$ is called a *smooth family of variations* of the control $\overline{v}(\cdot)$ if there exist smooth mappings

$$A : \mathbb{R}^k \longrightarrow L^1(\mathbb{R}_+; \Delta^r),\ a : \mathbb{R}^k \longrightarrow L^1(\mathbb{R}_+; \mathbb{R}_+),\ A(0) = \overline{v}(\cdot),\ a(0)(t) = t\ (t \in [0, t_1]),$$

such that $\Phi(z)(t) = A(z)(\int\limits_0^t a(\tau)d\tau),\ \forall z \in \mathbb{R}^k,\ t \geq 0.$
The dimension of the family k can be arbitrary.

If Φ is a smooth family of variations of $\overline{v}(\cdot)$ then $\overline{\lambda}(t_1),\ (t_1, 0)$ is a normal Lagrange point of the mapping

$$F_\Phi : (t, z) \mapsto (\varphi(t, \Phi(z)), f(t, \Phi(z))),\ t > 0, z \in \mathbb{R}^k.$$

Definition. *Positive index* of an extremal $\overline{\lambda}_\tau,\ \tau \in [0, t_1]$, is called the supremum of the positive indices of inertia of Hessians of the mappings F_Φ at the Lagrange point $\lambda(t_1),\ (t_1, 0)$, over all smooth families Φ of variations of the control $\overline{v}(\cdot)$.

The positive index of the extremal $\overline{\lambda}_\tau$ is denoted by $ind^+\overline{\lambda}$. It is a nonnegative integer or $+\infty$.

Let $\overline{x}(\tau) = x(\tau; \overline{v}(\cdot)) = \pi(\overline{\lambda}_\tau),\ 0 \leq \tau \leq t_1$. Denote by $\Psi(\overline{v})$ the set of all Pontryagin extremals λ_τ satisfying the relation $\pi(\lambda_\tau) = \overline{x}(\tau),\ \tau \in [0, t_1]$. Then $\Psi(\overline{v})$ is a convex closed subset of dimension not greater than n in the space of sections of the bundle $T^*M\Big|_{\overline{x}(\cdot)}$.

Proposition 1. *Suppose* $\overline{v}(\cdot)$ *is locally optimal control in the* L_1*–norm of the problem (1), (2) on the segment* $[0, t_1]$*. If* $\Psi(\overline{v}) = \{\overline{\lambda}.\}$ *then* $ind^+\overline{\lambda}. = 0$*; if* $\Psi(\overline{v})$ *is compact and* $\dim \Psi(\overline{v}) = m > 0$*, then there exists a* $\lambda. \in \Psi(\overline{v})$ *such that* $ind^+\lambda. \leq m-1$*.*

Remark. If $dim\, \Psi(\overline{v}) > 1$ then there exist deeper conditions of optimality which take into account the dependence of $ind^+\lambda.$ from $\lambda. \in \Psi(\overline{v})$, cf. [2].

2. According to a procedure already tested, the index of the extremal $\overline{\lambda}.$ should be computed through the Maslov index or its corresponding generalization for the case of a possibly nonsmooth Lagrangian submanifold, which contains $\overline{\lambda}.$ and is contained in $H^{-1}(0)$. All we need to know about the Lagrangian submanifold to compute the index is the knowledge of its tangent spaces at the points $\overline{\lambda}_\tau$. For the particular case of bang–bang controls the corresponding computations were done in §6, n^o2. Note that explicit formulas (6.1), (6.2) for tangent Lagrangian spaces were dependent only on linearizations of Hamiltonian flows corresponding to Hamiltonians h_τ along the extremal. Furthermore, these formulas correctly define a family in τ of Lagrangian subspaces, independently from the existence of the appropriate Lagrangian submanifold: we remind that the existence of a Lagrangian submanifold was proved under sufficiently strong conditions of regularity. We can not guarantee that an arbitrary (not necessarily a bang–bang) extremal $\overline{\lambda}.$ is contained in an appropriate Lagrangian submanifold in $H^{-1}(0)$. But it turns out that if $ind^+\overline{\lambda}. < +\infty$ then there exists a corresponding family of Lagrangian subspaces $\overline{\Lambda}_\tau \subset T_{\overline{\lambda}_\tau}(T^*M)$ "tangent" to $H^{-1}(0)$. Construction of these subspaces and an explicit expression for $ind^+\overline{\lambda}.$ generalize formulas (6.3)–(6.6).

Let $P_t \in Symp(T^*M),\ 0 \leq t \leq t_1$, be a Hamiltonian flow defined by a nonstationary Hamiltonian $\overline{h}_t,\ P_0 = id$. Instead of constructing the Lagrangian Subspaces $\overline{\Lambda}_t$ directly, we shall describe subspaces $\Lambda_t = P_{t^*}^{-1}\overline{\Lambda}_t$ contained in the fixed Lagrange

Grassmannian $L(T_{\lambda_0}(T^*M))$ for $\forall t \in [0, t_1]$. Denote

$$h_j(\lambda) = \lambda g_j(\pi(\lambda)) - g^0(\pi(\lambda)),\ \lambda \in T^*M,\ j = 0, 1, \dots, r,$$
$$\overline{\Gamma}_t = span\left\{\vec{h}_j(\overline{\lambda}_t)\middle|\overline{v}^j(t) > 0,\ 0 \le j \le r\right\};$$

The subspace $\overline{\Gamma}_t$ is contained in $T_{\overline{\lambda}_t}(T^*M),\ 0 \le t \le t_1$.

Let $\mathcal{T} \subset [0, t_1]$ be the set of density points of the measurable vector–function $\overline{v}(\tau)$ on the interval $(0, t_1)$, hence the set $[0, t_1] \setminus \mathcal{T}$ is of measure zero.

Proposition 2. *If $ind^+\overline{\lambda}. < +\infty$ then $\overline{\Gamma}_t$ is an isotropic subspace in $T_{\overline{\lambda}_t}(T^*M)$ for $\forall t \in \mathcal{T}$.*

From here on we suppose that $\overline{\Gamma}_t,\ t \in \mathcal{T}$ are isotropic subspaces. Put $\Gamma_t = P_{t^*}^{-1}\overline{\Gamma}_t \subset T_{\overline{\lambda}_0}(T^*M)$. Since the transformations P_{t^*} are symplectic Γ_t is isotropic for $\forall t \in \mathcal{T}$.

Further developments will take place in a fixed symplectic space $\Sigma = T_{\overline{\lambda}_0}(T^*M)$. Among the Lagrange subspaces of Σ a special role is plaid by the tangent spaces to the fiber at the point $\overline{\lambda}_0$. We shall denote it by $\Pi = T_{\overline{\lambda}_0}(T^*_{x_0}M)$. Let Λ be a Lagrange subspace, Γ an isotropic subspace in Σ. Put

$$\Lambda^\Gamma = (\Lambda + \Gamma) \cap \Gamma^\angle = \Lambda \cap \Gamma^\angle + \Gamma$$

. It is easily seen that Λ^Γ is a Lagrange subspace in Σ, and the mapping $\Lambda \mapsto \Lambda^\Gamma$ is a projection of $L(\Sigma)$ on the submanifold in $L(\Sigma)$ consisting of all Lagrange subspaces containing Γ. The mapping $\Lambda \mapsto \Lambda^\Gamma$ is discontinuous on $L(\Sigma)$, but its restriction on every submanifold of the form $\left\{\Lambda \in L(\Sigma)\middle| dim(\Lambda \cap \Gamma) = const\right\}$ is smooth.

We go now to the construction of Lagrangian subspaces Λ_t. We shall describe the curve $t \mapsto \Lambda_t$ in $L(\Sigma)$ using special piecewise–constant approximations of the curve. Let $D = \{\tau_1, \dots, \tau_k\} \subset \mathcal{T}$, where $\tau_1 < \dots < \tau_k, \tau_{k+1} = t_1$. Define a piecewise–constant curve $\Lambda(D)_t,\ 0 \le t \le t_1$, in $L(\Sigma)$ by

$$\Lambda(D)_t = \Pi,\ 0 \le t \le \tau_1;\ \Lambda(D)_t = \Lambda(D)_{\tau_i}^{\Gamma_{\tau_i}},\ \tau_i < t \le \tau_{i+1},\ i = 1, \dots, k,$$

and put

$$\begin{aligned} Ind_\Pi\Lambda(D) &= \sum_{i=1}^{k} ind_\Pi(\Lambda(D)_{\tau_i}, \Lambda(D)_{\tau_{i+1}}), \\ r(D) &= dim(\pi_* \sum_{i=1}^{k}\Gamma_{\tau_i}) - \frac{1}{2}dim(\pi_*\Lambda(D)_{t_1}). \end{aligned} \tag{3}$$

It is easily seen that $0 \le r(D) \le \frac{n}{2}$. The expression (3) coincides with Ind_Π of a continuous curve in $L(\Sigma)$, which is obtained by successively connecting the values of $L(D).$ with simple nondecreasing curves, cf. Proposition 6.3. Denote by $\mathcal{D}$ the directed set of finite subsets of $\mathcal{T}$ with inclusions of subsets as the partial order.

Theorem 1. *The following relations hold:*

(1) $Ind_\Pi\Lambda(D_1). - r(D_1) \le Ind_\Pi\Lambda(D_2). - r(D_2),\ \forall D_1 \subset D_2 \in \mathcal{D}.$
(2) $ind^+\overline{\lambda}. = \sup_{D\in\mathcal{D}}(Ind_\Pi\Lambda(D). - r(d)).$

Theorem 2. *Suppose that* $ind^{+}\overline{\lambda}. < +\infty$ *Then*

(1) *For* $\forall t \in [0, t_1]$ *the limit* $\mathcal{D} - lim\, \Lambda(D)_t = \Lambda_t$ *exists.*
(2) *The curve* $t \mapsto \Lambda_t$ *in* $L(\Sigma)$ *has at most denumerable set of points of discontinuity and for every* $t \in (0, t_1]$ $(t \in [0, t_1))$ *the limit* $\underset{\tau \to t-0}{lim} \Lambda_\tau$, $(\underset{\tau \to t+0}{lim} \Lambda_\tau)$ *exists.*
(3) *The curve* $\Lambda.$ *is differentiable almost everywhere on* $[0, t_1]$, *and at every point of differentiability* $\frac{d\Lambda_t}{dt} \geq 0$.

.

Definition. Suppose $ind^{+}\overline{\lambda}. < +\infty$. The curve Λ_t in $L(\Sigma)$, which exists according to Theorem 2, is called a *Jacobi curve* corresponding to the extremal $\overline{\lambda}.$.

It follows from Theorem 2 that a Jacobi curve has properties similar to those of a monotone real–valued function. We already have dealt with smooth monotone curves on $L(\Sigma)$. Using the properties of the invariant (6.2), it is possible to describe a wider class of curves for which a theory could be developed very similar to that of monotone real–valued functions. For example, the assertion (1) of Theorem 2 is based on a “Lagrange” analogue of compactness principle of Helly. Basic facts of this method are briefly described in [4]. We call Λ_t a Jacobi curve since it generalizes solutions of the Jacobi equation of the classical calculus of variations. Under sufficiently week assumptions of regularity every Jacobi curve turns out to be piecewise smooth and on the intervals of differentiability it satisfies to differential equation for which the right–hand side can be explicitely expressed through Γ_t, cf. [2], [3]. Put $\overline{\Lambda}_t = P_{t*}\Lambda_t$. If $\overline{v}(\cdot)$ is a bang–bang control, i.e. if $\overline{v}(\tau) \equiv v^{\theta}(\tau)$ for some θ, cf. §6, we obtain $\overline{\Lambda}_t \equiv \Lambda^{\theta}_t$.

3. All our basic constructions until now were invariant under smooth coordinate transformations in M. As a final theme we shall discuss the action on Jacobi curves of another important class of transformations of Control systems, the so–called feedback transformations. We remind the basic definition.

If the inequalities $v^j \geq 0$ are omited in (1) we come to the system

$$\dot{x} = \sum_{j=0}^{r} v^j g_j(x), \ \sum_{j=0}^{r} v^j = 1, \ x(0) = x_0, \tag{4}$$

for which the admissible velocities form affine subspaces. Let $b_{ij}(x), \ x \in M, \ i, j = 0, 1, \ldots, r$, be smooth functions on M, where $\sum\limits_{i=0}^{r} b_{ij}(x) \equiv 1, \ j = 0, 1, \ldots, r$, and the $(1 + r) \times (1 + r)$–matrix $b(x) = (b_{ij}(x))^r_{i,j=0}$ is nondegenerate for $\forall x \in M$. Put $v^j = \sum\limits_{j=0}^{r} b_{ij} u^j, \ j = 0, 1, \ldots, r$, where the u^j are considered as new control parameters, we come to the system

$$\dot{x} = \sum_{j=0}^{r} u^j (\sum_{i=0}^{r} b_{ij} g_i(x)), \ \sum_{j=0}^{r} u^j = 1, \ x(0) = x_0. \tag{5}$$

The system (5) is said to be obtained from (4) by feedback transformation. It is clear that the admissible trajectories of systems (4) and (5) coincide, so that they are equivalent indeed. After the feedback transformation the functional (2) takes the form

$$\int_o^t \sum_{j=0}^r u^j \left(\sum_{i=0}^r b_{ij} g_i^0(x(\tau; u(\cdot))) \right) d\tau. \qquad (6)$$

It is easily seen that every (normal) Pontryagin extremal of the problem (4), (2) is a (normal) Pontryagin extremal of the problem (5), (6) and conversely.

Let $\overline{\lambda}_t$, $0 \le t \le t_1$, be a normal Pontryagin extremal, $\overline{x}(t) = \pi(\overline{\lambda}_t)$. Then $h_j(\overline{\lambda}_t) = h_j^b(\overline{\lambda}_t) = 0$, $0 \le t \le t_1$, $j = 0, 1, \dots, r$, where $h_j(\lambda) = \lambda g_j(\pi(\lambda)) - g^0(\pi(\lambda))$, $h_j^b(\lambda) = \sum_{i=0}^r b_{ij} h_i(\lambda)$. An elementary calculation shows that $\vec{h}_j^b(\overline{\lambda}_t) = \sum_{i=0}^r b_{ij}(\overline{x}(t)) \vec{h}_i(\overline{\lambda})$. Put

$$\overline{\Gamma}_t = span \left\{ \vec{h}_j(\overline{\lambda}_t) \Big| \, j = 0, \dots, r \right\} = span \left\{ \vec{h}_j^b(\overline{\lambda}_t) \Big| \, j = 0, \dots, r \right\}.$$

We suppose that, as in $n^o 2$, $\overline{\Gamma}_t$ is an isotropic subspace in $T_{\overline{\lambda}_t}(T^*M)$, i.e. $\{h_i, h_j\}(\overline{\lambda}_t) = 0$, $i, j = 0, \dots, r$, $t \in [0, t_1]$.

Suppose

$$\frac{d}{dt} \overline{x} = \sum_{j=0}^r \overline{v}^j(t) g_j(\overline{x}) = \sum_{j=0}^r \overline{u}^j(t) \sum_{j=0}^r b_{ij}(\overline{x}) g_i(\overline{x}),$$

and put $\overline{h}_t = \sum_{j=0}^r \overline{v}^j(t) h_j$, $\overline{h}_t^b = \sum_{j=0}^r u^j(t) h_j^b$. Let P_t, $P_t^b \in Symp(T^*M)$, $0 \le t \le t_1$, be Hamiltonian flows generated by nonstationary Hamiltonians $\overline{h}_t$ and $\overline{h}_t^b$, $P_0 = P_0^b = id$, and suppose $\Gamma_t = P_{t*}^{-1} \overline{\Gamma}_t$, $\Gamma_t^b = (P_t^b)_*^{-1} \overline{\Gamma}_t$ are isotropic subspaces in $T_{\overline{\lambda}_0}(T^*M) = \Sigma$, $0 \le t \le t_1$.

Applying to the families of isotropic subspaces the same procedure as in $n^o 2$ we can correspond to every finite subset $D \subset (0, t_1)$ locally–constant curves $\Lambda(D)_t$ and $\Lambda(D)_t^b$ in $L(\Sigma)$. It is very probable that the following assertion holds:

If $\Lambda_t = \mathcal{D} - lim \Lambda(D)_t$ exists for every $t \in [0, t]$ then $\Lambda_t^b = \mathcal{D} - lim \Lambda(D)_t^b$ also exists for $\forall t \in [0, t]$, and

$$P_{t^*} \Lambda_t = P_{T*}^b \Lambda_t^b, \; t \in [0, t_1],$$

i.e. the curve $\overline{\Lambda}_t \stackrel{\text{def}}{=} P_{t*} \Lambda_t$ is preserved by the feedback transformation.

We cannot prove the assertion in the formulated generality, but under some additional assumptions of regularity it is possible not only to prove the assertion, but also represent Λ_t as a solution of an explicitly written linear Hamiltonian system. We shall give here the corresponding calculations under sufficiently strong conditions of regularity.

Put $h_{jt} = h_j \circ P_t$, $h_t = \overline{h}_t \circ P_t$. Then $\Gamma_t = span \left\{ \vec{h}_{jt}(\overline{\lambda}_0) \Big| \, j = 0, \dots, r \right\}$. Finally, let $h_{jt}^- = \frac{\partial}{\partial t} h_{jt} = \{h_t, h_{jt}\}$. Consider the family of $(r+1) \times (r+1)$–matrices $A(t) = \| a_{ij}(t) \|$, $t \in [0, t_1]$, where

$$a_{ij}(t) = \{h_i, \{\overline{h}_t, h_j\}\}(\overline{\lambda}_t) = \{h_{it}, h_{jt}^-\}(\overline{\lambda}_0), \; i, j = 0, \dots, r.$$

Differentiating with respect to t of the identity $\{h_i, h_j\}(\overline{\lambda}_t) = 0$ gives the relations $a_{ij}(t) = a_{ji}(t),\ \forall i, j,\ A(t)\overline{v}(t) \equiv 0$. Hence the matrix $A(t)$ is symmetric and of rank not greater than r. We call an extremal $\overline{\lambda}_t$ *nondegenerate* if $rank\, A(t) = r,\ 0 \le t \le t_1$. It is easily seen that the property of an extremal to be nondegenerate is preserved under the feedback transformations.

Proposition 3. *If the extremal $\overline{\lambda}_t$ is nondegenerate then for $\forall t \in [0, t_1]$ the limit $\Lambda_t = \mathcal{D} - lim \Lambda(D)_t$ exists, where the curve $t \mapsto \Lambda_t$ in $L(\Sigma)$is Lipschitz on the half–interval $(0, t_1]$, and the curve $\overline{\Lambda}_t = P_{t*}\Lambda_t$ is preserved under the feedback transformations.*

We call the curve Λ_t the *Jacobi curve* associated with the nondegenerate extremal $\overline{\lambda}_t$ of the problem (4), (2). To find the Hamiltonian system which is satisfied by the Jacobi curve Λ_t we remark that since $\overline{\lambda}_t$ is nondegenerate the kernel of the symmetric matrix $A(t)$ coincides with the strait line in $\mathbb{R}^{r+1}$ through the vector $\overline{v}(t)$. Hence a symmetric $(r+1) \times (r+1)$–matrix $\mathcal{A}(t) = \parallel \alpha_{ij}(t) \parallel$ exists which satisfies the condition $(\mathcal{A}(t)A(t)w - w) \in \mathbb{R}\overline{v}(t)$ for $\forall w \in \mathbb{R}^{r+1}$.

Let, as above, $\Pi = T_{\overline{\lambda}_0}(T^*_{x_0}M)$ be a Lagrangian submanifold in the symplectic space $\Sigma = T_{\overline{\lambda}_0}(T^*M)$ with the symplectic form σ.

Proposition 4. *Under the assumptions of Proposition 3 we have*

$$\Lambda_t = \left\{ z_t \in \Sigma \Big| \frac{d}{d\tau} z_\tau = \sum_{i,j=0}^{r} \alpha_{ij}(\tau) \sigma(\overrightarrow{h}_{i\tau}^{-}(\overline{\lambda}_0), z_\tau) \overrightarrow{h}_{j\tau}^{-}(\overline{\lambda}_0),\ 0 \le \tau \le t,\ z_0 \in \Pi^{\Gamma_0} \right\},$$

$$t \in (0, t_1],\ \Lambda_0 = \Pi.$$

In other words, $\Lambda_t = Q_t \Pi^{\Gamma_0}$, where $Q_\tau \in Sp(\Sigma)$ is a linear Hamiltonian flow in Σ corresponding to the quadratic nonstationary Hamiltonian

$$q_\tau(z) = \frac{1}{2} \sum_{i,j=0}^{r} \alpha_{ij}(\tau)(dh_{i\tau}^{-})z(dh_{j\tau}^{-})z,\ Q_0 = id.$$

Let Λ_t^b be a Jacobi curve associated with the same extremal $\overline{\lambda}_t$ for the problem (5), (6), which is obtained from the problem (4), (2) by the feedback transformation. Proposition 3 implies that $P_{t*}^b \Lambda_t^b = P_{t*}\Lambda_t,\ 0 \le t \le t_1$. Thus $\Lambda_t = (P_t^{-1} \circ P_t^b)_* \Lambda_t^b$. Observe that $(P_t^{-1} \circ P_t^b)_*$ is a symplectic transformation of Σ which preserves Π for $\forall t \in [0, t_1]$. Therefore the following Corollary holds.

Corollary. *Let $\overline{\lambda}_t$ be a nondegenerate Pontryagin extremal and $\Lambda_t,\ 0 \le t \le t_1$, be the associated Jacobi curve. The expressions $Ind_\Pi(\Lambda\Big|_{[\tau,t]}),\ 0 < \tau < t \le t_1$, are invariants of feedback transformations, as well as of arbitrary smooth variable substitutions in M.*

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