# The Curvature and Hyperbolicity of Hamiltonian Systems

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**Abstract**

Curvature-type invariants of Hamiltonian systems generalize sectional curvatures of the Riemannian manifolds: negativity of the curvatures is an indicator of the hyperbolic behavior of the Hamiltonian flow. In this paper, we give a self-contained description of the related constructions and facts; they lead to a natural extension of classical results about Riemannian geodesic flows and indicate some new phenomena.

## Introduction

This paper is especially written to the 70th anniversary of Dmitrij Anosov. One of the goals of the paper is to explain that classical Anosov's results about geodesic flows of the negative curvature Riemannian manifolds can be actually applied to the essentially larger class of flows than it is normally expected.

Needless to say, I am not at all expert in the hyperbolic dynamics, but I was obliged, as a member of the MIAN's department of differential equations, to attend the seminar guided by professor Anosov. I learned first definitions and took some hyperbolic flavor following presentations in this seminar and Anosov's comments to them. Then I realized that the curvature of general Hamiltonian systems originally discovered in the quite different context could serve for a test of hyperbolicity. Of course, I first expressed this fact in the Anosov seminar and now present the consistent text.

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The object to study in this paper is a Hamiltonian system on a symplectic manifold equipped with a Lagrangian vector distribution. We impose certain regularity assumption which guarantees that the action of the Hamiltonian flow on the distribution is nondegenerate. This action generates a one-parametric family of Lagrangian distributions. Evaluating these distributions at a fixed point of the manifold we obtain a one-parametric family of Lagrangian subspaces of the tangent space to the manifold at the given point. We call this family the "Jacobi curve" by analogy with the Jacobi fields associated to the Riemannian geodesic flows. This is essentially content of Section 1.

Jacobi curves are curves in the Lagrange Grassmannians and Sections 2–7 are devoted to the basic differential geometry of such curves. Geometry of Jacobi curves provides us with fundamental differential invariants of the Hamiltonian systems; these invariants are described and computed in Sections 8–10. Main invariants are the curvature form and the reduced curvature forms. In Sections 11–12 we consider the situation when one of these forms is negative. Strict negativity of the reduced curvature form implies the hyperbolic behavior of the flow that is a natural generalization of the classical fact about Riemannian geodesic flows in the case of the negative sectional curvature.

Strong negativity of the (not reduced) curvature form implies very strong consequences for the asymptotic behavior of the flow described in Theorem 3 and Corollary 6. This phenomenon does not occur in geodesic flows but is easily realized for the flows generated by natural mechanical systems. I do not know classical predecessors of these results.

# 1 Regular Hamiltonian systems

Smooth objects are supposed to be $C^\infty$ in this paper; the results remain valid for the class $C^k$ with a finite and not large $k$ but we prefer not to specify the minimal possible $k$.

Let $N$ be a $2n$-dimensional symplectic manifold endowed with a symplectic form $\sigma$. A *Lagrange distribution* $\Delta \subset TN$ is a smooth vector subbundle of $TN$ such that each fiber $\Delta_z = \Delta \cap T_zN,\ z \in N$, is a Lagrange subspace of the symplectic space $T_zN$; in other words, $\dim \Delta_z = n$ and $\sigma_z(\xi, \eta) = 0\ \forall \xi, \eta \in \Delta_z$.

Basic examples are cotangent bundles endowed with the standard symplectic structure and the "vertical" distribution:

$$N = T^*M, \ \Delta_z = T_z(T_q^*M), \quad \forall z \in T_q^*M, \ q \in M. \tag{1}$$

Let $h \in C^\infty(N)$; then $\vec{h} \in \mathrm{Vec}N$ is the associated to $h$ Hamiltonian vector field: $dh = \sigma(\cdot, \vec{h})$. We assume that $\vec{h}$ is a complete vector field, i.e. solutions of the Hamiltonian system $\dot{z} = \vec{h}(z)$ are defined on the whole time axis. We may assume that without a lack of generality since we are going to study dynamics of the Hamiltonian system on compact subsets of $N$ and may reduce the general case to the complete one by the usual cut-off procedure.

The generated by $\vec{h}$ Hamiltonian flow is denoted by $e^{t\vec{h}}, \ t \in \mathbb{R}$. Other notations: $\bar{\Delta} \subset \mathrm{Vec}N$ is the space of sections of the Lagrange distribution $\Delta$; $[v_1, v_2] \in \mathrm{Vec}N$ is the Lie bracket (the commutator) of the fields $v_1, v_2 \in \mathrm{Vec}N$, $[v_1, v_2] = v_1 \circ v_2 - v_2 \circ v_1$.

**Definition 1** *We say that $\vec{h}$ is regular with respect to the Lagrange distribution $\Delta$ if $\{[\vec{h}, v](z) : v \in \bar{\Delta}\} = T_zN$ for any $z \in N$.*

An effective version of Definition 1 is as follows: Let $v_i \in \bar{\Delta}, \ i = 1, \ldots, n$ be such that the vectors $v_1(z), \ldots, v_n(z)$ form a basis of $\Delta_z$; then $\vec{h}$ is regular at $z$ with respect to $\Delta$ if and only if the vectors

$$v_1(z), \ldots, v_n(z), [\vec{h}, v_1](z), \ldots, [\vec{h}, v_n](z)$$

form a basis of $T_zN$.

We define a bilinear mapping $\beta^h : \bar{\Delta} \times \bar{\Delta} \to C^\infty(N)$ by the formula:

$$\beta^h(v_1, v_2) = \sigma([\vec{h}, v_1], v_2).$$

**Lemma 1** *$\beta^h(v_2, v_1) = \beta^h(v_1, v_2), \ \forall v_1, v_2 \in \bar{\Delta}$ and $\beta^h(v_1, v_2)(z)$ depends only on $v_1(z), v_2(z)$.*

**Proof.** Hamiltonian flows preserve $\sigma$ and $\sigma$ vanishes on $\bar{\Delta}$. Using these facts, we obtain:

$$0 = \sigma(v_1, v_2) = \left(e^{t\vec{h}*}\sigma\right)(v_1, v_2) = \sigma(e_*^{t\vec{h}} v_1, e_*^{t\vec{h}} v_2).$$

Differentiation of the identity $0 = \sigma(e_*^{t\vec{h}} v_1, e_*^{t\vec{h}} v_2)$ with respect to $t$ at $t = 0$ gives: $0 = \sigma([\vec{h}, v_1], v_2) + \sigma(v_1, [\vec{h}, v_2])$. Now the anti-symmetry of $\sigma$ implies the symmetry of $\beta^h$. Moreover, $\beta^h$ is $C^\infty(M)$-linear with respect to each argument, hence $\beta^h(v_1, v_2)(z)$ depends only on $v_1(z), v_2(z)$. $\square$

Let $z \in N,\ \xi_i \in \Delta_z,\ \xi_i = v_i(z),\ v_i \in \Delta,\ i = 1, 2$. We set $\beta^h_z(\xi_1, \xi_2) = \beta^h(v_1, v_2)(z)$. According to Lemma 1, $g^h_z$ is a well-defined symmetric bilinear form on $\Delta_z$. It is easy to see that the regularity of $h$ at $z$ is equivalent to the nondegeneracy of $g^h_z$.

If $N = T^*M$ and $\Delta$ is the vertical distribution (see (1)), then $\beta^h_z = D^2_z(h|_{T^*_q M})$, where $z \in T^*_q M$. The last equation can be easily checked in local coordinates. Indeed, local coordinates defined on a neighborhood $O \subset M$ provide the identification of $T^*M\big|_O$ with $\mathbb{R}^n \times \mathbb{R}^n = \{(p, q) : p, q \in \mathbb{R}^n\}$ such that $T^*_q M$ is identified with $\mathbb{R}^n \times \{q\}$, the form $\sigma$ is identified with $\sum\limits_{i=1}^n dp_i \wedge dq_i$ and the field $\vec{h}$ with $\sum\limits_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\right)$. The fields $\frac{\partial}{\partial p_i}$ form a basis of the vertical distribution and

$$\beta^h\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = -\left\langle dq_j, \left[\sum_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\right), \frac{\partial}{\partial p_i}\right]\right\rangle = \frac{\partial^2 h}{\partial p_i \partial p_j}.$$

**Definition 2** *We say that a regular Hamiltonian field $\vec{h}$ is monotone with respect to $\Delta$ if $\beta^h_z$ is a sign-definite form for any $z \in N$.*

If $N = T^*M$ and $\Delta$ is the vertical distribution, then the monotonicity of $\vec{h}$ is equivalent to the strong convexity or concavity of the restrictions of $h$ to the fibers $T^*_q M$, $q \in M$.

We are going to study the action of the Hamiltonian flow $e^{t\vec{h}}$ on the vertical distribution $\Delta$. Namely, for any $z \in N$ consider the family of subspaces $J_z(t) = e^{-t\vec{h}}_* \Delta_{e^{t\vec{h}}(z)} \subset T_z M$, $t \in \mathbb{R}$; in particular, $J_z(0) = \Delta_z$. Let $G_n(T_z N)$ be the the Grassmann manifold (Grassmannian) consisting of all $n$-dimensional subspaces of the $2n$-dimentional space $T_z N$. Then $t \mapsto J_z(t)$ is a smooth curve in $G_n(T_z N)$, we call it the *Jacobi curve* associated to the pair $\vec{h}$, $\Delta$. Elementary differential geometry of Jacobi curves will provide us with desired curvature-type invariants. To introduce them, we need some basic facts on the geometry of Grassmannians.

## 2 The cross-ratio

Let $\Sigma$ be a $2n$-dimensional vector space, $v_0, v_1 \in G_n(\Sigma)$, $v_0 \cap v_1 = 0$. Than $\Sigma = v_0 + v_1$. We denote by $\pi_{v_0v_1} : \Sigma \to v_1$ the projector of $\Sigma$ onto $v_1$ parallel to $v_0$. In other words, $\pi_{v_0v_1}$ is a linear operator on $\Sigma$ such that $\pi_{v_0v_1}\big|_{v_0} = 0$, $\pi_{v_0v_1}\big|_{v_1} = \mathrm{id}$. Surely, there is a one-to-one correspondence between pairs of transversal $n$-dimensional subspaces of $\Sigma$ and rank $n$ projectors in $\mathrm{gl}(\Sigma)$.

**Lemma 2** *Let $v_0 \in G_n(\Sigma)$; we set $v_0^{\pitchfork} = \{v \in G_n(\Sigma) : v \cap v_0 = 0\}$, an open dense subset of $G_n(\Sigma)$. Then $\{\pi_{vv_0} : v \in v_0^{\pitchfork}\}$ is an affine subspace of $gl(\Sigma)$.*

Indeed, any operator of the form $\alpha\pi_{vv_0} + (1-\alpha)\pi_{wv_0}$, where $\alpha \in \mathbb{R}$, takes values in $v_0$ and its restriction to $v_0$ is the identity operator. Hence $\alpha\pi_{vv_0} + (1-\alpha)\pi_{wv_0}$ is the projector of $\Sigma$ onto $v_0$ along some subspace.

The mapping $v \mapsto \pi_{vv_0}$ thus serves as a local coordinate chart on $G_n(\Sigma)$. These charts indexed by $v_0$ form a natural atlas on $G_n(\Sigma)$.

Projectors $\pi_{vw}$ satisfy the following basic relations:

$$\pi_{v_0v_1} + \pi_{v_1v_0} = id, \quad \pi_{v_0v_2}\pi_{v_1v_2} = \pi_{v_1v_2}, \quad \pi_{v_0v_1}\pi_{v_0v_2} = \pi_{v_0v_1}, \tag{2}$$

where $v_i \in G_n(\Sigma)$, $v_i \cap v_j = 0$ for $i \neq j$. If $n = 1$, then $G_n(\Sigma)$ is just the projective line $\mathbb{RP}^1$; basic geometry of $G_n(\Sigma)$ is somehow similar to geometry of the projective line for arbitrary $n$ as well. The group $\mathrm{GL}(\Sigma)$ acts transitively on $G_n(\Sigma)$. Let us consider its standard action on $(k+1)$-tuples of points in $G_n(\Sigma)$:

$$A(v_0, \dots, v_k) \stackrel{def}{=} (Av_0, \dots, Av_k), \quad A \in \mathrm{GL}(\Sigma),\ v_i \in G_n(\Sigma).$$

It is an easy exercise to check that the only invariants of a triple $(v_0, v_1, v_2)$ of points of $G_n(\Sigma)$ for such an action are dimensions of the intersections: $\dim(v_i \cap v_j)$, $0 \le i \le 2$, and $\dim(v_0 \cap v_1 \cap v_2)$. Quadruples of points possess a more interesting invariant: a multidimensional version of the classical cross-ratio.

**Definition 3** *Let $v_i \in G_n(\Sigma)$, $i = 0, 1, 2, 3$, and $v_0 \cap v_1 = v_2 \cap v_3 = 0$. The cross-ratio of $v_i$ is the operator $[v_0, v_1, v_2, v_3] \in gl(v_1)$ defined by the formula:*

$$[v_0, v_1, v_2, v_3] = \pi_{v_0v_1}\pi_{v_2v_3}\big|_{v_1}.$$

*Remark.* We do not lose information when restrict the product $\pi_{v_0v_1}\pi_{v_2v_3}$ to $v_1$; indeed, this product takes values in $v_1$ and its kernel contains $v_0$.

For $n = 1$, $v_1$ is a line and $[v_0, v_1, v_2, v_3]$ is a real number. For general $n$, the Jordan form of the operator provides numerical invariants of the quadruple $v_i,\ i = 0, 1, 2, 3$.

We will mainly use an infinitesimal version of the cross-ratio that is an invariant $[\xi_0, \xi_1] \in \mathrm{gl}(v_1)$ of a pair of tangent vectors $\xi_i \in T_{v_i}G_n(\Sigma),\ i = 0, 1$, where $v_0 \cap v_1 = 0$. Let $\gamma_i(t)$ be curves in $G_n(\Sigma)$ such that $\gamma_i(0) = v_i,\ \frac{d}{dt}\gamma_i(t)\big|_{t=0} = \xi_i,\ i = 0, 1$. Then the cross-ratio: $[\gamma_0(t), \gamma_1(0), \gamma_0(\tau), \gamma_1(\theta)]$ is a well defined operator on $v_1 = \gamma_1(0)$ for all $t, \tau, \theta$ close enough to 0. Moreover, it follows from (2) that $[\gamma_0(t), \gamma_1(0), \gamma_0(0), \gamma_1(0)] = [\gamma_0(0), \gamma_1(0), \gamma_0(t), \gamma_1(0)] = [\gamma_0(0), \gamma_1(0), \gamma_0(0), \gamma_1(t)] = id$. We set

$$[\xi_0, \xi_1] = \frac{\partial^2}{\partial t \partial \tau}[\gamma_0(t), \gamma_1(0), \gamma_0(0), \gamma_1(\tau)]\big|_{v_1}\Big|_{t=\tau=0} \tag{3}$$

It is easy to check that the right-hand side of (3) depends only on $\xi_0, \xi_1$ and that $(\xi_0, \xi_1) \mapsto [\xi_0, \xi_1]$ is a bilinear mapping from $T_{v_0}G_n(\Sigma) \times T_{v_1}G_n(\Sigma)$ onto $\mathrm{gl}(v_1)$.

**Lemma 3** *Let $v_0, v_1 \in G_n(\Sigma),\ v_0 \cap v_1 = 0,\ \xi_i \in T_{v_i}G_n(\Sigma)$, and $\xi_i = \frac{d}{dt}\gamma_i(t)\big|_{t=0},\ i = 0, 1$. Then $[\xi_0, \xi_1] = \frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)}\big|_{v_1}\Big|_{t=\tau=0}$ and $v_1, v_0$ are invariant subspaces of the operator $\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)}\big|_{v_1}\Big|_{t=\tau=0}$.*

**Proof.** According to the definition, $[\xi_0, \xi_1] = \frac{\partial^2}{\partial t\partial\tau}(\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(0)\gamma_1(\tau)})\big|_{v_1}\Big|_{t=\tau=0}$. The differentiation of the identities $\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(t)\gamma_1(\tau)} = \pi_{\gamma_0(t)\gamma_1(0)}$, $\pi_{\gamma_0(t)\gamma_1(\tau)}\pi_{\gamma_0(0)\gamma_1(\tau)} = \pi_{\gamma_0(0)\gamma_1(\tau)}$ gives the equalities:

$$\frac{\partial^2}{\partial t\partial\tau}(\pi_{\gamma_0(t)\gamma_1(0)}\pi_{\gamma_0(0)\gamma_1(\tau)})\Big|_{t=\tau=0} = -\pi_{v_0v_1}\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(t)\gamma_1(\tau)}\Big|_{t=\tau=0}$$

$$= -\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(t)\gamma_1(\tau)}\Big|_{t=\tau=0}\pi_{v_0v_1}.$$

It remains to mention that $\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_1(t)\gamma_0(\tau)} = -\frac{\partial^2}{\partial t\partial\tau}\pi_{\gamma_0(\tau)\gamma_1(t)}$. $\square$

## 3 Coordinate setting

Given $v_i \in G_n(\Sigma)$, $i = 0, 1, 2, 3$, we coordinatize $\Sigma = \mathbb{R}^n \times \mathbb{R}^n = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}^n\}$ in such a way that $v_i \cap \{(0, y) : y \in \mathbb{R}^n\} = 0$. Then there exist $n \times n$-matrices $S_i$ such that

$$v_i = \{(x, S_i x) : x \in \mathbb{R}^n\}, \quad i = 0, 1, 2, 3. \tag{4}$$

The relation $v_i \cap v_j = 0$ is equivalent to $\det(S_i - S_j) \neq 0$. If $S_0 = 0$, then the projector $\pi_{v_0 v_1}$ is represented by the $2n \times 2n$-matrix $\begin{pmatrix} 0 & S_1^{-1} \\ 0 & I \end{pmatrix}$. In general, we have

$$\pi_{v_0 v_1} = \begin{pmatrix} S_{01}^{-1} S_0 & -S_{01}^{-1} \\ S_1 S_{01}^{-1} S_0 & -S_1 S_{01}^{-1} \end{pmatrix},$$

where $S_{01} = S_0 - S_1$. Relation (4) provides coordinates $\{x\}$ on the spaces $v_i$. In these coordinates, the operator $[v_0, v_1, v_2, v_3]$ on $v_1$ is represented by the matrix:

$$[v_0, v_1, v_2, v_3] = S_{10}^{-1} S_{03} S_{32}^{-1} S_{21},$$

where $S_{ij} = S_i - S_j$.

We now compute the coordinate representation of the infinitesimal cross-ratio. Let $\gamma_0(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, $\gamma_1(t) = \{(x, S_{1+t} x) : x \in \mathbb{R}^n\}$ so that $\xi_i = \frac{d}{dt}\gamma_i(t)\big|_{t=0}$ is represented by the matrix $\dot{S}_i = \frac{d}{dt} S_t\big|_{t=i}$, $i = 0, 1$. Then $[\xi_0, \xi_1]$ is represented by the matrix

$$\frac{\partial^2}{\partial t \partial \tau} S_{1t}^{-1} S_{t\tau} S_{\tau 0}^{-1} S_{01}\Big|_{\substack{t=0 \\ \tau=1}} = \frac{\partial}{\partial t} S_{1t}^{-1} \dot{S}_1\Big|_{t=0} = S_{01}^{-1} \dot{S}_0 S_{01}^{-1} \dot{S}_1.$$

So

$$[\xi_0, \xi_1] = S_{01}^{-1} \dot{S}_0 S_{01}^{-1} \dot{S}_1. \tag{5}$$

There is a canonical isomorphism $T_{v_0} G_n(\Sigma) \cong \mathrm{Hom}(v_0, \Sigma/v_0)$; it is defined as follows. Let $\xi \in T_{v_0} G_n(\Sigma)$, $\xi = \frac{d}{dt}\gamma(t)|_{t=0}$, and $z_0 \in v_0$. Take a smooth curve $z(t) \in \gamma(t)$ such that $z(0) = z_0$. Then the residue class $(\dot{z}(0) + v_0) \in \Sigma/v_0$ depends on $\xi$ and $z_0$ rather than on a particular choice of $\gamma(t)$ and $z(t)$. Indeed, let $\gamma'(t)$ be another curve in $G_n(\Sigma)$ whose velocity at $t = 0$ equals $\xi$. Take some smooth with respect to $t$ bases of $\gamma(t)$ and $\gamma'(t)$: $\gamma(t) = span\{e_1(t), \ldots, e_n(t)\}$, $\gamma'(t) = span\{e'_1(t), \ldots, e'_n(t)\}$, where $e_i(0) =$

$e'_i(0)$, $i = 1, \dots, n$; then $(\dot{e}_i(0) - \dot{e}'_i(0)) \in v_0$, $i = 1, \dots, n$. Let $z(t) = \sum_{i=1}^{n} \alpha_i(t) e_i(t)$, $z'(t) = \sum_{i=1}^{n} \alpha'_i(t) e'_i(t)$, where $\alpha_i(0) = \alpha'_i(0)$. We have:

$$\dot{z}(0) - \dot{z}'(0) = \sum_{i=1}^{n} ((\dot{\alpha}_i(0) - \dot{\alpha}'_i(0)) e_i(0) + \alpha'_i(0)(\dot{e}_i(0) - \dot{e}'_i(0))) \in v_0,$$

i.e. $\dot{z}(0) + v_0 = \dot{z}'(0) + v_0$.

We associate to $\xi$ the mapping $\bar{\xi} : v_0 \to \Sigma/v_0$ defined by the formula $\bar{\xi} z_0 = \dot{z}(0) + v_0$. The fact that $\xi \mapsto \bar{\xi}$ is an isomorphism of the vector spaces $T_{v_0} G_n(\Sigma)$ and $\mathrm{Hom}(v_0, \Sigma/v_0)$ can be easily checked in coordinates. The matrices $\dot{S}_i$ above are actually coordinate presentations of $\bar{\xi}_i$, $i = 0, 1$.

The standard action of the group $\mathrm{GL}(\Sigma)$ on $G_n(\Sigma)$ induces the action of $\mathrm{GL}(\Sigma)$ on the tangent bundle $TG_n(\Sigma)$. It is easy to see that the only invariant of a tangent vector $\xi$ for this action is $\mathrm{rank}\,\bar{\xi}$ (tangent vectors are just "double points" or "pairs of infinitesimaly close points" and number $(n - \mathrm{rank}\,\bar{\xi})$ is the infinitesimal version of the dimension of the intersection for a pair of points in the Grassmannian). Formula (5) implies:

$$\mathrm{rank}[\xi_0, \xi_1] \leq \min\{\mathrm{rank}\,\bar{\xi}_0, \mathrm{rank}\,\bar{\xi}_1\}.$$

# 4 Curves in the Grassmannian

Let $t \mapsto v(t)$ be a germ at $\bar{t}$ of a smooth curve in the Grassmannian $G_n(\Sigma)$.

**Definition 4** *We say that the germ $v(\cdot)$ is ample if $v(t) \cap v(\bar{t}) = 0$ $\forall t \neq \bar{t}$ and the operator-valued function $t \mapsto \pi_{v(t)v(\bar{t})}$ has a pole at $\bar{t}$. We say that the germ $v(\cdot)$ is regular if the function $t \mapsto \pi_{v(t)v(\bar{t})}$ has a simple pole at $\bar{t}$. A smooth curve in $G_n(\Sigma)$ is called ample (regular) if all its germs are ample (regular).*

Assume that $\Sigma = \{(x, y) : x, y \in \mathbb{R}^n\}$ is coordinatized in such a way that $v(\bar{t}) = \{(x, 0) : x \in \mathbb{R}^n\}$. Then $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, where $S(\bar{t}) = 0$ and $\pi_{v(t)v(\bar{t})} = \begin{pmatrix} I & -S_t^{-1} \\ 0 & 0 \end{pmatrix}$. The germ $v(\cdot)$ is ample if and only if the scalar function $t \mapsto \det S_t$ has a finite order root at $\bar{t}$. The germ $v(\cdot)$ is regular if and only if the matrix $\dot{S}_{\bar{t}}$ is not degenerate. More generally, the curve $\tau \mapsto \{(x, S_\tau x) : x \in \mathbb{R}^n\}$ is ample if and only if $\forall t$ the function $\tau \mapsto \det(S_\tau - S_t)$

has a finite order root at $t$. This curve is regular if and only if $\det \dot{S}_t \neq 0,\ \forall t$. The intrinsic version of this coordinate characterization of regularity reads: the curve $v(\cdot)$ is regular if and only if the map $\bar{\dot{v}}(t) \in \mathrm{Hom}(v(t), \Sigma/v(t))$ has rank $n,\ \forall t$.

Let $v(\cdot)$ be an ample curve in $G_n(\Sigma)$. We consider the Laurent expansions at $t$ of the operator-valued function $\tau \mapsto \pi_{v(\tau)v(t)}$,

$$\pi_{v(\tau)v(t)} = \sum_{i=-k_t}^{m} (\tau - t)^i \pi_t^i + O(\tau - t)^{m+1}.$$

Projectors of $\Sigma$ on the subspace $v(t)$ form an affine subspace of $\mathrm{gl}(\Sigma)$ (cf. Lemma 2). This fact implies that $\pi_t^0$ is a projector of $\Sigma$ on $v(t)$; in other words, $\pi_t^0 = \pi_{v^\circ(t)v(t)}$ for some $v^\circ(t) \in v(t)^\pitchfork$. We thus obtain another curve $t \mapsto v^\circ(t)$ in $G_n(\Sigma)$, where $\Sigma = v(t) \oplus v^\circ(t),\ \forall t$. The curve $t \mapsto v^\circ(t)$ is called the *derivative curve* of the ample curve $v(\cdot)$.

The affine space $\{\pi_{wv(t)} : w \in v(t)^\pitchfork\}$ is a translation of the linear space $\mathfrak{N}(v(t)) = \{\mathfrak{n} : \Sigma \to v(t) \mid \mathfrak{n}|_{v(t)} = 0\} \subset \mathrm{gl}(\Sigma)\}$ containing only nilpotent operators. It is easy to see that $\pi_t^i \in \mathfrak{N}(v(t))$ for $i \neq 0$.

The derivative curve is not necessary ample. Moreover, it may be nonsmooth and even discontinuous.

**Lemma 4** *If $v(\cdot)$ is regular then $v^\circ(\cdot)$ is smooth.*

**Proof.** We'll find the coordinate representation of $v^\circ(\cdot)$. Let $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$. Regularity of $v(\cdot)$ is equivalent to the nondegeneracy of $\dot{S}_t$. We have:

$$\pi_{v(\tau)v(t)} = \begin{pmatrix} S_{\tau t}^{-1} S_\tau & -S_{\tau t}^{-1} \\ S_t S_{\tau t}^{-1} S_\tau & -S_t S_{\tau t}^{-1} \end{pmatrix},$$

where $S_{\tau t} = S_\tau - S_t$. Then $S_{\tau t}^{-1} = (\tau - t)^{-1} \dot{S}_t^{-1} - \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} + O(\tau - t)$ as $\tau \to t$ and

$$\pi_{v(\tau)v(t)} = (\tau - t)^{-1} \begin{pmatrix} \dot{S}_t^{-1} S_t & -\dot{S}_t^{-1} \\ S_t \dot{S}_t^{-1} S_t & -S_t \dot{S}_t^{-1} \end{pmatrix} +$$

$$\begin{pmatrix} I - \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} S_t & \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} \\ S_t - \frac{1}{2} S_t \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} S_t & \frac{1}{2} S_t \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} \end{pmatrix} + O(\tau - t).$$

We set $A_t = -\frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1}$; then $\pi_{v^\circ(t)v(t)} = \begin{pmatrix} I + A_t S_t & -A_t \\ S_t + S_t A_t S_t & -S_t A_t \end{pmatrix}$ is smooth with respect to $t$. Hence $t \mapsto v^\circ(t)$ is smooth. We obtain:

$$v^\circ(t) = \{(A_t y, y + S_t A_t y) : y \in \mathbb{R}^n\}. \qquad (6)$$

# 5 The curvature

**Definition 5** *Let $v$ be an ample curve and $v^\circ$ be the derivative curve of $v$. Assume that $v^\circ$ is differentiable at $t$ and set $R_v(t) = [\dot{v}^\circ(t), \dot{v}(t)]$. The operator $R_v(t) \in gl(v(t))$ is called the curvature of the curve $v$ at $t$.*

If $v$ is a regular curve, then $v^\circ$ is smooth, the curvature is well-defined and has a simple coordinate presentation. To find this presentation, we'll use formula (5) applied to $\xi_0 = \dot{v}^\circ(t),\ \xi_1 = \dot{v}(t)$. As before, we assume that $v(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$; in particular, $v(t)$ is transversal to the subspace $\{(0, y) : y \in \mathbb{R}^n\}$. In order to apply (5) we need an extra assumption on the coordinatization of $\Sigma$: the subspace $v^\circ(t)$ has to be transversal to $\{(0, y) : y \in \mathbb{R}^n\}$ for given $t$. The last property is equivalent to the nondegeneracy of the matrix $A_t$ (see (6)). It is important to note that the final expression for $R_v(t)$ as a differential operator of $S$ must be valid without this extra assumption since the definition of $R_v(t)$ is intrinsic! Now we compute: $v^\circ(t) = \{(x, (A_t^{-1} + S_t)x) : x \in \mathbb{R}^n\}, \quad R_v(t) = [\dot{v}^\circ(t), \dot{v}(t)] = A_t \frac{d}{dt}(A_t^{-1} + S_t) A_t \dot{S}_t = (A_t \dot{S}_t)^2 - \dot{A}_t \dot{S}_t = \frac{1}{4}(\dot{S}_t^{-1} \ddot{S}_t)^2 - \dot{A}_t \dot{S}_t$. We also have $\dot{A}\dot{S} = -\frac{1}{2}\frac{d}{dt}(\dot{S}^{-1}\ddot{S}\dot{S}^{-1})\dot{S} = (\dot{S}^{-1})^2 - \frac{1}{2}\dot{S}^{-1}\dddot{S}$. Finally,

$$R_v(t) = \frac{1}{2}\dot{S}_t^{-1}\dddot{S}_t - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2 = \frac{d}{dt}\left((2\dot{S}_t)^{-1}\ddot{S}_t\right) - \left((2\dot{S}_t)^{-1}\ddot{S}_t\right)^2, \tag{7}$$

the matrix version of the Schwartzian derivative.

Curvature operator is a fundamental invariant of the curve in the Grassmannian. One more intrinsic construction of this operator, without using the derivative curve, is provided by the following

**Proposition 1** *Let $v$ be a regular curve in $G_n(\Sigma)$. Then*

$$[\dot{v}(\tau), \dot{v}(t)] = (\tau - t)^{-2} id + \frac{1}{3}R_v(t) + O(\tau - t)$$

*as $\tau \to t$.*

**Proof.** It is enough to check the identity in some coordinates. Given $t$ we may assume that

$$v(t) = \{(x, 0) : x \in \mathbb{R}^n\}, \quad v^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}.$$

Let $v(\tau) = \{(x, S_\tau x : x \in \mathbb{R}^n\}$, then $S_t = \ddot{S}_t = 0$ (see (6)). Moreover, we may assume that the bases of the subspaces $v(t)$ and $v^\circ(t)$ are coordinated in such a way that $\dot{S}_t = I$. Then $R_v(t) = \frac{1}{2} \dddot{S}_t$ (see (7)). On the other hand, formula (5) for the infinitesimal cross-ratio implies:

$$[\dot{v}(\tau), \dot{v}(t)] = S_\tau^{-1} \dot{S}_\tau S_\tau^{-1} = -\frac{d}{d\tau}(S_\tau^{-1}) =$$

$$-\frac{d}{d\tau} \left( (\tau - t)I + \frac{(\tau - t)^3}{6} \dddot{S}_t \right)^{-1} + O(\tau - t) =$$

$$-\frac{d}{d\tau} \left( (\tau - t)^{-1} I - \frac{(\tau - t)}{6} \dddot{S}_t \right) + O(\tau - t) = (\tau - t)^{-2} I + \frac{1}{6} \dddot{S}_t + O(\tau - t).$$

□

Curvature operator is an invariant of the curves in $G_n(\Sigma)$ with fixed parametrizations. Asymptotic presentation obtained in Proposition 1 implies a nice chain rule for the curvature of the reparametrized curves.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a regular change of variables, i.e. $\dot{\varphi} \neq 0, \ \forall t$. The standard imbedding $\mathbb{R} \subset \mathbb{RP}^1 = G_1(\mathbb{R}^2)$ makes $\varphi$ a regular curve in $G_1(\mathbb{R}^2)$. As we know (see (7)), the curvature of this curve is the Schwartzian of $\varphi$:

$$R_\varphi(t) = \frac{\dddot{\varphi}(t)}{2\dot{\phi}(t)} - \frac{3}{4} \left( \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)} \right)^2 .$$

We set $v_\varphi(t) = v(\varphi(t))$ for any curve $v$ in $G_n(\Sigma)$.

**Proposition 2** *Let $v$ be a regular curve in $G_n(\Sigma)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a regular change of variables. Then*

$$R_{v_\varphi}(t) = \dot{\varphi}^2(t) R_v(\varphi(t)) + R_\varphi(t). \tag{8}$$

**Proof.** We have

$$[\dot{v}_\varphi(\tau), \dot{v}_\varphi(t)] = (\tau - t)^{-2} \mathrm{id} + \frac{1}{3} R_{v_\varphi}(t) + O(\tau - t).$$

On the other hand,

$$[\dot{v}_\varphi(\tau), \dot{v}_\varphi(t)] = [\dot{\varphi}(\tau)\dot{v}(\varphi(\tau)), \dot{\varphi}(t)\dot{v}(\varphi(t))] = \dot{\varphi}(\tau)\dot{\varphi}(t)[\dot{v}(\varphi(\tau)), \dot{v}(\varphi(t))] =$$

$$\dot{\varphi}(\tau)\dot{\varphi}(t) \left( (\varphi(\tau) - \varphi(t))^{-2} id + \frac{1}{3} R_v(\varphi(t)) + O(\tau - t) \right) =$$

$$\frac{\dot\varphi(\tau)\dot\varphi(t)}{((\varphi(\tau)-\varphi(t))^2}\mathrm{id}+\frac{\dot\varphi^2(t)}{3}R_v(\varphi(t))+O(\tau-t).$$

We treat $\varphi$ as a curve in $\mathbb{RP}^1 = G_1(\mathbb{R}^2)$. Then $[\dot\varphi(\tau),\dot\varphi(t)] = \frac{\dot\varphi(\tau)\dot\varphi(t)}{(\varphi(\tau)-\varphi(t))^2}$, see (5). The one-dimensional version of Proposition 1 reads:

$$[\dot\varphi(\tau),\dot\varphi(t)] = (t-\tau)^{-2}+\frac{1}{3}R_\varphi(t)+O(\tau-t).$$

Finally,

$$[\dot v_\varphi(\tau),\dot v_\varphi(t)] = (t-\tau)^{-2}+\frac{1}{3}\left(R_\varphi(t)+\dot\varphi^2(t)R_v(\varphi(t))\right)+O(\tau-t). \quad \square$$

The following identity is an immediate corollary of Proposition 2:

$$\left(R_{v_\varphi}-\frac{1}{n}(\mathrm{tr}R_{v_\varphi})\mathrm{id}\right)(t)=\dot\varphi^2(t)\left(R_v-\frac{1}{n}(\mathrm{tr}R_v)\mathrm{id}\right)(\varphi(t)). \quad (9)$$

**Definition 6** *An ample curve $v$ is called flat if $R_v(t)\equiv 0$.*

It follows from Proposition 1 that any small enough piece of a regular curve can be made flat by a reparametrization if and only if the curvature of the curve is a scalar operator, i.e. $R_v(t) = \frac{1}{n}(\mathrm{tr}R_v(t))\mathrm{id}$. In the case of a nonscalar curvature, one can use equality (9) to define a distinguished parametrization of the curve and then derive invariants which do not depend on the parametrization.

*Remark.* In this paper we are mainly focused on the regular curves. See paper [3] for the version of the chain rule which is valid for any ample curve and for basic invariants of unparametrized ample curves.

# 6 Structural equations

Assume that $v$ and $w$ are two smooth curves in $G_n(\Sigma)$ such that $v(t)\cap w(t)=0,\ \forall t$.

**Lemma 5** *For any $t$ and any $e\in v(t)$ there exists a unique $f_e\in w(t)$ with the following property: $\exists$ a smooth curve $e_\tau\in v(\tau)$, $e_t=e$, such that $\frac{d}{d\tau}e_\tau\big|_{\tau=t}=f_e$. Moreover, the mapping $\Phi_t^{vw}: e\mapsto f_e$ is linear and for any $e_0\in v(0)$ there exists a unique smooth curve $e(t)\in v(t)$ such that $e(0)=e_0$ and*

$$\dot e(t)=\Phi_t^{vw}e(t),\quad \forall t. \quad (10)$$

**Proof.** First we take any curve $\hat{e}_\tau \in v(\tau)$ such that $e_t = e$. Then $\hat{e}_\tau = a_\tau + b_\tau$ where $a_\tau \in v(t),\ b_\tau \in w(t)$. We take $x_\tau \in v(\tau)$ such that $x_t = \dot{a}_t$ and set $e_\tau = \hat{e}_\tau + (t-\tau)x_\tau$. Then $\dot{e}_t = \dot{b}_t$ and we put $f_e = \dot{b}_t$.

Let us prove that $\dot{b}_t$ depends only on $e$ and not on the choice of $e_\tau$. Computing the difference of two admissible $e_\tau$ we reduce the lemma to the following statement: if $z(\tau) \in v(\tau),\ \forall \tau$ and $z(t) = 0$, then $\dot{z}(t) \in v(t)$.

To prove the last statement we take smooth vector-functions $e^i_\tau \in v(\tau),\ i = 1, \dots, n$ such that $v(\tau) = span\{e^1_\tau, \dots, e^n_\tau\}$. Then $z(\tau) = \sum\limits_{i=1}^{n} \alpha_i(\tau) e^i_\tau,\ \alpha_i(t) = 0$. Hence $\dot{z}(t) = \sum\limits_{i=1}^{n} \dot{\alpha}_i(t) e^i_t \in v_t$.

Linearity of the map $\Phi^{vw}_t$ follows from the uniqueness of $f_e$. Indeed, if $f_{e^i} = \frac{d}{d\tau} e^i_\tau\big|_{\tau=t}$, then $\frac{d}{d\tau}(\alpha_1 e^1_\tau + \alpha_2 e^2_\tau)\big|_{\tau=t} = \alpha_1 f_{e^1} + \alpha_2 f_{e^2}$; hence $\alpha_1 f_{e^1} + \alpha_2 f_{e^2} = f_{\alpha_1 e^1 + \alpha_2 e^2},\ \forall e^i \in v(t),\ \alpha_i \in \mathbb{R},\ i = 1, 2$.

Now consider the smooth submanifold $V = \{(t, e) : t \in \mathbb{R},\ e \in v(t)\}$ of $\mathbb{R} \times \Sigma$. We have $(1, \Phi^{vw}_t e) \in T_{(t,e)} V$ since $(1, \Phi^{vw}_t e)$ is the velocity of a curve $\tau \mapsto (\tau, e_\tau)$ in $V$. So $(t, e) \mapsto (1, \Phi^{vw}_t e),\ (t, e) \in V$ is a smooth vector field on $V$. The curve $e(t) \in v(t)$ satisfies (10) if and only if $(t, e(t))$ is a trajectory of this vector field. Now the standard existence and uniqueness theorem for ordinary differential equations provides the existence of a unique solution to the Cauchy problem for small enough $t$ while the linearity of the equation guarantees that the solution is defined for all $t$. $\square$

It follows from the proof of the lemma that $\Phi^{vw}_t e = \pi_{v(t)w(t)} \dot{e}_\tau\big|_{\tau=t}$ for any $e_\tau \in v(\tau)$ such that $e_t = e$. Let $v(t) = \{(x, S_{vt}x) : x \in \mathbb{R}^n\},\ w(t) = \{(x, S_{wt}x) : x \in \mathbb{R}^n\}$; the matrix presentation of $\Phi^{vw}_t$ in coordinates $x$ is $(S_{wt} - S_{vt})^{-1} \dot{S}_{vt}$. Linear mappings $\Phi^{vw}_t$ and $\Phi^{wv}_t$ provide a factorization of the infinitesimal cross-ratio $[\dot{w}(t), \dot{v}(t)]$. Indeed, equality (5) implies:

$$[\dot{w}(t), \dot{v}(t)] = -\Phi^{wv}_t \Phi^{vw}_t. \tag{11}$$

Equality (10) implies one more useful presentation of the infinitesimal cross-ratio: if $e(t)$ satisfies (10), then

$$[\dot{w}(t), \dot{v}(t)] e(t) = -\Phi^{wv}_t \Phi^{vw}_t e(t) = -\Phi^{wv}_t \dot{e}(t) = -\pi_{w(t)v(t)} \ddot{e}(t). \tag{12}$$

Now let $w$ be the derivative curve of $v$, $w(t) = v^\circ(t)$. It happens that $\ddot{e}(t) \in v(t)$ in this case and (12) is reduced to the *structural equation*:

$$\ddot{e}(t) = -[\dot{v}^\circ(t), \dot{v}(t)] e(t) = -R_v(t) e(t),$$

where $R_v(t)$ is the curvature operator. More precisely, we have the following

**Proposition 3** *Assume that $v$ is a regular curve in $G_n(\Sigma)$, $v^\circ$ is its derivative curve, and $e(\cdot)$ is a smooth curve in $\Sigma$ such that $e(t) \in v(t)$, $\forall t$. Then $\dot{e}(t) \in v^\circ(t)$ if and only if $\ddot{e}(t) \in v(t)$.*

**Proof.** Given $t$, we take coordinates in such a way that $v(t) = \{(x, 0) : x \in \mathbb{R}^n\}$, $v^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}$. Then $v(\tau) = \{(x, S_\tau x) : x \in \mathbb{R}^n\}$ for $\tau$ close enough to $t$, where $S_t = \ddot{S}_t = 0$ (see (6)).

Let $e(\tau) = (x(\tau), S_\tau x(\tau))$. The inclusion $\dot{e}(t) \in v^\circ(t)$ is equivalent to the equality $\dot{x}(t) = 0$. Further,

$$\ddot{e}(t) = (\ddot{x}(t), \ddot{S}_t x(t) + 2\dot{S}_t \dot{x}(t) + S_t \ddot{x}(t)) = (\ddot{x}(t), 2\dot{S}\dot{x}) \in v(t).$$

Regularity of $v$ implies the nondegeneracy of $\dot{S}(t)$. Hence $\ddot{e}(t) \in v(t)$ if and only if $\dot{x}(t) = 0$. $\square$

Now equality (12) implies

**Corollary 1** *If $\dot{e}(t) = \Phi_t^{vv^\circ} e(t)$, then $\ddot{e}(t) + R_v(t)e(t) = 0$.*

Let us consider invertible linear mappings $V_t : v(0) \to v(t)$ defined by the relations $V_t e(0) = e(t)$, $\dot{e}(\tau) = \Phi_\tau^{vv^\circ} e(\tau)$, $0 \le \tau \le t$. It follows from the structural equation that the curve $v$ is uniquely reconstructed from $\dot{v}(0)$ and the curve $t \mapsto V_t^{-1} R_V(t)$ in $\mathrm{gl}(v(0))$. Moreover, let $v_0 \in G_n(\Sigma)$ and $\xi \in T_{v_0} G_n(\Sigma)$, where the map $\bar{\xi} \in \mathrm{Hom}(v_0, \Sigma/v_0)$ has rank $n$; then for any smooth curve $t \mapsto A(t)$ in $\mathrm{gl}(v_0)$ there exists a unique regular curve $v$ such that $\dot{v}(0) = \xi$ and $V_t^{-1} R_v(t) V_t = A(t)$. Indeed, let $e_i(0)$, $i = 1, \ldots, n$, be a basis of $v_0$ and $A(t)e_i(0) = \sum\limits_{j=1}^{n} a_{ij}(t) e_j(0)$. Then $v(t) = span\{e_1(t), \ldots, e_n(t)\}$, where

$$\ddot{e}_i(\tau) + \sum_{j=1}^{n} a_{ij}(\tau) e_j(\tau) = 0, \; 0 \le \tau \le t, \tag{13}$$

and $e_i(t)$ are uniquely defined by fixing the $\dot{v}(0)$.

The obtained classification of regular curves in terms of the curvature is particularly simple in the case of a scalar curvature operators $R_v(t) = \rho(t)\mathrm{id}$. Indeed, we have $A(t) = V_t^{-1} R_v(t) V_t = \rho(t)\mathrm{id}$ and system (13) is reduced to $n$ copies of the Hill equation $\ddot{e}(\tau) + \rho(\tau)e(\tau) = 0$.

Recall that all $\xi \in TG_n(\Sigma)$ such that $\mathrm{rank}\, \bar{\xi} = n$ are equivalent under the action of $\mathrm{GL}(\Sigma)$ on $TG_n(\Sigma)$ induced by the standard action on the Grassmannian $G_n(\Sigma)$. We thus obtain

**Corollary 2** *For any smooth scalar function $\rho(t)$ there exists a unique, up to the action of $GL(\Sigma)$, regular curve $v$ in $G_n(\Sigma)$ such that $R_v(t) = \rho(t)id$.*

Another important special class is that of symmetric curves.

**Definition 7** *A regular curve $v$ is called symmetric if $V_t R_v(0) = R_v(t) V_t$, $\forall t$.*

In other words, $v$ is symmetric if and only if the curve $A(t) = V_t^{-1} R_v(t) V_t$ in $\mathrm{gl}(v(0))$ is constant and coincides with $R_v(0)$. The structural equation implies

**Corollary 3** *For any $n \times n$-matrix $A_0$, there exists a unique, up to the action of $GL(\Sigma)$, symmetric curve $v$ such that $R_v(t)$ is similar to $A_0$.*

The derivative curve $v^\circ$ of a regular curve $v$ is not necessary regular. The formula $R_v(t) = \Phi_t^{v^\circ v} \Phi_t^{v v^\circ}$ implies that $v^\circ$ is regular if and only if the curvature operator $R_v(t)$ is nondegenerate for any $t$. Then we may compute the second derivative curve $v^{\circ\circ} = (v^\circ)^\circ$.

**Proposition 4** *A regular curve $v$ with nondegenerate curvature operators is symmetric if and only if $v^{\circ\circ} = v$.*

**Proof.** Let us consider system (13). We are going to apply Proposition 3 to the curve $v^\circ$ (instead of $v$) and the vectors $\dot{e}_i(t) \in v^\circ(t)$. According to Proposition 3, $v^{\circ\circ} = v$ if and only if $\frac{d^2}{dt^2}\dot{e}_i(t) \in v^\circ(t)$. Differentiating (13) we obtain that $v^{\circ\circ} = v$ if and only if the functions $\alpha_{ij}(t)$ are constant. The last property is none other than a characterization of symmetric curves. $\square$

# 7 Lagrange Grassmannians

We study curves in the Grassmannian keeping in mind the Jacobi curves $J_z(t)$ (see Sec. 1). Recall that $J_z(t)$ are subspaces of the symplectic space $T_z N$ endowed with the symplectic form $\sigma_z$. Moreover, $J_z(t)$ are Lagrange subspaces of $T_z N$. In other words, $t \mapsto J_z(t)$ is a curve in the Lagrange Grassmannian $L(T_z N)$ consisting of all Lagrange subspaces of the symplectic space.

In this section, we give few simple facts on Lagrangian Grassmannians to be used below (see [1, Sec. 4] for a consistent description of their geometry).

Let $(\Sigma, \bar{\sigma})$ be a $2n$-dimensional symplectic space and $\upsilon_0, \upsilon_1 \in L(\Sigma)$ be a pair of transversal Lagrangian subspaces, $\upsilon_0 \cap \upsilon_1 = 0$. Bilinear form $\bar{\sigma}$ induces a non degenerate pairing of the spaces $\upsilon_0$ and $\upsilon_1$ by the rule $(e, f) \mapsto \bar{\sigma}(e, f), \ e \in \upsilon_0, f \in \upsilon_1$. To any basis $e_1, \dots, e_n$ of $\upsilon_0$ we may associate a unique dual basis $f_1, \dots, f_n$ of $\upsilon_1$ such that $\bar{\sigma}(e_i, f_j) = \delta_{ij}$. The form $\bar{\sigma}$ is totally normalized in the basis $e_1, \dots, e_n, f_1, \dots, f_n$ of $\Sigma$, since $\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0$. It follows that symplectic group

$$\mathrm{Sp}(\Sigma) = \{A \in \mathrm{GL}(\Sigma) : \bar{\sigma}(Ae, Af) = \bar{\sigma}(e, f), \ e, f \in \Sigma\}$$

acts transitively on the pairs of transversal Lagrangian subspaces.

Next result is a 'symplectic specification' of Lemma 2 from Section 2.

**Lemma 6** *Let $\upsilon_0 \in L(\Sigma)$; then $\{\pi_{v\upsilon_0} : v \in \upsilon_0^{\pitchfork} \cap L(\Sigma)\}$ is an affine subspace of the affine space $\{\pi_{v\upsilon_0} : v \in \upsilon_0^{\pitchfork}\}$ characterized by the relation:*

$$v \in \upsilon_0^{\pitchfork} \cap L(\Sigma) \ \Leftrightarrow \ \bar{\sigma}(\pi_{v\upsilon_0} \cdot, \cdot) + \bar{\sigma}(\cdot, \pi_{v\upsilon_0} \cdot) = \bar{\sigma}(\cdot, \cdot).$$

**Proof.** Assume that $\upsilon_1 \in \upsilon_0^{\pitchfork} \cap L(\Sigma)$. Let $e, f \in \Sigma, \ e = e_0 + e_1, \ f = f_0 + f_1$ where $e_i, f_i \in \upsilon_i, \ i = 0, 1$; then

$$\bar{\sigma}(e, f) = \bar{\sigma}(e_0 + e_1, f_0 + f_1) = \bar{\sigma}(e_0, f_1) + \bar{\sigma}(e_1, f_0) =$$

$$\bar{\sigma}(e_0, f) + \bar{\sigma}(e, f_0) = \bar{\sigma}(\pi_{\upsilon_1\upsilon_0} e, f) + \bar{\sigma}(e, \pi_{\upsilon_1\upsilon_0} f).$$

Conversely, let $v \in \upsilon_0^{\pitchfork}$ is not a Lagrangian subspace. Then there exist $e, f \in v$ such that $\bar{\sigma}(e, f) \neq 0$, while $\bar{\sigma}(\pi_{v\upsilon_0} e, f) = \bar{\sigma}(e, \pi_{v\upsilon_0} f) = 0$. $\square$

**Corollary 4** *Let $v(\cdot)$ be a regular curve in $G_n(\Sigma)$ and $v^\circ(\cdot)$ be the derivative curve of $v(\cdot)$. If $v(t) \in L(\Sigma), \ \forall t$, then $v^\circ(t) \in L(\Sigma)$.*

**Proof.** The derivative curve $v^\circ$ was defined in Section 4. Recall that $\pi_{v^\circ(t)v(t)} = \pi_t^0$, where $\pi_t^0$ is the free term of the Laurent expansion

$$\pi_{v(\tau)v(t)} \approx \sum_{i=-1}^{\infty} (\tau - t)^i \pi_t^i.$$

The free term $\pi_t^0$ belongs to the affine hull of $\pi_{v(\tau)v(t)}$, when $\tau$ runs a neighborhood of $t$. Since $\pi_{v(\tau)v(t)}$ belongs to the affine space $\{\pi_{v\upsilon_0} : v \in \upsilon_0^{\pitchfork} \cap L(\Sigma)\}$, then $\pi_t^0$ belongs to this affine space as well. $\square$

It is clearly seeing in coordinates how Lagrange Grassmanian is sitting in the usual one. Let $\Sigma = \mathbb{R}^{n*} \times \mathbb{R}^n = \{(\eta, y) : \eta \in \mathbb{R}^{n*}, y \in \mathbb{R}^n\}$. Then any $v \in (\{0\} \times \mathbb{R}^n)^{\pitchfork}$ has a form $v = \{(y^\top, Sy) : y \in \mathbb{R}^n\}$, where $S$ is an $n \times n$-matrix. It is easy to see that $v$ is a Lagrangian subspace if and only if $S$ is a symmetric matrix, $S = S^\top$.

It happens that any tangent vector to $L(\Sigma)$ at the point $v \in L(\Sigma)$ can be naturally identified with a quadratic form on $v$. Her we use the fact that $v$ is not just a point in the Grassmannian but an $n$-dimensional vector space. To associate a quadratic form on $v(t)$ to the velocity $\dot{v}(t) \in T_{v(t)}L(\Sigma)$ of a smooth curve $v(\cdot)$ we proceed as follows: given $z \in v(t)$ we take a smooth curve $\tau \mapsto z(\tau)$ in $\Sigma$ in such a way that $z(\tau) \in v(\tau)$, $\forall \tau$ and $z(t) = z$. Then we define a quadratic form $\underline{\dot{v}}(t)(z)$, $z \in v(t)$, by the formula $\underline{\dot{v}}(t)(z) = \sigma(z, \dot{z}(t))$.

The point is that $\sigma(z, \dot{z}(t))$ does not depend on the freedom in the choice of the curve $\tau \mapsto z(\tau)$, although $\dot{z}(t)$ depends on this choice. Let us check the required property in the coordinates. Assume that $v(\tau) = \{(y^\top, S_\tau y) : y \in \mathbb{R}^n\}$ We have $z = (y^\top, S_t y)$ for some $y \in \mathbb{R}^n$ and $z(\tau) = (y(\tau)^\top, S_\tau y(\tau))$. Then

$$\sigma(z, \dot{z}(t)) = y^\top(\dot{S}_t y + S_t \dot{y}) - \dot{y}^\top S_t y = y^\top \dot{S}_t y;$$

vector $\dot{y}$ does not show up. We have obtained a coordinate presentation of $\underline{\dot{v}}(t)$:

$$\underline{\dot{v}}(t)(y^\top, S_t y) = y^\top \dot{S}_t y,$$

which implies that $\dot{v} \mapsto \underline{\dot{v}}$, $\dot{v} \in T_v L(\Sigma)$ is an isomorphism of $T_v L(\Sigma)$ on the vector space of quadratic forms on $v$.

It is easy to see that a curve $v(\cdot)$ in $L(\Sigma)$ is regular if and only if the quadratic forms $\underline{\dot{v}}(t)$ are nondegenerate for all $t$. We say that such a curve is *monotone increasing (decreasing)* if $\underline{\dot{v}}(t)$ are positive definite (negative definite) forms. In both cases we say that $v(\cdot)$ is *monotone*.

Given a regular monotone increasing (decreasing) curve $v(\cdot)$, the quadratic form $\underline{\dot{v}}(t)$ defines an Euclidean structure $\langle \cdot, \cdot \rangle_{\dot{v}(t)}$ on $v(t)$ so that $\langle x, x \rangle_{\dot{v}(t)} = \underline{\dot{v}}(t)(x)$ $(= -\underline{\dot{v}}(t)(x))$. Let $R_v(t) \in \mathrm{gl}(v(t))$ be the curvature operator of the curve $v(\cdot)$; we define the *curvature quadratic form* $r_v(t)$ on $v(t)$ by the formula:

$$r_v(t)(x) = \langle R_v(t)x, x \rangle_{\dot{v}(t)}, \quad x \in v(t).$$

**Proposition 5** *The curvature operator $R_v(t)$ is a self-adjoint operator for the Euclidean structure $\langle \cdot, \cdot \rangle_{\dot{v}(t)}$. The form $r_v(t)$ is equivalent (up to linear changes of variables) to the form $\underline{\dot{v}}^\circ(t)$, where $v^\circ(\cdot)$ is the derivative curve.*

**Proof.** The statement is intrinsic and we may check it in any coordinates. Fix $t$ and take Darboux coordinates $\{(\eta, y) : \eta \in \mathbb{R}^{n*}, y \in \mathbb{R}^n\}$ in $\Sigma$ in such a way that $v(t) = \{(y^\top, 0) : y \in \mathbb{R}^n\}$, $v^\circ(t) = \{(0, y) : y \in \mathbb{R}^n\}$, $\underline{\dot{v}}(t)(y) = y^\top y$. Let $v(\tau) = \{(y^\top, S_\tau y) : y \in \mathbb{R}^n\}$, then $S_t = 0$. Moreover, $\dot{S}(t)$ is the matrix of the form $\underline{\dot{v}}(t)$ in given coordinates, hence $\dot{S}_t = I$. Recall that $v^\circ(\tau) = \{(y^\top A_\tau, y + S_\tau A_\tau y) : y \in \mathbb{R}^n\}$, where $A_\tau = -\frac{1}{2}\dot{S}_\tau^{-1}\ddot{S}_\tau\dot{S}_\tau^{-1}$ (see (6)). Hence $\ddot{S}_t = 0$. We have: $R_v(t) = \frac{1}{2}\dddot{S}_t$, $r_v(t)(y) = \frac{1}{2}y^\top \dddot{S}_t\, y$,

$$\underline{\dot{v}}^\circ(t)(y) = \sigma\Big((0, y), (y^\top \dot{A}_t, 0)\Big) = -y^\top \dot{A}_t y = \frac{1}{2} y^\top \dddot{S}_t\, y.$$

So $r_v(t)$ and $\underline{\dot{v}}^\circ(t)$ have equal matrices for our choice of coordinates in $v(t)$ and $v^\circ(t)$. The curvature operator is self-adjoint since it is presented by a symmetric matrix in coordinates where form $\underline{\dot{v}}(t)$ is the standard inner product. $\square$

Proposition 5 implies that the curvature operators of regular monotone curves in the Lagrange Grassmannian are diagonalizable and have only real eigenvalues.

# 8 Canonical connection

Now we apply the developed theory of curves in the Grassmannian to the Jacobi curves $J_z(t)$ (see Sec. 1).

**Proposition 6** *All Jacobi curves $J_z(\cdot)$, $z \in N$, associated to the given Hamiltonian field $\zeta = \vec{h}$ are regular (monotone) if and only if the field $\zeta$ is regular (monotone).*

**Proof.** The definition of the regular (monotone) field is actually the specification of the definition of the regular (monotone) germ of the curve in the Lagrange Grassmannian: general definition is applied to the germs at $t = 0$ of the curves $t \mapsto J_z(t)$. What remains is to demonstrate that other germs of these curves are regular (monotone) as soon as the germs at 0 are. The latter fact follows from the identity

$$J_z(t + \tau) = e_*^{-t\zeta} J_{e^{t\zeta}(z)}(\tau) \tag{14}$$

(which, in turn, is an immediate corollary of the identity $e_*^{-(t+\tau)\zeta} = e_*^{-t\zeta} \circ e_*^{-\tau\zeta}$). Indeed, (14) implies that the germ of $J_z(\cdot)$ at $t$ is the image of the germ of $J_{e^{t\zeta}(\tau)}(\cdot)$ at 0 under the fixed linear symplectic transformation $e_*^{-t\zeta}$ : $T_{e^{t\zeta}(z)}N \to T_zN$. The properties of the germs to be regular or monotone survive symplectic transformations since they are intrinsic properties. □

Let $\zeta$ be a regular field. Then the derivative curves $J_z^\circ(t)$ are well-defined. Moreover, identity (14) and the fact that the construction of the derivative curve is intrinsic imply:

$$J_z^\circ(t) = e_*^{-t\zeta} J_{e^{t\zeta}(z)}^\circ(0). \tag{15}$$

The value at 0 of the derivative curve provides the splitting $T_zM = J_z(0) \oplus J_z^\circ(0)$ where, recall, $J_z(0) = \Delta_z$.

The subspaces $J_z^\circ(0) \subset T_zN,\ z \in N$, form a smooth vector distribution, which is the direct complement to the 'vertical' distribution $\Delta$. Direct complements to the vertical distribution are sometimes called Ehresmann connections (or just nonlinear connections, even if linear connections are their special cases). The Ehresmann connection $\Delta^\zeta = \{J_z^\circ(0) : z \in N\}$ is called the *canonical connection* associated with $\zeta$ and the correspondent splitting $TN = \Delta \oplus \Delta^\zeta$ is called the *canonical splitting*. Our nearest goal is to give a simple intrinsic characterization of $\Delta^\zeta$ which does not require the integration of the equation $\dot{z} = \zeta(z)$ and is suitable for calculations not only in local coordinates but also in moving frames.

Let $\Xi = \{\Xi_z \subset T_zN : z \in N\}$ be an Ehresmann connection. Given a vector field $\xi$ on $N$ we denote $\xi_{ver}(z) = \pi_{\Xi z \Delta_z}\xi,\ \xi_{hor}(z) = \pi_{\Delta_z \Xi_z}\xi$, the "vertical" and the "horizontal" parts of $\xi(z)$. Then $\xi = \xi_{ver} + \xi_{hor}$, where $\xi_{ver}$ is a section of the distribution $\Delta$ and $\xi_{hor}$ is a section of the distribution $\Xi$. In general, sections of $\Delta$ are called vertical fields and sections of $\Xi$ are called horizontal fields.

**Proposition 7** *An Ehresmann connection $\Xi$ is a canonical one, i.e. $\Sigma = \Delta^\zeta$, if and only if the equality*

$$[\zeta, [\zeta, \nu]]_{hor} = 2[\zeta, [\zeta, \nu]_{ver}]_{hor} \tag{16}$$

*holds for any vertical vector field $\nu$. Here $[\ ,\ ]$ is Lie bracket of vector fields.*

**Proof.** The deduction of identity (16) is based on the following classical expression:

$$\frac{d}{dt} e_*^{-t\zeta}\xi = e_*^{-t\zeta}[\zeta, \xi], \tag{17}$$

for any vector field $\xi$.

Given $z \in N$, we take coordinates in $T_zN$ in such a way that $T_zN = \{(x,y) : x,y \in \mathbb{R}^n\}$, where $J_z(0) = \{(x,0) : x \in \mathbb{R}^n\}$, $J_z^\circ(0) = \{(0,y) : y \in \mathbb{R}^n\}$. Let $J_z(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, then $S_0 = \ddot{S}_0 = 0$ and $\det \dot{S}_0 \neq 0$ due to the regularity of the Jacobi curve $J_z$.

Let $\nu$ be a vertical vector field, $\nu(z) = (x_0, 0)$ and $\left(e_*^{-t\zeta}\nu\right)(z) = (x_t, y_t)$. Then $(x_t, 0) = \left(e_*^{-t\zeta}\nu\right)_{ver}(z)$, $(0, y_t) = \left(e_*^{-t\zeta}\nu\right)_{hor}(z)$. Moreover, $y_t = S_t x_t$ since $\left(e_*^{-t\zeta}\nu\right)(z) \in J_z(t)$. Differentiating the identity $y_t = S_t x_t$ we obtain: $\dot{y}_t = \dot{S}_t x_t + S_t \dot{x}_t$. In particular, $\dot{y}_0 = \dot{S}_0 x_0$. It follows from (17) that $(\dot{x}_0, 0) = [\zeta, \nu]_{ver}$, $(0, \dot{y}_0) = [\zeta, \nu]_{hor}$. Hence $(0, \dot{S}_0 x_0) = [\zeta, \nu]_{hor}(z)$, where, I recall, $\nu$ is any vertical field. Now we differentiate once more and evaluate the derivative at 0:

$$\ddot{y}_0 = \ddot{S}_0 x_0 + 2\dot{S}_0 \dot{x}_0 + S_0 \ddot{x}_0 = 2\dot{S}_0 \dot{x}_0. \tag{18}$$

The Lie bracket presentations of the left and right hand sides of (18) are: $(0, \ddot{y}_0) = [\zeta, [\zeta, \nu]]_{hor}$, $(0, \dot{S}_0 \dot{x}_0) = [\zeta, [\zeta, \nu]_{ver}]_{hor}$. Hence (18) implies identity (16).

Assume now that $\{(0,y) : y \in \mathbb{R}^n\} \neq J_z^\circ(0)$; then $\ddot{S}_0 x_0 \neq 0$ for some $x_0$. Hence $\ddot{y}_0 \neq 2\dot{S}_0 \dot{x}_0$ and equality (16) is violated. $\square$

Equality (16) can be equivalently written in the following form that is often more convenient for the computations:

$$\pi_*[\zeta, [\zeta, \nu]](z) = 2\pi_*[\zeta, [\zeta, \nu]_{ver}](z), \quad \forall z \in N. \tag{19}$$

Let $R_{J_z}(t) \in \mathrm{gl}(J_z(t))$ be the curvature of the Jacobi curve $J_z(t)$. Identity (14) and the fact that construction of the Jacobi curve is intrinsic imply that

$$R_{J_z}(t) = e_*^{-t\zeta} R_{J_{e^{t\zeta}(z)}}(0) e_*^{t\zeta}\big|_{J_z(t)}.$$

Recall that $J_z(0) = \Delta_z$; the operator $R_{J_z}(0) \in \mathrm{gl}(\Delta_z)$ is called *the curvature operator of the field* $\zeta$ at $z$. We introduce the notation: $R_\zeta(z) \stackrel{def}{=} R_{J_z}(0)$; then $R_\zeta = \{R_\zeta(z)\}_{z \in E}$ is an endomorphism of the 'vertical' vector bundle $\Delta$.

**Proposition 8** *Assume that $TN = \Delta \oplus \Delta^\zeta$ is the canonical splitting. Then*

$$R_\zeta \nu = -[\zeta, [\zeta, \nu]_{hor}]_{ver} \tag{20}$$

*for any vertical field $\nu$.*

**Proof.** Recall that $R_{J_z}(0) = [\dot{J}_z^\circ(0), \dot{J}_z(0)]$, where $[\cdot,\cdot]$ is the infinitesimal cross–ratio (not the Lie bracket!). The presentation (11) of the infinitesimal cross–ratio implies:

$$R_\zeta(z) = R_{J_z}(0) = -\Phi_0^{J_z^\circ J_z}\Phi_0^{J_z J_z^\circ},$$

where $\Phi_0^{vw}e = \pi_{v(0)w(0)}\dot{e}_0$ for any smooth curve $e_\tau \in v(\tau)$ such that $e_0 = e$. Equalities (15) and (17) imply: $\Phi_0^{J_z J_z^\circ}\nu(z) = [\zeta,\nu]_{ver}(z),\ \forall z \in M$. Similarly, $\Phi_0^{J_z^\circ J_z}\mu(z) = [\zeta,\mu]_{hor}(z)$ for any horizontal field $\mu$ and any $z \in M$. Finally,

$$R_\zeta(z)\nu(z) = -\Phi_0^{J_z^\circ J_z}\Phi_0^{J_z J_z^\circ} = -[\zeta,[\zeta,\nu]_{hor}]_{ver}(z). \qquad \square$$

# 9 Coordinate presentation

We restrict ourselves to the case of the involutive Lagrange distribution $\Delta$; then integral submanifolds of $\Delta$ form a Lagrange foliation of $N$. According to the standard Darboux–Weinstein theorem (see [7]) all Lagrange foliations are locally equivalent. More precisely, this theorem states that any $z \in M$ possesses a neighborhood $O_z$ and local coordinates which turn the restriction of the Lagrange foliation to $O_z$ into the trivial bundle $\mathbb{R}^n \times \mathbb{R}^n = \{(x,y) : x,y \in \mathbb{R}^n\}$ and, simultaneously, turn $\sigma|_{O_z}$ into the form $\sum_{i=1}^n dx_i \wedge dy_i$. In this special coordinates, the fibers become coordinate subspaces $\mathbb{R}^n \times \{y\},\ y \in \mathbb{R}^n$. Below we use abridged notations: $\frac{\partial}{\partial x_i} = \partial_{x_i},\ \frac{\partial \varphi}{\partial x_i} = \varphi_{x_i}$ etc. We also use the standard summation agreement for repeating indices.

Now consider a Hamiltonian vector field $\zeta = -h_{y_i}\partial_{x_i} + h_{x_i}\partial_{y_i}$, where $h$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$ (a Hamiltonian). The field $\zeta$ is regular if and only if the matrix $h_{xx} = \left(h_{x_i x_j}\right)_{i,j=1}^n$ is non degenerate. We are going to characterize the canonical connection associated with $\zeta$.

Vector fields $\partial_{x_i},\ i = 1,\ldots,n$, provide a basis of the space of vertical fields. As soon as coordinates are fixed, any Ehresmann connection has a unique basis of the form:

$$(\partial_{y_i})_{hor} = \partial_{y_i} + c_i^j \partial_{x_j},$$

where $c_i^j,\ i,j = 1,\ldots,n$, are smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$. To characterize a connection in coordinates thus means to find functions $c_i^j$. In the case of the canonical connection of a regular vector field, the functions $c_i^j$ can be easily recovered from identity (19) applied to $\nu = \partial_{x_i},\ i = 1,\ldots,n$.

Let $C = \left(c_i^j\right)_{i,j=1}^n$; the straightforward computation reduces identity (19) to the following equalities:

$$2\left(h_{xx}Ch_{xx}\right)_{ij} = h_{x_k}h_{x_ix_jy_k} - h_{y_k}h_{x_ix_jx_k} - h_{x_iy_k}h_{x_kx_j} - h_{x_ix_k}h_{y_kx_j}$$

or, in the matrix form:

$$2h_{xx}Ch_{xx} = \{h, h_{xx}\} - h_{xy}h_{xx} - h_{xx}h_{yx},$$

where $\{h, h_{xx}\}$ is the Poisson bracket: $\{h, h_{xx}\}_{ij} = \{h, h_{x_ix_j}\} = h_{x_k}h_{x_ix_jy_k} - h_{y_k}h_{x_ix_jx_k}$.

Note that matrix $C$ is symmetric (indeed, $h_{xx}h_{yx} = (h_{xy}h_{xx})^\top$) that is a coordinate expression of the fact that canonical Ehresmann connection is a Lagrange distribution.

As soon as we found the canonical connection, formula (20) gives us the presentation of the curvature operator although the explicit coordinate expression can be bulky. Let us specify the vector field more. In the case of the Hamiltonian of a natural mechanical system in $\mathbb{R}^n$,

$$h(x, y) = \frac{1}{2}|x|^2 + U(y), \tag{21}$$

the canonical connection is trivial: $c_i^j = 0$; the matrix of the curvature operator is just $U_{yy}$.

Hamiltonian vector field associated to the Hamiltonian $h(x, y) = g^{ij}(y)x_ix_j$ with a non degenerate symmetric matrix $(g^{ij})_{i,j=1}^n$ generates a (pseudo-)Riemannian geodesic flow. Canonical connection in this case is classical Levi Civita connection and the curvature operator is Ricci operator of (pseudo-)Riemannian geometry (see [2, Sec. 5] for details). Finally, Hamiltonian $h(x, y) = g^{ij}(y)x_ix_j + U(y)$ (the energy of a natural mechanical system on the (pseudo-)Riemannian manifold) has the same connection as Hamiltonian $h(x, y) = g^{ij}(y)x_ix_j$ while its curvature operator is sum of Ricci operator and second covariant derivative of $U$.

More generally, let $\vec{h}$ be a regular Hamiltonian field on $T^*M$ and $U$ a smooth function on $M$ (a potential). We can treat $U$ as a constant on the fibers function on $T^*M$. Then $\vec{U}$ is a vertical field, the Hamiltonian $h + U$ has the same canonical connection as $h$, while $R_{\vec{h}+\vec{U}}\nu = R_{\vec{h}} - [\vec{u}, [\vec{h}, \nu]_{hor}]_{ver}$ according to the formula (20). The second term of this sum can be seeing as the second covariant derivative of $U$ in virtue of given Ehresmann connection.

# 10 Reduction

We consider a Hamiltonian system $\dot{z} = \vec{h}(z)$ on a symplectic manifold $N$ endowed with a fixed Lagrange distribution $\Delta$. Assume that $g : N \to \mathbb{R}$ is a first integral of our Hamiltonian system, i.e. $\{h, g\} = 0$.

**Lemma 7** *Let $z \in N,\ g(z) = c$. The subspace $\Delta_z$ is transversal to $g^{-1}(c)$ at $z$ if and only if $\vec{g}(z) \notin \Delta_z$.*

**Proof.** Hypersurface $g^{-1}(c)$ is not transversal to $\Delta_z$ at $z$ if and only if

$$d_z g(\Delta_z) = 0 \ \Leftrightarrow\ \sigma(\vec{g}(z), \Delta_z) = 0 \ \Leftrightarrow\ \vec{g}(z) \in \Delta_z^{\angle} = \Delta_z,$$

where $\mathcal{A}^{\angle}$ is orthogonal complement with respect to the symplectic form of the subset $\mathcal{A}$ in the symplectic space. $\square$

If all points of some level $g^{-1}(c)$ satisfy conditions of Lemma 7, then $g^{-1}(c)$ is a $(2n-1)$-dimensional manifold endowed with the rank $(n-1)$ distribution $\Delta_z^g \stackrel{def}{=} T_z\left(E_z \cap g^{-1}(c)\right)$. Note that $\mathbb{R}\vec{g}(z) = \ker \sigma\big|_{T_z g^{-1}(c)}$, hence $\Sigma_z^g \stackrel{def}{=} T_z g^{-1}(c)/\mathbb{R}\vec{g}(z)$ is a $2(n-1)$-dimensional symplectic space and $\Delta_z^g$ is a Lagrangian subspace in $\Sigma_z^g$, i.e. $\Delta_z^g \in L(\Sigma_z^g)$.

The submanifold $g^{-1}(c)$ is invariant for the flow $e^{t\vec{h}}$. Moreover, $e_*^{t\vec{h}}\vec{g} = \vec{g}$. Hence $e_*^{t\vec{h}}$ induces a symplectic transformation $e_*^{t\vec{h}} : \Sigma_z^g \to \Sigma_{e^{t\vec{h}}(z)}^g$. Set $J_z^g(t) = e_*^{-t\vec{h}}\Delta_{e^{t\vec{h}}(z)}^g$. The curve $t \mapsto J_z^g(t)$ in the Lagrange Grassmannian $L(\Sigma_z^g)$ is called a *reduced Jacobi curve* for the Hamiltonian field $\vec{h}$ at $z \in N$.

The reduced Jacobi curve can be easily reconstructed from the Jacobi curve $J_z(t) = e_*^{-t\vec{h}}\Delta_{e^{t\vec{h}}(z)} \in L(T_z N)$ and vector $\vec{g}(z)$. An elementary calculation shows that

$$J_z^g(t) = J_z(t) \cap \vec{g}(z)^{\angle} + \mathbb{R}\vec{g}(z).$$

Now we can temporary forget the symplectic manifold and Hamiltonians and formulate everything in terms of the curves in the Lagrange Grassmannian. So let $v(\cdot)$ be a smooth curve in the Lagrange Grassmannian $L(\Sigma)$ and $\gamma$ a one-dimensional subspace in $\Sigma$. We set $v^{\gamma}(t) = v(t) \cap \gamma^{\angle} + \gamma$, a Lagrange subspace in the symplectic space $\gamma^{\angle}/\gamma$. If $\gamma \not\subset v(t)$, then $v^{\gamma}(\cdot)$ is smooth and $\underline{\dot{v}}^{\gamma}(t) = \underline{\dot{v}}(t)\big|_{\gamma(t)\cap\gamma^{\angle}}$ as it easily follows from the definitions. In particular, regularity and monotonicity of $\Lambda(\cdot)$ implies regularity and monotonicity of $v^{\gamma}(\cdot)$. The curvatures of $v(\cdot)$ and $v^{\gamma}(\cdot)$ are related in a more complicated way.

**Theorem 1** *Let $v(t),\ t \in [t_0, t_1]$ be a regular monotone curve in $L(\Sigma)$ and $\gamma$ a one-dimensional subspace of $\Sigma$ such that $\gamma \not\subset v(t),\ \forall t \in [t_0, t_1]$. Then $r_{v^\gamma}(t) \geq r_v(t)\big|_{v(t)\cap\gamma^\angle}$ and $\operatorname{rank}\left(r_{v^\gamma}(t) - r_v(t)\big|_{v(t)\cap\gamma^\angle}\right) \leq 1$. More precisely,*

$$r_{v^\gamma}(t)(x) = r_v(t)(x) + \frac{3\sigma(\ddot{a}_e(t), x)^2}{4\sigma(a_e(t), e)}, \quad x \in v(t) \cup \gamma^\angle,$$

*where $e \in \gamma \setminus 0$ and vector $a_e(t) \in v(t)$ is defined by the relation $(\dot{a}_e(t) - e) \in v(t)$.*

**Proof.** It is sufficient to compute curvatures at $t = 0$. Take coordinates in $\Sigma$ in such a way that $\Sigma \cong \mathbb{R}^{n*} \times \mathbb{R}^n = \{(\eta, y) : \eta \in \mathbb{R}^{n*}, y \in \mathbb{R}^n\},\ v(0) = \mathbb{R}^{n*} \times \{0\}$ and $\gamma$ is a coordinate axis in $\{0\} \times \mathbb{R}^n$. Then $v(t) = \{(y^\top, S_t y) : y \in \mathbb{R}^n\}$, $v^\gamma(t) = \{(y_\gamma^\top, S_t^\gamma y_\gamma) : y_\gamma \in \mathbb{R}^{n-1}\}$, where $S_t^\gamma$ is obtained from $S_t$ by the elimination of the row and column corresponding to the axis $\gamma$. We may also assume that $v(\cdot)$ is monotone increasing and $\dot{S}_0 = I$. The coordinate expression of the curvature via the Schwartzian derivative implies:

$$r_v(0)(y) = \frac{1}{2}\langle \dddot{S}_0\, y, y\rangle - \frac{3}{4}\langle \ddot{S}_0 y, \ddot{S}_0 y\rangle,$$

$$r_{v^\gamma}(0)(y_\gamma) = \frac{1}{2}\langle \dddot{S}_0^\gamma\, y_\gamma, y_\gamma\rangle - \frac{3}{4}\langle \ddot{S}_0^\gamma y_\gamma, \ddot{S}_0^\gamma y_\gamma\rangle.$$

It is now obvious that $r_{v^\gamma}(0) - r_v(0)\big|_{v(t)\cap\gamma^\angle}$ is a nonnegative quadratic form whose rank is not greater than 1. Let us write this form in the way which does not depend on the special choice of the basis in $\mathbb{R}^n$:

$$r_{v^\gamma}(0)(y) - r_v(0)(y) = \frac{3\langle \ddot{S}_0 a_0, y\rangle^2}{4\langle a_0, e\rangle},$$

where $e$ is any nonzero element of the line $\gamma \subset \mathbb{R}^n$ and $a_0 = \dot{S}_0^{-1} e$.

Moreover, we have: $a_e(t) = \left(y_e^\top(t), S_t y_e(t)\right)$, where $y_e(0) = a_0,\ \ddot{S} a_0 = -\dot{S}_0 \dot{y}_e(0)$. Hence $(\ddot{a}(0) + \ddot{S}_0 a_0) \in v(0)$ and $\langle \ddot{S} a_0, y\rangle = -\sigma(\ddot{a}_e(0), y),\ \langle a_0, e\rangle = \sigma(a_e(0), e)$. $\square$

The inequality $r_{v^\gamma}(t) \geq r_v(t)\big|_{v(t)\cap\gamma^\angle}$ turns into the equality if $\gamma \subset v^\circ(t),\ \forall t$. Then $\gamma \subset \ker \underline{\ddot{v}}^\circ(t)$. According to Proposition 5, to $\gamma$ there corresponds a one-dimensional subspace in the kernel of $r_v(t)$; in particular, $r_v(t)$ is degenerate.

Return to the Jacobi curves $J_z(t)$ of a monotone Hamiltonian field $\vec{h}$. Quadratic form $r_z \stackrel{def}{=} r_{J_z}(0)$ on $\Delta_z$ is called *the curvature form of* $h$ at $z \in N$. We have, $r_z(\xi) = \beta_z^h(R_z\xi, \xi)\operatorname{sgn}\beta^h$, $\xi \in \Delta_z$, where $\beta_z^h$ is the symmetric bilinear form defined in Section 1. Let $g$ be a first integral of $\vec{h}$ and $d_z g \neq 0$; then the monotonicity of $\vec{h}$ implies $\vec{g} \notin \Delta_z$. Quadratic form $r_z^g \stackrel{def}{=} r_{J_z^g}(0)$ is called the *reduced by* $g$ *curvature form of* $h$ at $z$. The following identity is a simple corollary of Theorem 1:

$$r_z^g(\xi) = r_z(\xi) + \frac{3\sigma_z([\vec{h}, [\vec{h}, \bar{a}]](z), \xi)^2}{4|\beta_z^h(\bar{a}, \bar{a})|}, \quad \xi \in \Delta_z \cap \ker d_z h,$$

where the vector field $\bar{a}$ is determined by the identity $\beta(\xi, \bar{a}) = \sigma(\xi, \vec{g})$, $\forall \xi \in \Delta$.

There always exists at least one first integral: the Hamiltonian $h$ itself. In general, $\vec{h}(z) \notin J_z^\circ(0)$ and the reduction procedure has a nontrivial influence on the curvature. For instance, in the case of a natural mechanical system in $\mathbb{R}^n$ (see (21)) we obtain: $r_{(x,y)}^h(\xi) = r_{(x,y)}(\xi) + \frac{3}{|x|^2}\left\langle \frac{dU}{dy}, \xi \right\rangle^2$. More generally, for a natural mechanical system on the Riemannian manifold, with the potential energy $U$ we have:

$$r_{x,y}^h(\xi) = r_{x,y}(\xi) + \frac{3\beta^h(d_y U, \xi)^2}{2(h(x,y) - U(y))} \tag{22}$$

Still, there is an important class of Hamiltonians and Lagrange foliations for which the relation $\vec{h}(z) \in J_z^\circ(0)$ holds $\forall z$. These are homogeneous on fibers Hamiltonians on cotangent bundles (in particular, those generating Riemannian or Finsler geodesic flows). In this case the generating homotheties of the fibers Euler vector field belongs to the kernel of the curvature form.

# 11 Hyperbolicity

In this section we show that negativity of the curvature forms implies hyperbolic behavior of the flow generated by the monotone Hamiltonian field. This is a natural extension of the classical result about hyperbolicity of the Riemannian geodesic flow in the case of the negative sectional curvatures.

Main tool is the structural equation derived in Section 6. First we'll show that this equation is well coordinated with the symplectic structure.

Let $\Lambda(t)$, $t \in \mathbb{R}$, be a regular curve in $L(\Sigma)$ and $\Sigma = \Lambda(t) \oplus \Lambda^\circ(t)$ the correspondent canonical splitting. Consider the structural equation

$$\ddot{e}(t) + R_\Lambda(t)e(t) = 0, \quad \text{where } e(t) \in \Lambda(t),\ \dot{e}(t) \in \Lambda^\circ(t), \tag{23}$$

(see Corollary 1).

**Lemma 8** *The mapping $e(0)\oplus\dot{e}(0) \mapsto e(t)\oplus\dot{e}(t)$, where $e(\cdot)$ and $\dot{e}(\cdot)$ satisfies (23), is a symplectic transformation of $\Sigma$.*

**Proof.** We have to check that $\sigma(e_1(t), e_2(t))$, $\sigma(\dot{e}_1(t), \dot{e}_2(t))$, $\sigma(e_1(t), \dot{e}_2(t))$ do not depend on $t$ as soon as $e_i(t), \dot{e}_i(t)$, $i = 1, 2$, satisfy (23). First two quantities vanish since $\Lambda(t)$ and $\Lambda^\circ(t)$ are Lagrangian subspaces. The derivative of the third quantity vanishes as well since $\ddot{e}_i(t) \in \Lambda(t)$. $\square$

Let $\vec{h}$ be a regular monotone field on the symplectic manifold $N$ equipped with a Lagrange distribution $\Delta$. As before, we denote by $J_z(t)$ the Jacobi curves of $\vec{h}$ and by $J_z^h(t)$ the reduced to the level of $h$ Jacobi curves (see previous Section). Let $R(z) = R_{J_z}(0)$, $R^h(z) = R_{J_z^h}(0)$ be the correspondent curvature operators and $r_z$, $r_z^h$ the curvature forms. We say that the Hamiltonian field $\vec{h}$ has a negative curvature at $z$ (with respect to $\Delta$) if all eigenvalues of $R(z)$ are negative or, equivalently, $r_z < 0$. We say that $\vec{h}$ has a negative reduced curvature at $z$ if all eigenvalues of $R_z^h$ are negative, in other words, if $r_z^h < 0$.

**Proposition 9** *Let $z_0 \in N$, $z_t = e^{t\vec{h}}(z)$. Assume that $\overline{\{z_t : t \in \mathbb{R}\}}$ is a compact subset of $N$ and that $N$ is endowed with a Riemannian structure. If $\vec{h}$ has a negative curvature at any $z \in \overline{\{z_t : t \in \mathbb{R}\}}$, then there exist a constant $\alpha > 0$ and a splitting $T_{z_t}N = \Delta_{z_t}^+ \oplus \Delta_{z_t}^-$, where $\Delta_{z_t}^\pm$ are Lagrangian subspaces of $T_{z_t}N$ such that $e_*^{\tau\vec{h}}(\Delta_{z_t}^\pm) = \Delta_{z_{t+\tau}}^\pm$ $\forall\, t, \tau \in \mathbb{R}$ and*

$$\|e_*^{\pm\tau\vec{h}}\zeta_\pm\| \geq e^{\alpha\tau}\|\zeta_\pm\| \quad \forall\, \zeta \in \Delta_{z_t}^\pm,\ \tau \geq 0,\ t \in \mathbb{R}. \tag{24}$$

*Similarly, if $\vec{h}$ has a negative reduced curvature at any $z \in \overline{\{z_t : t \in \mathbb{R}\}}$, then there exists a splitting $T_{z_t}(h^{-1}(c)/\mathbb{R}h(z_t)) = \hat{\Delta}_{z_t}^+ \oplus \hat{\Delta}_{z_t}^-$, where $c = h(z_0)$ and $\hat{\Delta}_{z_t}^\pm$ are Lagrangian subspaces of $T_{z_t}(h^{-1}(c)/\mathbb{R}h(z_t))$ such that $e_*^{\tau\vec{h}}(\hat{\Delta}_{z_t}^\pm) = \hat{\Delta}_{z_{t+\tau}}^\pm$ $\forall\, t, \tau \in \mathbb{R}$ and $\|e_*^{\pm\tau\vec{h}}\zeta_\pm\| \geq e^{\alpha\tau}\|\zeta_\pm\|$ $\forall\, \zeta \in \hat{\Delta}_{z_t}^\pm$, $\tau \geq 0$, $t \in \mathbb{R}$.*

**Proof.** Obviously, the desired properties of $\Delta^{\pm}_{z_t}$ and $\hat{\Delta}^{\pm}_{z_t}$ do not depend on the choice of the Riemannian structure on $N$. We'll introduce a special Riemannian structure determined by $h$. The Riemannian structure is a smooth family of inner products $\langle\cdot,\cdot\rangle_z$ on $T_zN$, $z\in N$. We have $T_zN = J_z(0)\oplus J^{\circ}_z(0)$, where $J_z(0)=\Delta_z$. Replacing $h$ with $-h$ if necessary we may assume that $\beta^h_z$ is a positive definite form and set $\langle\cdot,\cdot\rangle_z\big|_{J_z(0)}=\beta^h_z$. Symplectic form $\sigma$ induces a nondegenerate pairing of $J_z(0)$ and $J^{\circ}_z(0)$. In particular, for any $\zeta\in J_z(0)$ there exists a unique $\zeta^{\circ}\in J^{\circ}_z(0)$ such that $\sigma(\zeta^{\circ},\cdot)\big|_{J_z(0)}=\langle\zeta,\cdot\rangle_z\big|_{J_z(0)}$. There exists a unique extension of the inner product $\langle\cdot,\cdot\rangle_z$ from $J_z(0)$ to the whole $T_zN$ with the following properties:

- $J^{\circ}_z(0)$ is orthogonal to $J_z(0)$ with respect to $\langle\cdot,\cdot\rangle_z$;
- $\langle\zeta_1,\zeta_2\rangle_z=\langle\zeta_1^{\circ},\zeta_2^{\circ}\rangle_z\ \forall\,\zeta_1,\zeta_2\in J_z(0)$.

We'll need the following classical fact from Hyperbolic Dynamics (see, for instance, [10, Sec. 17.6]).

**Lemma 9** *Let $A(t)$, $t\in\mathbb{R}$, be a bounded family of symmetric $n\times n$-matrices whose eigenvalues are all negative and uniformly separated from 0. Let $\Gamma(t,\tau)$ be the fundamental matrix of the $2n$-dimensional linear system $\dot{x}=-y$, $\dot{y}=A(t)x$, where $x,y\in\mathbb{R}^n$, i.e.*

$$\frac{\partial}{\partial t}\Gamma(t,\tau)=\begin{pmatrix}0&-I\\A&0\end{pmatrix}\Gamma(t,\tau),\quad \Gamma(\tau,\tau)=\begin{pmatrix}I&0\\0&I\end{pmatrix}. \tag{25}$$

*Then there exist closed conic neighborhoods $C^+_\Gamma$, $C^-_\Gamma$, where $C^+_\Gamma\cap C^-_\Gamma=0$, of some $n$-dimensional subspaces of $\mathbb{R}^{2n}$ and a constant $\alpha>0$ such that*

$$\Gamma(t,\tau)C^+_\Gamma\subset C^+_\Gamma,\quad |\Gamma(t,\tau)\xi_+|\geq e^{\alpha(\tau-t)}|\xi_+|,\ \forall\,\xi_+\in C^+_\Gamma,\ t\leq\tau,$$

*and*

$$\Gamma(t,\tau)C^-_\Gamma\subset C^-_\Gamma,\quad |\Gamma(t,\tau)\xi_-|\geq e^{\alpha(t-\tau)}|\xi_-|,\ \forall\,\xi_-\in C^-_\Gamma,\ t\geq\tau.$$

*The constant $\alpha$ depends only on upper and lower bounds of the eigenvalues of $A(t)$.* □

**Corollary 5** *Let $C^{\pm}_\Gamma$ be the cones described in Lemma 9; then $\Gamma(0,\pm t)C^{\pm}_\Gamma\subset\Gamma(0;\pm\tau)C^{\pm}_\Gamma$ for any $t\geq\tau\geq 0$, and the subsets $K^{\pm}_\Gamma=\bigcap\limits_{t\geq0}\Gamma(0,t)C^{\pm}_\Gamma$ are Lagrangian subspaces of $\mathbb{R}^n\times\mathbb{R}^n$ equipped with the standard symplectic structure.*

**Proof.** The relations $\Gamma(\tau,t)C_\Gamma^+ \subset C_\Gamma^+$ and $\Gamma(-\tau,-t)C_\Gamma^- \subset C_\Gamma^-$ imply:

$$\Gamma(0,\pm t)C_\Gamma^\pm = \Gamma(0,\pm\tau)\Gamma(\pm\tau,\pm t)C_\Gamma^\pm \subset \Gamma(0,\pm\tau)C_\Gamma^\pm.$$

In what follows we'll study $K_\Gamma^+$; the same arguments work for $K_\Gamma^-$. Take vectors $\zeta,\zeta' \in K_\Gamma^+$; then $\zeta = \Gamma(0,t)\zeta_t$ and $\zeta' = \Gamma(0,t)\zeta_t'$ for any $t \geq 0$ and some $\zeta_t,\zeta_t' \in C_\Gamma^+$. Then, according to Lemma 9, $|\zeta_t| \leq e^{-\alpha t}|\zeta|$, $|\zeta_t'| \leq e^{-\alpha t}|\zeta'|$, i.e. $\zeta_t$ and $\zeta_t'$ tend to 0 as $t \to +\infty$. On the other hand,

$$\sigma(\zeta,\zeta') = \sigma(\Gamma(0,t)\zeta_t,\Gamma(0,t)\zeta_t') = \sigma(\zeta_t,\zeta_t') \quad \forall t \geq 0$$

since $\Gamma(0,t)$ is a symplectic matrix. Hence $\sigma(\zeta,\zeta') = 0$.

We have shown that $K_\Gamma^+$ is an isotropic subset of $\mathbb{R}^n \times \mathbb{R}^n$. On the other hand, $K_\Gamma^+$ contains an $n$-dimensional subspace since $C_\Gamma^+$ contains one and $\Gamma(0,t)$ are invertible linear transformations. Isotropic $n$-dimensional subspace is equal to its skew-orthogonal complement, therefore $K_\Gamma^+$ is a Lagrangian subspace. $\square$

Take now a regular monotone curve $\Lambda(t)$, $t \in \mathbb{R}$ in the Lagrange Grassmannian $L(\Sigma)$. We may assume that $\Lambda(\cdot)$ is monotone increasing, i.e. $\dot\Lambda(t) > 0$. Recall that $\underline{\dot\Lambda}(t)(e(t)) = \sigma(e(t),\dot e(t))$, where $e(\cdot)$ is an arbitrary smooth curve in $\Sigma$ such that $e(\tau) \in \Lambda(\tau)$, $\forall\tau$. Differentiation of the identity $\sigma(e_1(\tau),e_2(\tau)) = 0$ implies: $\sigma(e_1(t),\dot e_2(t)) = -\sigma(\dot e_1(t),e_2(t)) = \sigma(e_2(t),\dot e_1(t))$ if $e_i(\tau) \in \Lambda(\tau)$, $\forall\tau$, $i = 1,2$. Hence the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot\Lambda(t)}$ defined by the quadratic form $\underline{\dot\Lambda}(t)$ reads: $\langle e_1(t),e_2(t)\rangle_{\dot\Lambda(t)} = \sigma(e_1(t),\dot e_2(t))$.

Take a basis $e_1(0),\dots,e_n(0)$ of $\Lambda(0)$ such that the form $\underline{\dot\Lambda}(0)$ has the unit matrix in this basis, i.e. $\sigma(e_i(0),\dot e_j(0)) = \delta_{ij}$. In fact, vectors $\dot e_j(0)$ are defined modulo $\Lambda(0)$; we can normalize them assuming that $\dot e_i(0) \in \Lambda^\circ(0)$, $i = 1,\dots,n$. Then $e_1(0),\dots,e_n(0),\dot e_1(0),\dots,\dot e_n(0)$ is a Darboux basis of $\Sigma$. Fix coordinates in $\Sigma$ using this basis: $\Sigma = \mathbb{R}^n \times \mathbb{R}^n$, where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^n \times \mathbb{R}^n$ is identified with $\sum\limits_{j=1}^n (x^j e_j(0) + y^j \dot e_j(0)) \in \Sigma$, $x = (x^1,\dots,x^n)^\top$, $y = (y^1,\dots,y^n)^\top$.

We claim that there exists a smooth family $A(t)$, $t \in \mathbb{R}$, of symmetric $n \times n$ matrices such that $A(t)$ has the same eigenvalues as $R_\Lambda(t)$ and

$$\Lambda(t) = \Gamma(0,t)\left(\begin{smallmatrix} \mathbb{R}^n \\ 0 \end{smallmatrix}\right), \quad \Lambda^\circ(t) = \Gamma(0,t)\left(\begin{smallmatrix} 0 \\ \mathbb{R}^n \end{smallmatrix}\right), \quad \forall t \in \mathbb{R}$$

in the fixed coordinates, where $\Gamma(t,\tau)$ satisfies (25). Indeed, let $e_i(t)$, $i = 1,\dots,n$, be solutions to the structural equation (23). Then

$$\Lambda(t) = span\{e_1(t),\dots,e_n(t)\}, \quad \Lambda^\circ(t) = span\{\dot e_1(t),\dots,\dot e_n(t)\}.$$

Moreover, $\ddot{e}_i(t) = -\sum\limits_{i=1}^{n} a_{ij}(t)e_j(t)$, where $A(t) = \{a_{ij}(t)\}_{i,j=1}^{n}$ is the matrix of the operator $R_\Lambda(t)$ in the 'moving' basis $e_1(t), \dots, e_n(t)$. Lemma 8 implies that $\langle e_i(t), e_j(t)\rangle_{\dot{\Lambda}(t)} = \sigma(e_i(t), \dot{e}_j(t)) = \delta_{ij}$. In other words, the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ has unit matrix in the basis $e_1(t), \dots, e_n(t)$. Operator $R_\Lambda(t)$ is self-adjoint for the Euclidean structure $\langle\cdot,\cdot\rangle_{\dot{\Lambda}(t)}$ (see Proposition 5). Hence matrix $A(t)$ is symmetric.

Let $e_i(t) = \left(\begin{smallmatrix} x_i(t) \\ y_i(t) \end{smallmatrix}\right) \in \mathbb{R}^n \times \mathbb{R}^n$ in the fixed coordinates. Make up $n \times n$-matrices $X(t) = (x_1(t), \dots, x_n(t))$, $Y(t) = (y_1(t), \dots, y_n(t))$ and a $2n \times 2n$-matrix $\left(\begin{smallmatrix} X(t) & \dot{X}(t) \\ Y(t) & \dot{Y}(t) \end{smallmatrix}\right)$. We have

$$\frac{d}{dt}\begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(t) = \begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(t)\begin{pmatrix} 0 & -A(t) \\ I & 0 \end{pmatrix}, \quad \begin{pmatrix} X & \dot{X} \\ Y & \dot{Y} \end{pmatrix}(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence $\left(\begin{smallmatrix} X & \dot{X} \\ Y & \dot{Y} \end{smallmatrix}\right)(t) = \Gamma(t,0)^{-1} = \Gamma(0,t)$.

Let now $\Lambda(\cdot)$ be the Jacobi curve, $\Lambda(t) = J_{z_0}(t)$. Set $\xi_i(z_t) = e_*^{t\vec{h}} e_i(t)$, $\eta_i(z_t) = e_*^{t\vec{h}}\dot{e}_i(t)$; then

$$\xi_1(z_t), \dots, \xi_n(z_t), \eta_1(z_t), \dots, \eta_n(z_t) \tag{26}$$

is a Darboux basis of $T_{z_t}N$, where $J_{z_t}(0) = span\{\xi_1(z_t), \dots, \xi_n(z_t)\}$, $J^{\circ}_{z_t}(0) = span\{\eta_1(z_t), \dots, \eta_n(z_t)\}$. Moreover, the basis (26) is orthonormal for the inner product $\langle\cdot,\cdot\rangle_{z_t}$ on $T_{z_t}N$.

The intrinsic nature of the structural equation implies the translation invariance of the construction of the frame (26): if we would start from $z_s$ instead of $z_0$ and put $\Lambda(t) = J_{z_s}(t)$, $e_i(0) = \xi_i(z_s)$, $\dot{e}_i(0) = \eta_i(z_s)$ for some $s \in \mathbb{R}$, then we would obtain $e_*^{t\vec{h}}e_i(t) = \xi_i(z_{s+t})$, $e_*^{t\vec{h}}\dot{e}_i(t) = \eta_i(z_{s+t})$.

The frame (26) gives us fixed orthonormal Darboux coordinates in $T_{z_s}N$ for $\forall\, s \in \mathbb{R}$ and the correspondent symplectic $2n \times 2n$-matrices $\Gamma_{z_s}(\tau,t)$. We have: $\Gamma_{z_s}(\tau,t) == \Gamma_{z_0}(s+\tau, s+t)$; indeed, $\Gamma_{z_s}(\tau,t)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is the coordinate presentation of the vector

$$e_*^{(\tau-t)\vec{h}} \sum_i \left(x^i\xi^i(z_{s+t}) + y^i\eta_i(z_{s+t})\right)$$

in the basis $\xi_i(z_{s+\tau})$, $\eta_i(z_{s+\tau})$. In particular,

$$|\Gamma_{z_s}(0,t)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| = \left\| e_*^{-t\vec{h}} \sum_i \left(x^i\xi^i(z_{s+t}) + y^i\eta_i(z_{s+t})\right)\right\|_{z_s}. \tag{27}$$

Recall that $\xi_1(z_\tau), \ldots, \xi_n(z_\tau), \eta_1(z_\tau), \ldots, \eta_n(z_\tau)$ is an orthonormal frame for the scalar product $\langle\cdot,\cdot\rangle_{z_\tau}$ and $\|\zeta\|_{z_\tau} = \sqrt{\langle\zeta,\zeta\rangle_{z_\tau}}$.

We introduce the notation:

$$\lfloor W \rfloor_{z_s} = \left\{ \sum_i \left( x^i \xi^i(z_s) + y^i \eta_i(z_s) \right) : \left( \begin{smallmatrix} x \\ y \end{smallmatrix} \right) \in W \right\},$$

for any $W \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $C^\pm_{\Gamma_{z_0}}$ be the cones from Lemma 9. Then

$$e_*^{-\tau\vec{h}} \lfloor \Gamma_{z_s}(0,t) C^\pm_{\Gamma_{z_0}} \rfloor_{z_{s-\tau}} = \lfloor \Gamma_{z_{s-\tau}}(0, t+\tau) C^\pm_{\Gamma_{z_0}} \rfloor_{z_{s-\tau}}, \quad \forall\, t, \tau, s. \tag{28}$$

Now set $K^+_{\Gamma_{z_s}} = \bigcap_{t \geq 0} C^+_{\Gamma_{z_0}}$, $K^-_{\Gamma_{z_s}} = \bigcap_{t \leq 0} C^-_{\Gamma_{z_0}}$ and $\Delta^\pm_{z_s} = \lfloor K^\mp_{\Gamma_{z_s}} \rfloor_{z_s}$. Corollary 5 implies that $\Delta^\pm_{z_s}$ are Lagrangian subspaces of $T_{z_s}N$. Moreover, it follows from (28) that $e_*^{t\vec{h}} \Delta^\pm_{z_s} = \Delta^\pm_{z_{s+t}}$, while (28) and (27) imply inequalities (24).

This finishes the proof of the part of Proposition 9 which concerns Jacobi curves $J_z(t)$. We leave to the reader a simple adaptation of this proof to the case of reduced Jacobi curves $J^h_z(t)$. $\square$

**Remark.** Constant $\alpha$ depends, of course, on the Riemannian structure on $N$. In the case of the special Riemannian structure defined at the beginning of the proof of Proposition 9 this constant depends only on the upper and lower bounds for the eigenvalues of the curvature operators and reduced curvature operators correspondently (see Lemma 9 and further arguments).

Let $e^{tX}$ $, t \in \mathbb{R}$ be the flow generated by the the vector field $X$ on a manifold $M$. Recall that a compact invariant subset $W \subset M$ of the flow $e^{tX}$ is called a hyperbolic set if there exists a Riemannian structure in a neighborhood of $W$, a positive constant $\alpha$, and a continuous splitting $T_zM = E^+_z \oplus E^-_z \oplus \mathbb{R}X(z)$, $z \in W$, such that $e_*^{tX} E^\pm_z = E^\pm_{e^{tX}(z)}$ and $\|e_*^{\pm tX}\zeta^\pm\| \geq e^{\alpha t}\|\zeta^\pm\|$, $\forall t \geq 0$, $\zeta^\pm \in E^\pm_z$. Just the fact some invariant set is hyperbolic implies a rather detailed information about asymptotic behavior of the flow in a neighborhood of this set (see [10] for the introduction to Hyperbolic Dynamics). The flow $e^{tX}$ is called an Anosov flow if the entire manifold $M$ is a hyperbolic set.

The following result is an immediate corollary of Proposition 9 and the above remark.

**Theorem 2** *Let $\vec{h}$ be a regular monotone field on $N$, $c \in \mathbb{R}$, $W \subset h^{-1}(c)$ a compact invariant set of the flow $e^{t\vec{h}}$, $t \in \mathbb{R}$, and $d_zh \neq 0$, $\forall z \in W$. If $\vec{h}$ has* *a negative reduced curvature at every point of* $W$*, then* $W$ *is a hyperbolic set of the flow* $e^{t\vec{h}}\big|_{h^{-1}(c)}$. $\square$

This theorem generalizes a classical result about geodesic flows on compact Riemannian manifolds with negative sectional curvatures. Indeed, if $N$ is the cotangent bundle of a Riemannian manifold and $e^{t\vec{h}}$ is the geodesic flow, then negativity of the reduced curvature of $\vec{h}$ means simply negativity of the sectional Riemannian curvature. In this case, the Hamiltonian $h$ is homogeneous on the fibers of the cotangent bundle and the restrictions $e^{t\vec{h}}\big|_{h^{-1}(c)}$ are equivalent for all $c > 0$.

The situation changes if $h$ is the energy function of a general natural mechanical system on the compact Riemannian manifold. In this case, the flow and the reduced curvature depend on the energy level. Still, negativity of the sectional curvature implies negativity of the reduced curvature at $h^{-1}(c)$ for all sufficiently big $c$. In particular, $e^{t\vec{h}}\big|_{h^{-1}(c)}$ is an Anosov flow for any sufficiently big $c$; this can be easily derived from formula (22).

Theorem 2 concerns only the reduced curvature while the next result deals with the (not reduced) curvature of $\vec{h}$.

**Theorem 3** *Let* $h$ *be a regular monotone Hamiltonian and* $W$ *a compact invariant set of the flow* $e^{t\vec{h}}$*. If* $\vec{h}$ *has a negative curvature at any point of* $W$*, then* $W$ *is a finite set and each point of* $W$ *is a hyperbolic equilibrium of the field* $\vec{h}$.

**Proof.** Let $z \in W$; the trajectory $z_t = e^{t\vec{h}}(z),\ t \in \mathbb{R}$, satisfies conditions of Proposition 9. Take the correspondent splitting $T_{z_t}N = \Delta^+_{z_t} \oplus \Delta^-_{z_t}$. In particular, $\vec{h}(z_t) = \vec{h}^+(z_t) + \vec{h}^-(z_t)$, where $\vec{h}^\pm(z_t) \in \Delta^\pm_{z_t}$.

We have $e^{\tau\vec{h}}_*\vec{h}(z_t) = \vec{h}(z_{t+\tau})$. Hence

$$\|\vec{h}(z_{t+\tau})\| = \|e^{\tau\vec{h}}_*\vec{h}(z_t)\| \geq \|e^{\tau\vec{h}}_*\vec{h}^+(z_t)\| - \|e^{\tau\vec{h}}_*\vec{h}^-(z_t)\|$$

$$\geq e^{\alpha\tau}\|\vec{h}^+(z_t)\| - e^{-\alpha\tau}\|\vec{h}^-(z_t)\|, \quad \forall \tau \geq 0.$$

Compactness of $\overline{\{z_t : t \in \mathbb{R}\}}$ implies that $\vec{h}^+(z_t)$ is uniformly bounded; hence $\vec{h}^+(z_t) = 0$. Similarly, $\|\vec{h}(z_{t-\tau}\| \geq e^{\alpha\tau}\|\vec{h}^-(z_t)\| - e^{-\alpha\tau}\|\vec{h}^+(z_t)\|$ that implies the equality $\vec{h}^-(z_t) = 0$. Finally, $\vec{h}(z_t) = 0$. In other words, $z_t \equiv z$ is an equilibrium of $\vec{h}$ and $T_zN = \Delta^+_z \oplus \Delta^-_z$ is the splitting of $T_zN$ into the repelling and attracting invariant subspaces for the linearization of the flow $e^{t\vec{h}}$ at $z$.

Hence $z$ is a hyperbolic equilibrium; in particular, $z$ is an isolated equilibrium of $\vec{h}$. $\square$

We say that a subset of a finite dimensional manifold is bounded if it has a compact closure.

**Corollary 6** *Assume that $h$ is a regular monotone Hamiltonian and $\vec{h}$ has everywhere negative curvature. Then any bounded semi-trajectory of the system $\dot{z} = \vec{h}(z)$ converges to an equilibrium with the exponential rate while another semi-trajectory of the same trajectory must be unbounded.* $\square$

Typical Hamiltonians which satisfy conditions of Corollary 6 are energy functions of natural mechanical systems in $\mathbb{R}^n$ with a strongly concave potential energy. Indeed, in this case, the second derivative of the potential energy is equal to the matrix of the curvature operator in the standard Cartesian coordinates (see Sec. 9).

# 12 Entropy

It is easy to see from what was done in the previous section that structural equation (23) allows to almost automatically generalize the arguments normally applied to Riemannian geodesic flows and to make them working for the flows generated by an arbitrary monotone Hamiltonian fields. This is true not only for the hyperbolicity property but also for the measure theoretic entropy. We finish the paper with an estimate for the entropy that is a direct generalization of the result obtained in [8]; see paper [9] for the detailed proof of this generalization.

Assume that $\vec{h}$ is a monotone Hamiltonian field, $h^{-1}(c)$ is a compact level set and $\vec{h}(z) \notin \Delta_z$, $\forall z \in h^{-1}(c)$. The normalized Liouville measure on $h^{-1}(c)$ is defined by the form $\mu = \frac{1}{C} i_X \sigma^n \big|_{h^{-1}(c)}$, where $\langle dh, X \rangle = 1$ and $C = \int\limits_{h^{-1}(c)} i_X \sigma^n$. It is easy to check that $\mu$ does not depend on the freedom in the choice of the vector field $X$.

The Hamiltonian flow $e^{t\vec{h}}$ preserves symplectic form $\sigma$ while the flow $e^{t\vec{h}}\big|_{h^{-1}(c)}$ preserves $\mu$. Let $\mathfrak{h}_\mu$ be the measure theoretic entropy of the last flow.

**Theorem 4 (Ballmann–Wojtkowski–Chittaro)** *Assume that the reduced* *curvature operator* $R^h_z$ *is nonpositive for any* $z \in h^{-1}(c)$. *Then*

$$\mathfrak{h}_\mu \geq \int\limits_{h^{-1}(c)} \operatorname{tr}\sqrt{-R^h_z}\,\mu(z). \tag{29}$$

*Remark.* The estimate is sharp: inequality (29) turns into the equality if all reduced Jacobi curves $J^h_z(\cdot)$ are symmetric curves (see Definition 7); the flows generated by such "symmetric" Hamiltonians provide a natural generalization of the geodesic flows of symmetric Riemannian manifolds.

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