Curvature of Conformally Hamiltonian Systems

A. Agrachev

SISSA, Trieste, Italy

Recall that the geodesic flow on a Riemannian manifold M treated as a flow on T^*M is the Hamiltonian flow for the Hamiltonian function

$$h(p,q) = \max_{v \in T_qM} (\langle p, v\rangle - \frac{1}{2}|v|_q), \qquad q \in M,\ p \in T^*_qM.$$

Specification of our curvature to the geodesic flow is just the Riemannian sectional curvature.

The curvature is going to be a differential symplectic invariant. Invariant of what? Indeed, a Hamiltonian systems without equilibria can be locally rectified and they do not have differential invariants.

The curvature is an invariant of the couple: a Hamiltonian systems and a Lagrangian vector distribution. Let (N, σ) be a symplectic manifold, $\Pi \subset TN$ a Lagrangian distribution: $\Pi = \bigcup_{z \in N} \Pi_z$, $\Pi_z \in L(T_z N)$, where

$$L(T_z N) = \{\Lambda \subset T_z N : \Lambda^{\angle} = \Lambda\}$$

is the Lagrange Grassmannian.

In the case of the cotangent bundle, standard Lagrangian distribution is tangent to the fibers:

$$T^*M = \{(p, q) : p \in T_q^*M,\ q \in M\}, \quad \Pi_{(p,q)} = T_{(p,q)}(T_q^*M).$$

Given a Hamiltonian $h : N \to \mathbb{R}$, we consider the action of the flow $e^{t\vec{h}} : N \to N$ on Π. Let

$$\Lambda_z^t = \left(e_*^{-t\vec{h}}\Pi\right)_z, \quad \Lambda_z^t \in L(T_zN).$$

The curvature at $z \in N$ is a basic differential invariant of the curve $t \mapsto \Lambda_z^t$ in the Lagrange Grassmannian.

In this talk, we deal with regular monotone curves. What is it?

Let (Σ, σ) be a $2n$-dimensional symplectic space and $L(\Sigma)$ be the Lagrange Grassmannian, $L(\Sigma) \subset Gr_n(\Sigma)$.

We have $T_\Lambda Gr(\Sigma) \cong Hom(\Lambda, \Sigma/\Lambda)$. Moreover, $\Sigma/\Lambda \cong \Lambda^*$ and $T_\Lambda L(\Sigma) \cong Sym(\Lambda, \Lambda^*) \subset Hom(\Lambda, \Lambda^*)$.

In other words, $T_\Lambda L(\Sigma)$ is naturally isomorphic to the space of quadratic forms on Λ.

The curve $t \mapsto \Lambda(t)$ is regular if $\dot{\Lambda}(t)$ are nondegenerate quadratic forms and is monotone if $\dot{\Lambda}(t)$ are sign-definite quadratic forms.

If $\Lambda(t) = \left(e_*^{-t\vec{h}}\Pi\right)_{(p_0,q_0)}$, where $\Pi = T(T_q^*M),\ h : T^*M \to \mathbb{R}$, then

$$\dot{\Lambda}(t) \approx -\frac{\partial^2 h}{\partial p^2}(p_t, q_t), \quad \text{where } (p_t, q_t) = e^{t\vec{h}}(p_0, q_0).$$

Let $\Lambda, \Delta \in L(\Sigma)$, $\Lambda \cap \Delta = 0$, then $\Sigma = \Lambda \oplus \Delta$ and $\Delta \cong \Lambda^*$.

Let $\xi \in T_\Lambda L(\Sigma) = Sym(\Lambda, \Lambda^*)$, $\eta \in T_\Delta L(\Sigma) = Sym(\Lambda^*, \Lambda)$; then

$$\eta \circ \xi : \Lambda \to \Lambda$$

is a linear operator, the cross-ratio of the "double points" ξ, η.

If ξ is a sign-definite quadratic form, then $\eta \circ \xi$ is symmetric for the Euclidean structure on Λ defined by ξ.

Differential geometry of a regular curve in $L(\Sigma)$.

Let $\Lambda^{\pitchfork} = \{\Delta \in L(\Sigma) : \Lambda \cap \Delta = 0\}$; this is an affine subspace of the vector space $Sym(\Lambda^*, \Lambda)$.

Indeed, $\Lambda^* \cong \Sigma/\Lambda$; the "difference" $\Delta_1 - \Delta_0$ of two elements of $\Lambda^{\pitchfork}$ is identified with the linear map $(\Delta_1 - \Delta_0) : \Sigma/\Lambda \to \Lambda$, which send the residue class $s+\Lambda$ to the vector $(s+\Lambda)\cap\Delta_1 - (s+\Lambda)\cap\Delta_0$.

If $\Lambda(\cdot)$ is a regular curve, then $\Lambda(\tau) \cap \Lambda(t) = 0$ for τ close to t, $\tau \neq t$. Moreover, the curve $\tau \mapsto \Lambda(\tau)$ in the affine space $\Lambda^{\pitchfork}$ has a simple pole as $\tau \to t$.

In the coordinates: $\Sigma = \{(p,q)\}$, $\sigma = dp \wedge dq$, $\Lambda(t) = \{(p,0)\}$; then $\Lambda(\tau) = \{(p, S_\tau p\}$, where $S_\tau = S_\tau^*$, $S_t = 0$. The curve $\Lambda(\cdot)$ is regular if and only if $\dot{S}_\tau$ is invertible.

Coordinate presentation of the curve $\Lambda(\cdot)$ in the affine space $\Lambda(t)^\pitchfork$ is the matrix curve $\tau \mapsto S_\tau^{-1} + B$, where B is a constant symmetric matrix whose choice fix the origin in the affine space. Moreover,

$$S_\tau^{-1} = (\tau - t)^{-1} \dot{S}_t^{-1} - \frac{1}{2} \dot{S}_t^{-1} \ddot{S}_t \dot{S}_t^{-1} + O(\tau - t).$$

Free term of the Laurent expansion is a point of the affine space $\Lambda(t)^{\pitchfork}$, all other terms are elements of the vector space $Sym(\Lambda_t^*, \Lambda_t)$. We have

$$\Lambda(\tau) = (\tau - t)^{-1}\dot{\Lambda}(t)^{-1} + \Lambda^{\circ}(t) + O(\tau - t).$$

The curve $t \mapsto \Lambda^{\circ}(t)$ in $L(\Sigma)$ is the *derivative curve* of the curve $\Lambda(\cdot)$. Recall that $\Lambda(t) \cap \Lambda^{\circ}(t) = 0$.

The curvature:

$$R_{\Lambda}^{t} : \Lambda(t) \to \Lambda(t), \quad R_{\Lambda}^{t} = \dot{\Lambda}^{\circ}(t) \circ \dot{\Lambda}(t).$$

Coordinate expression:

$$R^t_\Lambda = \frac{1}{2}\dot{S}_t^{-1}\ \dddot{S}_t - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2,$$

the matrix Schwartzian.

We also have:

$$\dot{\Lambda}(\tau) \circ \dot{\Lambda}(t) = (\tau - t)^{-2} I + \frac{1}{3} R^t_\Lambda + O(\tau - t).$$

Let $\Sigma = T_{(p,q)}(T^*M)$,

$$\Pi_{(p,q)} = T_{(p,q)}(T_q^*M), \quad \Lambda_{(p,q)}(t) = \left(e_*^{-t\vec{h}}\Pi\right)_{(p,q)},$$

then the Lagrangian distribution

$$\Lambda_h^\circ = \bigcup_{(p,q)\in T^*M} \Lambda_{(p,q)}^\circ(0)$$

is a complement to the "vertical" distribution Π in T^*M.

In other words, Λ_h° is an Ehresmann connection on the cotangent bundle T^*M.

Assume that M is equipped with a Riemannian structure $G_q : T_q M \to T_q^* M,\ |v|^2 = \langle G_q v, v\rangle,\ |p|^2 = \langle p, G_q^{-1} p\rangle$, and

$$h(p,q) = \frac{1}{2}|p|^2 + V(q);$$

then $G : TM \to T^*M$ transforms the Levi-Civita connection in the connection Λ_h°.

Moreover, the curvature operator $R_{(p,q)}^h \doteq R^0_{\Lambda_{(p,q)}}$ has a form:

$$G_q^{-1} R_{(p,q)}^h \xi = \mathfrak{R}(v,w)w + (\nabla_q^2 V)v, \quad \xi \in \Pi_{(p,q)},$$

where $\mathfrak{R}$ is the Riemannan curvature and $v = G_q^{-1}\xi,\ w = G_q^{-1}p$.

The curvature is an indicator of the loss of information on the initial state when moving along the flow. Bigger curvature - the information is better preserved. Indeed:

- Hamiltonian reduction of the system to a level of a first integral increases the curvature.

- If the curvature of reduction of the system to the compact energy level $h = c$ is negative, then the system is hyperbolic on the energy level. Moreover, the entropy of the system with respect to the normalized Liouville measure μ satisfies the inequality

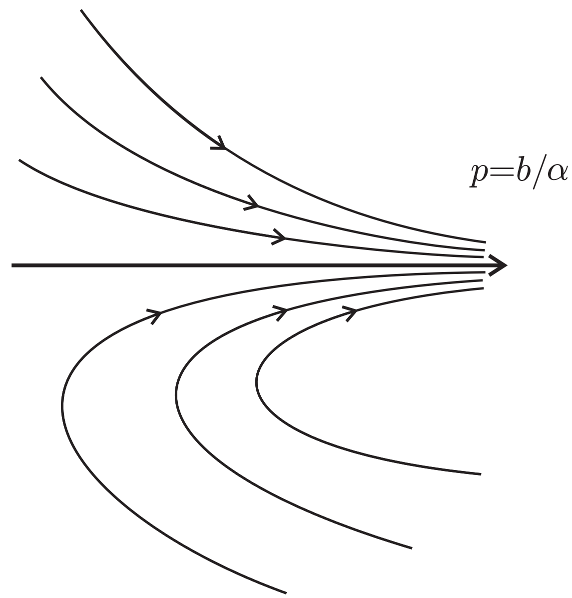
$$\eta \geq \int\limits_{h^{-1}(c)} \mathrm{tr}\sqrt{-R_{(p,q)}}\; d\mu(p,q).$$

Isotropic friction.

Let e be the Euler vector field on T^*M, $e = \langle p, \frac{\partial}{\partial p} \rangle$, and $\alpha > 0$ be the friction coefficient. The field $\vec{h}^\alpha = \vec{h} - \alpha e$ is conformally Hamiltonian. In particular, the flow $e^{t\vec{h}^\alpha}$ transforms Lagrangian submanifolds in the Lagrangian ones and we may define the Ehresmann connection $\Lambda^\circ_{h^\alpha}$ and the curvature R^{h^α} as we did it for Hamiltonian systems.

It happens that the curvature is negative for big enough α and tends to $-\infty$ as $\alpha \to \infty$. Moreover, at least in the case of a mechanical Hamiltonian $h(p,q) = \frac{1}{2}|p|^2 + V(q)$, negativity of the curvature implies that, in the long time scale, the system $(p,q)^\cdot = \vec{h}^\alpha(p,q)$ on T^*M behaves like a system on M; the "second order" system tends to a first order one.

The simplest case: $M = \mathbb{R}$, $V(q) = -bq$. If we apply a constant force, then we are eventually moving with a constant velocity.



This is a universal phenomenon: if the friction is strong enough to guaranty negativity of the curvature, then we are eventually moving with a prescribed velocity profile. The profile depends on the Hamiltonian; it is not constant, in general.

In what follows, $h = \frac{1}{2}|p|^2 + V(q)$; the connection $\Lambda^\circ_{h^\alpha}$ and the curvature R^{h^α} depend on α in a very simple way in this case.

We have $T_{(p,q)}(T^*M) = T^*_q M \oplus (\Lambda^\circ_{h^\alpha})_{(p,q)}$. Let $\pi : (p,q) \mapsto q$ be the standard projection and $v \in T_q M$. We denote by J^α_v the horizontal lift of v induced by the connection: $J^\alpha_v(p,q) \in (\Lambda^\circ_{h^\alpha})_{(p,q)}$ and $\pi_* J^\alpha_v(p,q) = v$. Then

$$J^\alpha_v = J^0_v + \frac{\alpha}{2} G_q^{-1} v, \qquad R^{h^\alpha} = R^{h^0} - \frac{\alpha^2}{4} I.$$

Given $u \in C^2(M)$, we say that the gradient vector field ∇u is a *potential stationary flow* for $\vec{h}^\alpha$ if $\{d_q u : q \in M\} \subset T^*M$ is an invariant submanifold of the system $(p,q)^\cdot = \vec{h}^\alpha(p,q)$. Note that restriction of the system to this invariant submanifold projects to the gradient system $\dot{q} = \nabla_q u$.

Assume that M is a complete Riemannian manifold, $\mathfrak{R}$ and $\nabla^2 V$ are uniformly bounded and $\Omega_c = \{(p,q) \in T^*M : |p| \leq c\}$.

Theorem. *If $R^{h^\alpha}_{(p,q)} < 0$, $\forall\,(p,q)$ s.t. $H(p,q) \leq \max V$, then $\exists$ a potential stationary flow ∇u s.t.*

$$e^{t\vec{h}^\alpha}(\Omega_c) \to \{d_q u : q \in M\} \ as \ t \to +\infty$$

with an exponential rate, $\forall\, c > 0$.

$\{d_q u : q \in M\}$ is a normally stable submanifold of $e^{t\vec{h}^\alpha}$.

If M is compact and $R^{h^\alpha}_{(p,q)} < -\left(\frac{(k-2)\alpha}{2k}\right)^2 I$, then $u \in C^k(M)$.

The map $(h,\alpha) \mapsto u$ is continuous in the C^2-topology.

The least action principle:

$$u(q) = -\inf\left\{\int_{-\infty}^{0} e^{\alpha t}\left(\frac{1}{2}|\dot{\gamma}(t)|^2 - V(\gamma(t))\right)\,dt : \gamma(0) = q\right\}.$$

The modified Hamilton–Jacobi equation:

$$H(d_q u, q) + \alpha u(q) = 0.$$

Why negative curvature implies (partial) hyperbolicity?

Let $\Lambda(\cdot)$ be a regular monotone curve in $L(\Sigma)$ and $\Lambda^\circ(\cdot)$ be its derivative curve; then $\Lambda(t) \cap \Lambda^\circ(t) = 0$. Recall that

$$R^t_\Lambda = \dot{\Lambda}^\circ(t) \circ \dot{\Lambda}(t).$$

The curvature R^t_Λ is nonpositive if and only if $\dot{\Lambda}^\circ(t)$ is a monotone curve whose monotonicity direction is opposite to one of $\dot{\Lambda}(t)$. Nonpositivity of the curvature implies the existence of

$$\Lambda(\pm\infty) = \lim_{t\to\pm\infty} \Lambda(t), \quad \text{where } \Lambda(+\infty) \cap \Lambda(-\infty) = 0.$$

Assume that $\Lambda(\cdot)$ is increasing and $\Lambda^\circ(\cdot)$ is decreasing and take $t_0 \in \mathbb{R}$. Symplectic group acts transitively on the pairs of transversal Lagrangian subspaces and we may assume that

$$\Lambda(t_0) = \{(p, 0)\}, \quad \Lambda^\circ(t_0) = \{(p, p)\}.$$

Let

$$\Lambda(t) = \{(p, S_t)\}, \quad \Lambda^\circ(t) = \{(p, S_t^\circ p)\},$$

then $\dot{S}_t > 0$, $\dot{S}_t^\circ \leq 0$, and $S_t - S_t^\circ$ is nondegenerate for any t.

We see that S_t is a monotone increasing family of quadratic forms and $S_t < I$, $\forall\, t \geq t_0$. Hence there exists $S_{+\infty} = \lim\limits_{t \to +\infty} S_t$.

Similarly for $t \to -\infty$.

Let f be a conformally Hamiltonian vector field on a symplectic manifold N equipped with a Lagrangian distribution Π. We set $\Lambda_z^f(t) = \left(e_*^{-tf}\Pi\right)_z$, $z \in N$, and $\Lambda_z^f(\pm\infty) = \lim\limits_{t\to\pm\infty} \Lambda_z^f(t)$; then

$$\Lambda^f(\pm\infty) = \bigcup_{z\in N} \Lambda_z^f(\pm\infty)$$

are e^{tf}-invariant Lagrangian distributions on N and

$$TN = \Lambda^f(+\infty) \oplus \Lambda^f(-\infty).$$

In other words, TN splits in a kind of "expanding" and "contracting" invariant distributions.

All details see in my papers:

- The curvature and hyperbolicity of Hamiltonian systems. Proceed. Steklov Math. Inst., 2007, v.256, 26–46

- Well-posed infinite horizon variational problems. Proceed. Steklov Math. Inst., 2010, v.268, 17–31

- Invariant Lagrange submanifolds of dissipative systems. Russian Math. Surveys, 2010, v.65, 222–223

Updated files are in the webpage: https://people.sissa.it/ agrachev/