BISHOP AND LAPLACIAN COMPARISON THEOREMS ON THREE DIMENSIONAL CONTACT SUBRIEMANNIAN MANIFOLDS WITH SYMMETRY

ANDREI AGRACHEV AND PAUL W.Y. LEE

ABSTRACT. We prove a Bishop volume comparison theorem and a Laplacian comparison theorem for three dimensional contact subriemannian manifolds with symmetry.

1. INTRODUCTION

Recently, there are numerous progress in the understanding of curvature type invariants in subriemannian geometry and their applications to PDE [12, 14, 15, 1, 2, 3, 4, 16]. In this paper, we continue to investigate some consequences on bounds of these curvature invariants. More precisely, we prove a Bishop comparison theorem and a Laplacian comparison theorem for three dimensional contact subriemannian manifolds with symmetry (also called Sasakian manifolds). Weaker results of volume comparison on Sasakian manifolds have previously been obtained in [10]. We would like to thank Professor Chanillo for pointing this out.

The paper is organized as follows. In section 2, we recall various notions in subriemannian geometry needed in this paper. In particular, we recall the definition of curvature R_{11} and R_{22} for three dimensional contact subriemannian manifolds introduced in [14, 15, 1]. In section 3, we show that the curvature R_{11} is closely related to the Tanaka-Webster curvature in CR geometry. In section 4, we collect various results on the cut loci of Sasakian manifold with constant Tanaka-Webster curvature (also called Sasakian space forms). In section 5, we give an estimate for the volume of subriemannian balls. In section 6, we prove the subriemannian Bishop theorem which compares the volume of subriemannian balls of a Sasakian manifold and a Sasakian space form. We introduce the subriemannian Hessian and sub-Laplacian in section 7 and give the formula for the Laplacian of the subriemannian

1

distance in Sasakian space form in section 8. We prove a subriemannian Hessian and a subriemannian Laplacian comparison theorem in section 9. As an application, we give a lower bound of the solution to the subriemannian heat equation in section 10. Finally, the approach of this paper can also be used in the higher dimensional cases with much more lengthy computations. This generalization will be discussed in a forthcoming paper.

Acknowledgment

The authors would like to thank N. Garofalo for stimulating discussions.

2. Subriemannian Geometry

In this section, we recall various notions in subriemannian geometry needed in this paper. A subriemannian manifold is a triple (M, Δ, g), where M is a smooth manifold, Δ is a distribution (a vector subbundle of the tangent bundle of M), and g is a fibrewise inner product defined on the distribution Δ. The inner product g is also called a subriemannian metric. An absolutely continuous curve $\gamma : [0, 1] \to M$ on the manifold M is called horizontal if it is almost everywhere tangent to the distribution Δ. We can use the inner product g to define the length $l(\gamma)$ of a horizontal curve γ by

$$l(\gamma) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt.$$

The distribution Δ is called bracket-generating if vector fields contained in Δ together with their iterated Lie brackets span the whole tangent bundle. More precisely, let Δ_1 and Δ_2 be two distributions on a manifold M, and let $\mathfrak{X}(\Delta_i)$ be the space of all vector fields contained in the distribution Δ_i. Let $[\Delta_1, \Delta_2]$ be the distribution defined by

$$[\Delta_1, \Delta_2]_x = \text{span}\{[w_1, w_2](x) | w_i \in \mathfrak{X}(\Delta_i)\}.$$

We define inductively the following distributions: $[\Delta, \Delta] = \Delta^2$ and $\Delta^k = [\Delta, \Delta^{k-1}]$. A distribution Δ is called bracket generating if $\Delta^k = TM$ for some k. Under the bracket generating assumption, we have the following famous Chow-Rashevskii Theorem (see [17] for a proof):

Theorem 2.1. *(Chow-Rashevskii) Assume that the manifold M is connected and the distribution Δ is bracket generating, then there is a horizontal curve joining any two given points.*

Assuming the distribution Δ is bracket generating, we can define the subriemannian or Carnot-Caratheodory distance $d(x, y)$ between two points x and y on the manifold M is defined by

$$(2.1) \qquad\qquad d(x, y) = \inf l(\gamma),$$

where the infimum is taken over all horizontal curves which start from x and end at y.

The horizontal curves which realize the infimum in (2.1) are called length minimizing geodesics. From now on all manifolds are assumed to be a complete metric space with respect to a given subriemannian distance. In particular, a version of Hopf-Rinow theorem for subriemannian manifolds ([5]) guarantees that there is at least one geodesic joining any two given points.

Next we discuss the geodesic equation in the subriemannian setting. Let α be a covector in the cotangent space $T_x^* M$ at the point x. By nondegeneracy of the metric g, we can define a vector v in the distribution Δ_x such that $g(v, \cdot)$ coincides with $\alpha(\cdot)$ on Δ_x. The subriemannian Hamiltonian H corresponding to the subriemannian metric g is defined by

$$H(\alpha) := \frac{1}{2} g(v, v).$$

Note that this construction defines the usual kinetic energy Hamiltonian in the Riemannian case.

Let $\pi : T^*M \to M$ be the projection map. The tautological one form θ on T^*M is defined by

$$\theta_\alpha(V) = \alpha(d\pi(V)),$$

where α is in the cotangent bundle T^*M and V is a tangent vector on the manifold T^*M at α.

Let $\omega = d\theta$ be the symplectic two form on T^*M. The Hamiltonian vector field $\vec{H}$ corresponding to the Hamiltonian H is defined by $\omega(\vec{H}, \cdot) = -dH(\cdot)$. By the non-degeneracy of the symplectic form ω, the Hamiltonian vector field $\vec{H}$ is uniquely defined. We denote the flow corresponding to the vector field $\vec{H}$ by $e^{t\vec{H}}$. If $t \mapsto e^{t\vec{H}}(\alpha)$ is a trajectory of the above Hamiltonian flow, then its projection $t \mapsto \gamma(t) = \pi(e^{t\vec{H}}(\alpha))$ is a locally minimizing geodesic. This means that sufficiently short segment of the curve γ is a minimizing geodesic between its endpoints. The minimizing geodesics obtained this way are called normal geodesics. In the special case where the distribution Δ is the whole tangent bundle TM, the distance function (2.1) is the usual Riemannian distance

and all geodesics are normal. However, this is not the case for subriemannian manifolds in general (see [17] and reference therein for more detail).

Next we restrict our attention to the three dimensional contact subriemannian manifold. Let Δ be a bracket generating distribution with two dimensional fibres on a three dimensional manifold M. Δ is a contact distribution if there exists a covector σ such that $\Delta = \{v|\sigma(v) = 0\}$ and the restriction of $d\sigma$ to Δ is non-degenerate. If we fix a subriemannian metric g, then we can choose σ so that the restriction of $d\sigma$ to the distribution Δ coincides with the volume form with respect to the subriemannian metric g.

Let $\{v_1, v_2\}$ be a local orthonormal frame in the distribution Δ with respect to the subriemannian metric g and let v_0 be the Reeb field defined by the conditions $\sigma(v_0) = 1$ and $d\sigma(v_0, \cdot) = 0$. This defines a frame $\{v_0, v_1, v_2\}$ in the tangent bundle TM and we let $\{\alpha_0 = \sigma, \alpha_1, \alpha_2\}$ be the corresponding dual co-frame in the cotangent bundle T^*M (i.e. $\alpha_i(v_j) = \delta_{ij}$).

The frame $\{v_0, v_1, v_2\}$ and the co-frame $\{\alpha_0, \alpha_1, \alpha_2\}$ defined above induce a frame in the tangent bundle TT^*M of the cotangent bundle T^*M. Indeed, let $\vec{\alpha}_i$ be the vector fields on the cotangent bundle T^*M defined by $i_{\vec{\alpha}_i}\omega = -\alpha_i$. Note that the symbol α_i in the definition of $\vec{\alpha}_i$ represents the pull back $\pi^*\alpha_i$ of the 1-form α on the manifold M by the projection $\pi : T^*M \to M$. This convention of identifying forms in the manifold M and its pull back on the cotangent bundle T^*M will be used for the rest of this paper without mentioning. Let $h_i : T^*M \to \mathbb{R}$ be the Hamiltonian lift of the vector fields v_i defined by $h_i(\alpha) = \alpha(v_i)$. Let ξ_1 and ξ_2 be the 1-forms defined by $\xi_1 = h_1\alpha_2 - h_2\alpha_1$ and $\xi_2 = h_1\alpha_1 + h_2\alpha_2$, respectively, and let $\vec{\xi}_i$ be the vector fields defined by $i_{\vec{\xi}_i}\omega = -\xi_i$. The vector fields $\vec{h}_0, \vec{h}_1, \vec{h}_2, \vec{\sigma}, \vec{\xi}_1, \vec{\xi}_2$ define a local frame for the tangent bundle TT^*M of the cotangent bundle T^*M. In the above notation the subriemannian Hamiltonian is given by $H = \frac{1}{2}((h_1)^2 + (h_2)^2)$ and the Hamiltonian vector field is $\vec{H} = h_1\vec{h}_1 + h_2\vec{h}_2$.

We also need the bracket relations of the vector fields v_0, v_1, v_2. Let a_{ij}^k be the functions on the manifold M defined by

$$(2.2) \qquad [v_i, v_j] = a_{ij}^0 v_0 + a_{ij}^1 v_1 + a_{ij}^2 v_2.$$

It is not hard to check that

$$(2.3) \qquad a_{01}^0 = a_{02}^0 = 0, \quad a_{12}^0 = -1, \quad a_{01}^1 + a_{02}^2 = 0.$$

Recall that a basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ in a symplectic vector space with a symplectic form ω is a Darboux basis if it satisfies $\omega(e_i, e_j) = $

$\omega(f_i, f_j) = 0$, and $\omega(f_i, e_j) = \delta_{ij}$. We recall the following theorem from [1].

Theorem 2.2. *For each fixed α in the manifold T^*M, there is a moving Darboux frame*

$$e_i(t) = (e^{t\vec{H}})_* e_i(0), \quad f_i(t) = (e^{t\vec{H}})_* f_i(0), \quad i = 1, 2, 3$$

in the symplectic vector space $T_\alpha T^*M$ and functions

$$R_t^{11} = (e^{t\vec{H}})^* R_0^{11}, R_t^{22} = (e^{t\vec{H}})^* R_0^{22} : T^*M \to \mathbb{R}$$

depending on time t such that the following structural equations are satisfied

$$
\begin{cases}
\dot{e}_1(t) = f_1(t), \\
\dot{e}_2(t) = e_1(t), \\
\dot{e}_3(t) = f_3(t), \\
\dot{f}_1(t) = -R_t^{11} e_1(t) - f_2(t), \\
\dot{f}_2(t) = -R_t^{22} e_2(t), \\
\dot{f}_3(t) = 0.
\end{cases}
$$

Moreover,

$$
\begin{cases}
e_1(0) = \frac{1}{\sqrt{2H}} \vec{\xi}_1, \\
e_2(0) = \frac{1}{\sqrt{2H}} \vec{\sigma}, \\
e_3(0) = -\frac{1}{\sqrt{2H}} (h_0 \vec{\alpha}_0 + h_1 \vec{\alpha}_1 + h_2 \vec{\alpha}_2), \\
f_1(0) = \frac{1}{\sqrt{2H}} [h_1 \vec{h}_2 - h_2 \vec{h}_1 + \chi_0 \vec{\alpha}_0 + (\vec{\xi}_1 h_{12}) \vec{\xi}_1 - h_{12} \vec{\xi}_2], \\
f_2(0) = \frac{1}{\sqrt{2H}} [2H \vec{h}_0 - h_0 \vec{H} - \chi_1 \vec{\alpha}_0 + (\vec{\xi}_1 a) \vec{\xi}_1 - a \vec{\xi}_2], \\
f_3(0) = -\frac{1}{\sqrt{2H}} \vec{H}, \\
R_0^{11} = h_0^2 + 2H\kappa - \frac{3}{2} \vec{\xi}_1 a, \\
R_0^{22} = R_0^{11} \vec{\xi}_1 a - 3\vec{H} \vec{\xi}_1 \vec{H} a + 3\vec{H}^2 \vec{\xi}_1 a + \vec{\xi}_1 \vec{H}^2 a.
\end{cases}
$$

where

$$
\begin{aligned}
&a = dh_0(\vec{H}), \\
&\chi_0 = h_2 h_{01} - h_1 h_{02} + \vec{\xi}_1 a, \\
&\chi_1 = h_0 a + 2\vec{H} \vec{\xi}_1 a - \vec{\xi}_1 \vec{H} a, \\
&\kappa = v_1 a_{12}^2 - v_2 a_{12}^1 - (a_{12}^1)^2 - (a_{12}^2)^2 - \tfrac{1}{2}(a_{01}^2 - a_{02}^1).
\end{aligned}
$$

3. Connection with the Tanaka-Webster Scalar Curvature

In this section, we show that κ defined in Theorem 2.2 is up to a constant the Tanaka-Webster scalar curvature in CR geometry.

Let g^R be the Riemannian metric on the manifold M such that the basis v_0, v_1, v_2 is orthonormal. Let $K(v, w)$ be the sectional curvature of the plane spanned by v and w and let $\mathfrak{Rc}(v)$ be the Ricci curvature of the vector v.

Theorem 3.1. *The invariant $\mathfrak{W}$ satisfies*

$$\mathfrak{W} = 2K(v_1, v_2) + \mathfrak{Rc}(v_0) + 1.$$

Remark 3.2. From now on, we called $\mathfrak{W}$ the Tanaka-Webster scalar curvature. It follows from the above theorem that $4\mathfrak{W}$ coincides with the definition of Tanaka-Webster scalar curvature in [19].

Proof. Let ∇ denotes the Riemannian connection of the Riemannian metric g^R. By Koszul's formula, we have the following.

$$\nabla_{v_1} v_1 = a_{01}^1 v_0 - a_{12}^1 v_2,$$

$$\nabla_{v_2} v_2 = a_{02}^2 v_0 - a_{21}^2 v_1,$$

$$\nabla_{v_1} v_2 = \frac{1}{2}(a_{02}^1 + a_{01}^2 - 1)v_0 + a_{12}^1 v_1,$$

$$\nabla_{v_2} v_1 = \frac{1}{2}(a_{01}^2 + a_{02}^1 + 1)v_0 - a_{12}^1 v_2,$$

$$\nabla_{v_1} v_0 = -a_{01}^1 v_1 + \frac{1}{2}(-a_{02}^1 - a_{01}^2 + 1)v_2,$$

$$\nabla_{v_2} v_0 = -a_{02}^2 v_2 + \frac{1}{2}(-a_{01}^2 - a_{02}^1 - 1)v_1,$$

$$\nabla_{v_0} v_1 = a_{01}^2 v_2 + \frac{1}{2}(-a_{02}^1 - a_{01}^2 + 1)v_2,$$

$$\nabla_{v_0} v_2 = a_{02}^1 v_1 + \frac{1}{2}(-a_{01}^2 - a_{02}^1 - 1)v_1.$$

Let $\mathfrak{R}$ be the Riemann curvature tensor of the Riemannian metric g^R. By definition of $\mathfrak{R}$, we also have

$$g^R(\mathfrak{R}(v_i, v_j)v_i, v_j)$$
$$= g^R(\nabla_{[v_i, v_j]} v_i - \nabla_{v_i} \nabla_{v_j} v_i + \nabla_{v_j} \nabla_{v_i} v_i, v_j).$$

It follows from the above that

$$\mathfrak{Rc}(v_0) = g^R(\mathfrak{R}(v_0, v_1)v_0, v_1) + g^R(\mathfrak{R}(v_0, v_2)v_0, v_2)$$
$$= -2(a_{01}^1)^2 - \frac{1}{2}(a_{01}^2 + a_{02}^1)^2 + \frac{1}{2}$$

and

$$K(v_1, v_2) = g^R(\mathfrak{R}(v_1, v_2)v_1, v_2)$$
$$= \frac{1}{2}(a_{02}^1 - a_{01}^2) - (a_{12}^1)^2 - (a_{12}^2)^2 + v_1 a_{12}^2$$
$$- v_2 a_{12}^1 + (a_{01}^1)^2 + \frac{1}{4}(a_{02}^1 + a_{01}^2)^2 - \frac{3}{4}.$$

$\square$

4. Examples and Sasakian Space Forms

A three dimensional contact subriemannian manifold is Sasakian if the Reeb field preserves the subriemannian metric. Using the notation of this paper, it is the same as $a = dh_0(\vec{H}) = 0$.

A three dimensional Sasakian manifold is a Sasakian space form if the Tanaka-Webster scalar curvature is constant. In this section, we collect various facts about the injectivity domain (see below for the definition) of Sasakian space forms including the recent results in [7].

Let (M, Δ, g) be a subriemannian manifold. Let H be the subriemannian Hamiltonian and let $e^{t\vec{H}}$ be the Hamiltonian flow. Let $\pi : T^*M \to M$ be the projection map and let us fix a point x in the manifold M. Let Ω_x be the set of all covectors α in the cotangent space T_x^*M such that the curve $\gamma : [0,1] \to M$ defined by $\gamma(t) = \pi(e^{t\vec{H}}(\alpha))$ is a length minimizing geodesic. We call $\Omega = \bigcup_x \Omega_x$ the injectivity domain of the subriemannian manifold. We also let Ω_x^R be the set of covectors in Ω_x such that the corresponding curve γ has length less than or equal to R. A point α in T_x^*M is a cut point if $\gamma(t) = \pi(e^{t\vec{H}}(\alpha))$ is minimizing geodesic on $[0, 1]$ and not minimizing on any larger interval. A point α is a conjugate point if the map $\pi(e^{1 \cdot \vec{H}})$ is singular at α.

The Heisenberg group $\mathbb{H}$ is a well-known example of a Sasakian manifold with vanishing Tanaka-Webster curvature. The manifold in this case is given by $\mathbb{R}^3$ and the distribution Δ is the span of two vector fields $\partial_x - \frac{1}{2}y\partial_z$ and $\partial_y + \frac{1}{2}x\partial_z$. These two vector fields also define a subriemannian metric for which they are orthonormal. In this case all cut points are conjugate points and Ω^R is given by

(4.1) $$\Omega_0^R = \{\alpha | \sqrt{2H(\alpha)} \le R, -2\pi \le h_0(\alpha) \le 2\pi\}.$$

Recall that $SU(2)$, the special unitary group, consists of 2×2 matrices with complex coefficients and determinant 1. The Lie algebra $su(2)$ consists of skew Hermitian matrices with trace zero. The left invariant

vector fields of the following two elements in $su(2)$

$$u_1 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}$$

span the standard distribution Δ on $SU(2)$. Let g^c be the subriemannian metric for which $g^c(cu_1, cu_2) = 1$. The Reeb field in this case is $c^2 u_0$, where

$$u_0 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}.$$

A computation shows that the Tanaka-Webster curvature is given by c^2. It follows from the result in [7] that all cut points are conjugate points in this case and Ω^R is given by

(4.2) $$\Omega_{id}^R = \{\alpha | \sqrt{h_0(\alpha)^2 + 2c^2 H(\alpha)} \leq 2\pi\}.$$

Remark 4.1. The notations in here and that of [7] are slightly different. It was shown in [7, Theorem 12] that all cut points are conjugate points in the case $SU(2)$ with $c = 1$. Moreover, the path $t \mapsto \pi(e^{t\vec{H}}(\alpha))$ hits the first conjugate point at the time $\frac{2\pi}{\sqrt{1+h_0(\alpha)^2}}$, where $H(\alpha) = \frac{1}{2}$. This is equivalent to (4.2). The rest of the cases with $c \neq 1$ follow from scaling.

The special linear group $SL(2)$ is the set of all 2×2 matrices with real coefficients and determinant 1. The Lie algebra $sl(2)$ is the set of all 2×2 real matrices with trace zero. The left invariant vector fields of the following two elements in $sl(2)$

$$u_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

span the standard distribution Δ on $SL(2)$. Let g^c be the subriemannian metric for which $g^c(cu_1, cu_2) = 1$. The Reeb field in this case is $c^2 u_0$, where

$$u_0 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}.$$

The Tanaka-Webster curvature is given by $-c^2$. The structure of the set of cut points in this case is much more complicated. However, the result in [7] and a computation shows the following.

Theorem 4.2. *Assume that a cut point α in the cotangent bundle of $SL(2)$ with subriemannian metric g^c is contained in Ω_{id}^R, where $R = \frac{2\sqrt{2}\pi}{c}$. Then it is a conjugate point. Moreover, it satisfies*

$$\sqrt{|h_0(\alpha)^2 - 2c^2 H(\alpha)|} = 2\pi$$

Proof. Let $\tau = h_0^2 - 2Hc^2$. From the proof of [7, Section 5.3, Theorem 15], α in the cotangent space $T_x^*SL(2)$ at a point x is a cut point if it satisfies

$$(4.3) \qquad \frac{\tan(h_0(\alpha)/2)}{h_0(\alpha)} = \frac{\tanh(\sqrt{-\tau(\alpha)}/2)}{\sqrt{-\tau(\alpha)}}$$

for $\tau(\alpha) < 0$,

$$(4.4) \qquad \frac{\tan(h_0(\alpha)/2)}{h_0(\alpha)} = \frac{\tan(\sqrt{\tau(\alpha)}/2)}{\sqrt{\tau(\alpha)}}$$

for $\tau(\alpha) > 0$, or

$$(4.5) \qquad \frac{\tan(h_0(\alpha)/2)}{h_0(\alpha)} = \frac{1}{2}$$

for $\tau(\alpha) = 0$.

Let r_1, r_2, r_3 be the infimum of $2H(\alpha)c^2$ where α runs over positive solutions of (4.3), (4.4),(4.5), respectively. The goal is to find the minimum of $\{r_1, r_2, r_3\}$.

Let $f(x) = \frac{\tan(\sqrt{x})}{\sqrt{x}}$ and $g(x) = \frac{\tanh(\sqrt{x})}{\sqrt{x}}$. Let F_1 be a branch of inverses of $f\big|_{[0,\infty)}$ and let G be the inverse of $g\big|_{[0,\infty)}$. Finding r_1 is the same as minimizing $4(F_1 + G)$.

A computation shows that that the derivatives of F_1 and G satisfy

$$(4.6) \qquad F_1'(x) = \frac{1 + x^2 F_1(x) - x}{2F_1(x)}, \quad G'(x) = \frac{1 - x^2 G(x) - x}{2G(x)}.$$

Since F_1 and G are nonnegative, $F_1' + G' = 0$ implies that $x = 1$. It follows that $r_1 \geq r_3$.

Let F_2 be another branch of inverses of $f\big|_{[0,\infty)}$ for which $F_2 > F_1$. We can assume that F_1 is the smalllest branch and F_2 is the second smallest branch. In this case, finding r_2 is the same as minimizing $4(F_2 - F_1)$. It follows from (4.6) that $F_2'(x) - F_1'(x) = 0$ implies $x = 1$. Therefore, there are two possibilities. Either the minimum of $F_2 - F_1$ occurs at $x = 1$ which implies that $r_2 \geq r_3$ or $4(F_2 - F_1)$ goes to the infimum as $x \to \infty$ which implies that $r_2 = 4\left(\frac{3\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2 = 8\pi^2 < r_3$.

The last assertion follows from [7]. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5. Volume of Subriemannian Balls

In this section, we give an estimate on the volume of subriemannian balls of Sasakian manifolds assuming the Tanaka-Webster curvature is bounded below. More precisely, let us fix a point x in the manifold and let v_1, v_2 be an orthonormal basis of the subriemannian metric around x. Let v_0 be the Reeb field and $\alpha_0, \alpha_1, \alpha_2$ be the dual coframe of the frame v_0, v_1, v_2 (i.e. $\alpha_i(v_j) = \delta_{ij}$). We use this coframe to introduce coordinates on the cotangent space T_x^*M and let $\mathfrak{m}$ be the corresponding volume form. We will also denote the corresponding Lebesgue measure by the same symbol. Let (r, θ, h) be the cylindrical coordinates on T_x^*M corresponding to the above coordinate system (i.e. $h = h_0(\alpha)$, $r^2 = 2H(\alpha)$, and $\tan(\theta) = \frac{h_2(\alpha)}{h_1(\alpha)}$). Recall that Ω_R denotes the set of all covectors α such that $\sqrt{2H(\alpha)} \leq R$ and the curve $t \mapsto \pi(e^{t\vec{H}}(\alpha)), 0 \leq t \leq 1$ is length minimizing. We use the coordinate system introduced above on T_x^*M to identify the set Ω_R with a subset in $\mathbb{R}^n$. Finally, let η be the volume form defined by the condition $\eta(v_0, v_1, v_2) = 1$. We denote the measure induced by η using the same symbol.

Theorem 5.1. *Assume that there exists a constant k_1 (resp. k_2) such that the Tanaka-Webster curvature $\mathfrak{W}$ of a three dimensional Sasakian manifold satisfies $\mathfrak{W} \geq k_1$ (resp. $\leq k_2$) on the ball $B(x, R)$ of radius R centered at the point x. Then*

$$\eta(B(x, R)) \leq \int_{\Omega_R} b^{k_1} d\mathfrak{m} \quad \left(resp. \geq \int_{\Omega_R} b^{k_2} d\mathfrak{m} \right),$$

where $b_k : T_x^*M \to \mathbb{R}$ is defined via the above mentioned cylindrical coordinates by

$$b^k = \begin{cases} \frac{r^2(2-2\cos(\tau)-\tau\sin(\tau))}{\sigma^2} & if \quad \sigma > 0, \\ \frac{r^2(2-2\cosh(\tau)+\tau\sinh(\tau)))}{\sigma^2} & if \quad \sigma < 0, \\ \frac{r^2}{12} & if \quad \sigma = 0, \end{cases}$$

$\sigma = h^2 + r^2 k$, and $\tau = \sqrt{|\sigma|}$.

As a corollary, we have a formula for the volume of subriemannian balls on Sasakian space forms. Remark that explicit formula for the set Ω_R in various examples are present in Section 4 (see also [7] for more details).

Corollary 5.2. *Assume that the three dimensional subriemannian manifold is a Sasakian space form with Tanaka-Webster curvature k. Then*

$$\eta(B(x, R)) = \int_{\Omega_R} b^k d\mathfrak{m}.$$

Proof of Theorem 5.1. Recall that η is the measure on M defined by $\eta(v_0, v_1, v_2) = 1$. Let $\psi_t : T_x^* M \to M$ be the map $\psi_t(\alpha) = \pi(e^{1 \cdot \vec{H}}(\alpha))$ and let $\rho_t : T_x^* M \to \mathbb{R}$ be the function defined by

$$(5.1) \qquad \qquad \psi_t^* \eta = \rho_t \mathfrak{m}.$$

Let us fix a covector α in $T_x^* M$. Let $e_1(t), e_2(t), e_3(t), f_1(t), f_2(t), f_3(t)$ be a canonical Darboux frame at α defined by Theorem 2.2. Let $a_{ij}(t)$ and $b_{ij}(t)$ be defined by

$$(5.2) \qquad \qquad e_i(0) = \sum_{j=1}^{3} (a_{ij}(t) e_j(t) + b_{ij}(t) f_j(t)).$$

Finally let A_t and B_t be the matrices with (i, j)-th entry equal to $a_{ij}(t)$ and $b_{ij}(t)$, respectively.

By definition of $\mathfrak{m}$, we have $\mathfrak{m}(e_1(0), e_2(0), e_3(0)) = 1$. It also follows from Theorem 2.2 that $\eta(f_1(0), f_2(0), f_3(0)) = 2H(\alpha)$. Therefore, (5.1) implies that

$$(5.3) \qquad \qquad \rho_t = 2H(\alpha) \det B_t.$$

It follows that from (5.3) that

$$(5.4) \quad \eta(B(x, r)) = \int_{\psi_1(\Omega_r)} d\eta = \int_{\Omega_r} |\rho_1| d\mathfrak{m} = \int_{\Omega_r} 2H(\alpha) |\det B_1| d\mathfrak{m}.$$

Let $E_t = (e_1(t), e_2(t), e_3(t))^T$ and let $F_t = (f_1(t), f_2(t), f_3(t))^T$. Here the superscript T denote matrix transpose. By the definition of the matrices A_t and B_t, we have

$$E_0 = A_t E_t + B_t F_t.$$

If we differentiate the above equation with respect to time t, we have

$$0 = \dot{A}_t E_t + A_t \dot{E}_t + \dot{B}_t F_t + B_t \dot{F}_t$$
$$= \dot{A}_t E_t + A_t C_1 E_t + A_t C_2 F_t + \dot{B}_t F_t + B_t(-R_t E_t - C_1^T F_t),$$

Since the manifold is Sasakian, $a = dh_0(\vec{H}) = 0$. Therefore, by Theorem 2.2, E_t and F_t satisfy the following equatoins

$$\dot{E}_t = C_1 E_t + C_2 F_t, \quad \dot{F}_t = -R_t E_t - C_1^T F_t,$$

where

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_t = \begin{pmatrix} R_t^{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} h_0^2(\alpha) + 2\mathfrak{W}_t H(\alpha) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\mathfrak{W}_t = \mathfrak{W}(\psi_t(\alpha))$.

It follows that the matrices A_t and B_t satisfy the following equations

(5.5) $\dot{A}_t + A_t C_1 - B_t R_t = 0, \quad \dot{B}_t + A_t C_2 - B_t C_1^T = 0$

with initial conditions $B_0 = 0$ and $A_0 = Id$.

If we set $S_t = B_t^{-1} A_t$ and $U_t = S_t^{-1} = A_t^{-1} B_t$, then they satisfy the following Riccati equations.

$$\dot{S}_t - S_t C_2 S_t + C_1^T S_t + S C_1 - R_t = 0$$

and

$$\dot{U}_t + U_t R_t U_t - C_1 U_t - U_t C_1^T + C_2 = 0$$

with initial condition $U_0 = 0$.

Let us fix a constant k and consider the following Riccati equation with constant coefficients

(5.6) $\dot{U}_t^k + U_t^k R^k U_t^k - C_1 U_t^k - U_t^k C_1^T + C_2 = 0$

and initial condition $U_0^k = 0$, where $R^k = \begin{pmatrix} h_0^2(\alpha) + 2kH(\alpha) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The solution of (5.6) can be found by the method in [13]. If $h_0^2 + 2H(\alpha)k > 0$, then

$$U_t^k = \begin{pmatrix} \frac{-t\sin(\tau_t)}{\tau_t \cos(\tau_t)} & \frac{t^2(\cos(\tau_t)-1)}{\tau_t^2 \cos(\tau_t)} & 0 \\ \frac{t^2(\cos(\tau_t)-1)}{\tau_t^2 \cos(\tau_t)} & \frac{t^3(\tau_t \cos(\tau_t)-\sin(\tau_t))}{\tau_t^3 \cos(\tau_t)} & 0 \\ 0 & 0 & -t \end{pmatrix}.$$

If $h_0^2 + 2H(\alpha)k < 0$, then

$$U_t^k = \begin{pmatrix} \frac{-t\sinh(\tau_t)}{\tau_t \cosh(\tau_t)} & \frac{t^2(1-\cosh(\tau_t))}{\tau_t^2 \cosh(\tau_t)} & 0 \\ \frac{t^2(1-\cosh(\tau_t))}{\tau_t^2 \cosh(\tau_t)} & \frac{t^3(\sinh(\tau_t)-\tau_t \cosh(\tau_t))}{\tau_t^3 \cosh(\tau_t)} & 0 \\ 0 & 0 & -t \end{pmatrix}.$$

If $h_0^2 + 2H(\alpha)k = 0$, then

$$U_t^k = \begin{pmatrix} -t & -\frac{t^2}{2} & 0 \\ -\frac{t^2}{2} & -\frac{t^3}{3} & 0 \\ 0 & 0 & -t \end{pmatrix},$$

where $\tau_t = t\sqrt{|h_0^2 + 2H(\alpha)k|}$.

If we call the inverse $S_t^k = (U_t^k)^{-1}$, then

$$S_t^k = \begin{pmatrix} \frac{\tau_t(\tau_t \cos(\tau_t) - \sin(\tau_t))}{t(2 - 2\cos(\tau_t) - \tau_t \sin(\tau_t))} & \frac{\tau_t^2(1 - \cos(\tau_t))}{t^2(2 - 2\cos(\tau_t) - \tau_t \sin(\tau_t))} & 0 \\ \frac{\tau_t^2(1 - \cos(\tau_t))}{t^2(2 - 2\cos(\tau_t) - \tau_t \sin(\tau_t))} & \frac{-\tau_t^3 \sin(\tau_t)}{t^3(2 - 2\cos(\tau_t) - \tau_t \sin(\tau_t))} & 0 \\ 0 & 0 & -\frac{1}{t} \end{pmatrix},$$

if $h_0^2 + 2H(\alpha)k > 0$.

$$S_t^k = \begin{pmatrix} \frac{\tau_t(\sinh(\tau_t) - \tau_t \cosh(\tau_t))}{t(2 - 2\cosh(\tau_t) + \tau_t \sinh(\tau_t))} & \frac{\tau_t^2(\cosh(\tau_t) - 1)}{t^2(2 - 2\cosh(\tau_t) + \tau_t \sinh(\tau_t))} & 0 \\ \frac{\tau_t^2(\cosh(\tau_t) - 1)}{t^2(2 - 2\cosh(\tau_t) + \tau_t \sinh(\tau_t))} & \frac{-\tau_t^3 \sinh(\tau_t)}{t^3(2 - 2\cosh(\tau_t) + \tau_t \sinh(\tau_t))} & 0 \\ 0 & 0 & -\frac{1}{t} \end{pmatrix},$$

if $h_0^2 + 2H(\alpha)k < 0$.

$$S_t^k = \begin{pmatrix} -\frac{4}{t} & \frac{6}{t^2} & 0 \\ \frac{6}{t^2} & -\frac{12}{t^3} & 0 \\ 0 & 0 & -\frac{1}{t} \end{pmatrix},$$

if $h_0^2 + 2H(\alpha)k = 0$.

By [18], if $k_1 \leq \mathfrak{W} \leq k_2$, then

$$(5.7) \qquad\qquad U_t^{k_2} \leq U_t \leq U_t^{k_1} \leq 0.$$

Therefore, $S_t^{k_1} \leq S_t \leq S_t^{k_2}$.

On the other hand, by (5.5) and the definition of S_t, we have

$$\dot{B}_t + B_t(S_t C_2 - C_1^T) = 0.$$

It follows that $\frac{d}{dt}\det B_t = \mathrm{tr}(C_1^T - S_t C_2)\det B_t = -\mathrm{tr}(S_t C_2)\det B_t$. If we replace the matrix R_t in (5.5) by R^k and denote the solution by A_t^k and B_t^k, then we have

$$\frac{\frac{d}{dt}\det B_t}{\det B_t} = -\mathrm{tr}(S_t C_2) \geq -\mathrm{tr}(S_t^{k_2} C_2) = \frac{\frac{d}{dt}\det B_t^{k_2}}{\det B_t^{k_2}}.$$

It follows that $\frac{\det B_t}{\det B_t^{k_2}}$ is nondecreasing. By definition of U_t and (5.7), we also have

$$\lim_{t\to 0}\frac{\det B_t}{\det B_t^{k_2}} = \lim_{t\to 0}\frac{\det A_t \det U_t}{\det A_t^{k_2}\det U_t^{k_2}} \geq \lim_{t\to 0}\frac{\det A_t}{\det A_t^{k_2}} = 1$$

Therefore, it follows that

$$\frac{\det B_t}{\det B_t^{k_2}} \geq \lim_{t \to 0} \frac{\det B_t}{\det B_t^{k_2}} \geq 1.$$

Similarly, we also have

$$\frac{\det B_t}{\det B_t^{k_1}} \leq 1.$$

A calculation gives

$$(5.8) \qquad |\det B_t^k| := b_t^k = \frac{t(2 - 2\cos(\tau_t) - \tau_t \sin(\tau_t))}{\tau_1^4}$$

if $h_0^2 + 2H(\alpha)k > 0$,

$$(5.9) \qquad b_t^k = \frac{t(2 - 2\cosh(\tau_t) + \tau_t \sinh(\tau_t))}{\tau_1^4}$$

if $h_0^2 + 2H(\alpha)k < 0$, and

$$(5.10) \qquad b_t^k = \frac{t^5}{12}$$

if $h_0^2 + 2H(\alpha)k = 0$.

It follows that $b_t^{k_2} \leq |\det B_t| \leq b_t^{k_1}$. Therefore, we have the following as claimed

$$\int_{\Omega_R} r^2 b_1^{k_2} d\mathfrak{m} \leq \eta(B(x,r)) \leq \int_{\Omega_R} r^2 b_1^{k_1} d\mathfrak{m}.$$

$\square$

6. Subriemannian Bishop Theorem

In this section, we prove a subriemannian analog of Bishop theorem for three dimensional Sasakian manifolds. Recall that η is the volume form defined by the condition $\eta(v_0, v_1, v_2)$. We denote the measure induced by η using the same symbol and let η^k be the corresponding measure in a Sasakian space form of curvature k. Let $B^k(R)$ be a subriemannian ball of radius R in one of the Sasakian space forms $SU(2)$, $\mathbb{H}$, or $SL(2)$ of curvature k (see Section 4 for a discussion of these space forms).

Theorem 6.1. *(Subriemannian Bishop Theorem) Assume that the Tanaka-Webster scalar curvature $\mathfrak{W}$ of a three dimensional Sasakian manifold satisfies $\mathfrak{W} \geq k$ on the ball $B(x, R)$ for some constant k. If $k \geq 0$, then*

$$(6.1) \qquad \eta(B(x,R)) \leq \eta^k(B^k(R))$$

and equality holds only if $\mathfrak{W} = k$ on $B(x, R)$. The same conclusion holds for $k < 0$ provided that $R \leq \frac{2\sqrt{2}\,\pi}{\sqrt{-k}}$.

Proof. By Theorem 5.1, we have

$$\eta(B(x, R)) \leq \int_{\Omega_R} b^k d\mathfrak{m}.$$

If we can show that

(6.2) $\quad \Omega_R \subseteq \Omega'_R := \{\alpha \in T_x^* M | \sqrt{|h_0(\alpha)^2 + kH(\alpha)|} \leq 2\pi\},$

then Corollary 5.2, together with (4.1), (4.2), and Theorem 4.2, implies that

$$\eta(B(x, R)) \leq \int_{\Omega'_R} b^k d\mathfrak{m} = \eta^k(B^k(R))$$

which is (6.1). Note that the condition $R \leq \frac{2\sqrt{2}\,\pi}{\sqrt{-k}}$ is needed when $k < 0$ because of Theorem 4.2.

To show (6.2) let us suppose that there is a covector α in Ω_R such that

$$\tau_1 := \sqrt{|h_0(\alpha)^2 + kH(\alpha)|} > 2\pi.$$

Using the notation in the proof of Theorem 5.1, we let

$$E_0 = A_t E_t + B_t F_t,$$

where $E_t = (e_0(t), e_1(t), e_2(t))^T$ and $F_t = (f_0(t), f_1(t), f_2(t))^T$ are canonical Darboux frame at α.

By the proof of Theorem 5.1, we have that $|\det B_t| \leq b_t^k$, where b_t^k is defined in (5.8), (5.9), and (5.10). Since $\tau_1 > 2\pi$, it follows that $\det B_t = 0$ for some $t < 1$. Therefore, $t\alpha$ is a conjugate point contradicting the fact that α is contained in Ω_R.

Next, suppose equality holds in (6.1) and $\mathfrak{W} > k$ on an open set $\mathcal{O}$ contained in the ball $B(x, R)$. For each point y in $\mathcal{O}$, let $\gamma(t) = \pi(e^{t\vec{H}}(\alpha))$ be a minimizing geodesic connecting x and y. It follows that $R_t > R_t^k$ for all t close enough to 1. By the result in [18] and a similar argument as in Theorem 5.1, we have $|\det B_1| < |\det B_1^k|$. It follows from (5.4) that $\eta(B(x, R)) < \eta^k(B^k(R))$ which is a contradiction. $\quad\square$

7. Subriemannian Hessian and Laplacian

In this section, we introduce subriemannian versions of Hessian and Laplacian. For the computation, we will also give an expression for it in the canonical Darboux frame.

Assume that a functon $f : M \to \mathbb{R}$ is twice differentiable at a point x in the manifold M. The canonical Darboux frame

$$\{e_1(t), e_2(t), e_3(t), f_1(t), f_2(t), f_3(t)\}$$

at df_x gives a splitting of the tangent space $T_{df_x} T^*M = \mathcal{H} \oplus \mathcal{V}$ defined by

$$\mathcal{H} = \mathbf{span}\{f_1(0), f_2(0), f_3(0)\}, \quad \mathcal{V} = \mathbf{span}\{e_1(0), e_2(0), e_3(0)\}.$$

The differential $d\pi$ of the projection map $\pi : T^*M \to M$ defines an identification between $\mathcal{H}$ and $T_x M$. On the other hand, the map $\iota : T_x^* M \to \mathcal{V}$ defined by $\iota(\alpha) = -\vec{\alpha}$ also gives an identification between $T_x^* M$ and $\mathcal{V}$.

Let us consider the differential $d(df_x) : T_x M \to T_{df_x} T^*M$ at x of the map $x \mapsto df_x$. It defines a three dimensional subspace $\Lambda := d(df_x)(T_x M)$ of the tangent space $T_{df_x} T^*M$. Since Λ is transvesal to the space $\mathcal{V}$, it defines a linear map S from $\mathcal{H}$ to $\mathcal{V}$ for which the graph is given by Λ. More precisely, if $w = w_h + w_v$ is a vector in the space Λ, where w_h and w_v are in $\mathcal{H}$ and $\mathcal{V}$, respectivly. Then $S(w_h) = w_v$. Under the identifications of the tangent space $T_x M$ with $\mathcal{H}$ and the cotangent space $T_x^* M$ with $\mathcal{V}$, we obtain a linear map $H^{SR} f(x) : T_x M \to T_x^* M$ called subriemannian Hessian. More precisely,

$$H^{SR} f(x)(d\pi(w)) = \iota^{-1} S w.$$

Proposition 7.1. *The subriemannian Hessian $H^{SR} f$ is symmetric i.e.* $\left\langle H^{SR} f(x) v, w \right\rangle = \left\langle H^{SR} f(x) w, v \right\rangle.$

Proof. Let w_1 and w_2 be two vectors in the subspace $\mathcal{H}$. Since the subspace Λ is a Lagrangian subspace, we have

$$\omega(w_1 + S(w_1), w_2 + S(w_2)) = 0.$$

Since both $\mathcal{H}$ and $\mathcal{V}$ are Lagrangian subspaces, we also have

$$\omega(w_1, S(w_2)) + \omega(S(w_1), w_2) = 0.$$

It follows from skew symmetry of ω and the definition of subriemannian Hessian that

$$\begin{aligned}
&\left\langle H^{SR} f(x) d\pi(w_2), d\pi(w_1) \right\rangle \\
&= \omega(w_1, S(w_2)) \\
&= \omega(w_2, S(w_1)) \\
&= \left\langle H^{SR} f(x) d\pi(w_1), d\pi(w_2) \right\rangle.
\end{aligned}$$

$$\square$$

Let $\iota_i = d\pi(f_i(0))$ and let $\mathfrak{H}f(x)$ be the subriemannian Hessian matrix with ij-th entry $\mathfrak{H}_{ij}f(x)$ defined by

$$\mathfrak{H}_{ij}f(x) = \langle H^{SR}f(x)\iota_i, \iota_j \rangle = \omega(Sf_i(0), f_j(0)).$$

Recall that v_0 denotes the Reeb field and v_1, v_2 be an orthonormal basis with respect to the subriemannian metric g.

Proposition 7.2. *The subriemannian Hessian matrix $\mathfrak{H}f$ satisfies the following*

$$\mathfrak{H}_{11}f = \frac{(v_1f)^2v_2^2f + (v_2f)^2v_1^2f - (v_1f)(v_2f)(v_1v_2f + v_2v_1f)}{(v_1f)^2 + (v_2f)^2}$$
$$+ a_{12}^1v_2f - a_{12}^2v_1f,$$
$$\mathfrak{H}_{12}f = (v_1f)v_2v_0f - (v_2f)v_1v_0f + (v_0f)\mathfrak{H}_{13}f + \vec{\xi}_1a(df),$$
$$\mathfrak{H}_{13}f = \frac{(v_1f)(v_2f)(v_1^2f - v_2^2f) - (v_1f)^2(v_2v_1f) + (v_2f)^2(v_1v_2f)}{(v_1f)^2 + (v_2f)^2},$$
$$\mathfrak{H}_{22}f = ((v_1f)^2 + (v_2f)^2)v_0^2f - (v_0f)(v_1f)v_1v_0f - (v_0f)(v_2f)v_2v_0f$$
$$+ v_0f(\mathfrak{H}_{23}f + a(df)) - \chi_1(df)$$
$$\mathfrak{H}_{23}f = (v_0f)\mathfrak{H}_{33}f - (v_1f)v_0v_1f - (v_2f)v_0v_2f,$$
$$\mathfrak{H}_{33}f = \frac{(v_1f)^2v_1^2f + (v_1f)(v_2f)(v_2v_1f + v_1v_2f) + (v_2f)^2v_2^2f}{(v_1f)^2 + (v_2f)^2}.$$

Proof. Let A be the matrix with the ij-th entry a_{ij} defined by

$$d(df_x)(\iota_i) = f_i(0) + \sum_{k=1}^{3} a_{ik}e_k(0),$$

By the definition of the linear map S, we have

$$S(f_i(0)) = \sum_{j=1}^{3} a_{ij}e_j(0).$$

It follows that

$$\mathfrak{H}_{ij}f(x) = \omega(S(f_i(0)), f_j(0)) = -a_{ij}.$$

Let us look at the case a_{11}. We have $2H(df) = (v_1f)^2 + (v_2f)^2$. Since $\pi(df_x) = x$, we also have

$$\pi^*\alpha_i(d(df_x))(\iota_1) = \alpha_i(\iota_1).$$

Therefore, by Theorem 2.2, we have

$$
\begin{aligned}
\mathfrak{H}_{ij} f(x) &= -a_{11} \\
&= -\omega(f_1(0), d(df_x)(\iota_1)) \\
&= \frac{(v_1 f)^2 v_2^2 f + (v_2 f)^2 v_1^2 f - (v_1 f)(v_2 f)(v_1 v_2 f + v_2 v_1 f)}{(v_1 f)^2 + (v_2 f)^2} \\
&\quad + a_{12}^1 v_2 - a_{12}^2 v_1.
\end{aligned}
$$

Similar calculations give the rest of the entries of A. $\square$

We define the horizontal gradient ∇_H by $g(\nabla_H f, v) = df(v)$ for all vectors v in the distribution Δ. Recall that η is the volume form defined by $\eta(v_0, v_1, v_2) = 1$. The sub-Laplacian Δ_H is defined by $\Delta_H f = \mathbf{div}_\eta \nabla_H f$. Here $\mathbf{div}_\eta$ denotes the divergence with respect to the volume form η. Let C_2 be the matrix defined by

$$
C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Corollary 7.3. *The subriemannian Hessian matrix $\mathfrak{H}$ and the sub-Laplacian satisfies*

$$
\boldsymbol{tr}(C_2 \mathfrak{H} f) = \Delta_H f,
$$

where $\boldsymbol{tr}$ denote the trace of the matrix.

Proof. A simple calculation using Proposition 7.2 shows that

$$
\mathbf{tr}(C_2 \mathfrak{H} f) = (v_1^2 + v_2^2 + a_{12}^1 v_2 - a_{12}^2 v_1) f = \Delta_H f.
$$

$\square$

Let d be the subriemannian distance function of a subriemannian manifold (M, Δ, g). Let us fix a point x_0 in the manifold M. Let $r : M \to \mathbb{R}$ be the function $r(x) = d(x, x_0)$ and let $\mathfrak{f}(x) = -\frac{1}{2} r^2(x)$. Finally, we show that the subriemannian Hessian of the function $\mathfrak{f}$ takes a very simple form.

Proposition 7.4. *The subriemannian Hessian matrix $\mathfrak{H}\mathfrak{f}$ satisfies the following wherever $\mathfrak{f}$ is twice differentiable*

$$
\begin{aligned}
\mathfrak{H}_{11}\mathfrak{f} &= \Delta_H \mathfrak{f} + 1, \\
\mathfrak{H}_{12}\mathfrak{f} &= (v_1 \mathfrak{f}) v_2 v_0 \mathfrak{f} - (v_2 \mathfrak{f}) v_1 v_0 \mathfrak{f} + \vec{\xi}_1 a(d\mathfrak{f}), \\
\mathfrak{H}_{13}\mathfrak{f} &= 0, \\
\mathfrak{H}_{22}\mathfrak{f} &= -2\mathfrak{f} v_0^2 \mathfrak{f} + (v_0 \mathfrak{f})^2 - \chi_1(d\mathfrak{f}), \\
\mathfrak{H}_{23}\mathfrak{f} &= 0, \\
\mathfrak{H}_{33}\mathfrak{f} &= -1.
\end{aligned}
$$

If we assume that the subriemannian manifold is Sasakian, then the above simplifies to

$$\mathfrak{H}_{11}\mathfrak{f} = \Delta_H\mathfrak{f} + 1,$$
$$\mathfrak{H}_{12}\mathfrak{f} = (v_1\mathfrak{f})v_2v_0\mathfrak{f} - (v_2\mathfrak{f})v_1v_0\mathfrak{f},$$
$$\mathfrak{H}_{13}\mathfrak{f} = 0,$$
$$\mathfrak{H}_{22}\mathfrak{f} = -2\mathfrak{f}\,v_0^2\mathfrak{f} + (v_0\mathfrak{f})^2,$$
$$\mathfrak{H}_{23}\mathfrak{f} = 0,$$
$$\mathfrak{H}_{33}\mathfrak{f} = -1.$$

Proof. The first formula follows from differentiating the following equation by v_0, v_1, and v_2

$$(v_1\mathfrak{f})^2 + (v_2\mathfrak{f})^2 = -2\mathfrak{f}$$

and combining them with Proposition 7.2.

The second follows from $a = 0$. $\qquad\square$

8. Sub-Laplacian of Distance Functions in Sasakian Space Forms

In this section, we give a formula for the sub-Laplacian of the subriemannian distance function of a Sasakian space form. Let d be the subriemannian distance function of a subriemannian manifold (M, Δ, g). Let us fix a point x_0 in the manifold M. Let $r : M \to \mathbb{R}$ be the function $r(x) = d(x, x_0)$ and let $\mathfrak{f}(x) = -\frac{1}{2}r^2(x)$.

Theorem 8.1. *Assume that the subriemannian manifold (M, Δ, g) is a three dimensional Sasakian space form of Tanaka-Webster curvature k. Then the subriemannian Hessian matrix $\mathfrak{H}\mathfrak{f}$ satisfies the following wherever $\mathfrak{f}$ is twice differentiable.*

$$\mathfrak{H}\mathfrak{f} = \begin{cases} -\begin{pmatrix} \frac{\tau(\sin\tau - \tau\cos\tau)}{2-2\cos\tau - \tau\sin\tau} & \frac{\tau^2(1-\cos\tau)}{2-2\cos\tau - \tau\sin\tau} & 0 \\ \frac{\tau^2(1-\cos\tau)}{2-2\cos\tau - \tau\sin\tau} & \frac{\tau^3\sin\tau}{2-2\cos\tau - \tau\sin\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma > 0, \\[6ex] -\begin{pmatrix} \frac{\tau(\tau\cosh\tau - \sinh\tau)}{2-2\cosh\tau + \tau\sinh\tau} & \frac{\tau^2(\cosh\tau - 1)}{2-2\cosh\tau + \tau_0\sinh\tau} & 0 \\ \frac{\tau^2(\cosh\tau - 1)}{2-2\cosh\tau + \tau_0\sinh\tau} & \frac{\tau^3\sinh\tau}{2-2\cosh\tau + \tau\sinh\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma < 0, \\[6ex] -\begin{pmatrix} 4 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma = 0, \end{cases}$$

where $\sigma = (v_0\mathfrak{f}(z))^2 - 2\mathfrak{f}(z)k$ and $\tau = \sqrt{|\sigma|}$.

By combining Theorem 8.1 and Proposition 7.3, we obtain the following.

Corollary 8.2. *Let d be the subriemannian distance function of a three dimensional Sasakian space form of Tanaka-Webster curvature k. Then the sub-Laplacian $\Delta_H r$ satisfies the following wherever r is twice differentiable.*

$$\Delta_H r = \begin{cases} \frac{\tau(\sin\tau - \tau\cos\tau)}{r(2 - 2\cos\tau - \tau\sin\tau)} & \text{if } \sigma > 0, \\ \frac{\tau(\tau\cosh\tau - \sinh\tau)}{r(2 - 2\cosh\tau + \tau\sinh\tau)} & \text{if } \sigma < 0, \\ \frac{4}{r} & \text{if } \sigma = 0, \end{cases}$$

where $\sigma = r(z)^2((v_0 r(z))^2 + k)$ and $\tau = \sqrt{|\sigma|}$.

Proof of Theorem 8.1. Let $\varphi_t(x) = \pi(e^{t\vec{H}}(d\mathfrak{f}_x))$. Assume that z is a point where $\mathfrak{f}$ is twice differentiable. Let Λ be the image of the linear map $d((d\mathfrak{f})_z) : T_z M \to T_{d\mathfrak{f}_z} T^* M$. Let $E_t = (e_1(t), e_2(t), e_3(t))^T$, $F_t = (f_1(t), f_2(t), f_3(t))^T$ be a Darboux frame at df_z and let $\iota_i = d\pi(f_i(0))$. Let A_t and B_t be the matrices with ij-th entry $a_{ij}(t)$ and $b_{ij}(t)$, respectively, defined by

$$d(d\mathfrak{f}_y)(\iota_i) = \sum_{j=1}^{3} (a_{ij}(t)e_j(t) + b_{ij}(t)f_j(t)).$$

We define the matrix S_t by $S_t = B_t^{-1} A_t$. Since $\pi(e^{1\cdot\vec{H}}(d\mathfrak{f}_x)) = x_0$ for all x, we have $\lim_{t\to 1} S_t^{-1} = 0$. The same argument as in Theorem 5.1 shows that

$$\dot{S}_t - R + S_t C_1 + C_1^T S_t - S_t C_2 S_t = 0,$$

where $R = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and σ is defined by

$$\sigma = h_0(d\mathfrak{f}_z)^2 + 2H(d\mathfrak{f}_z)k = (v_0\mathfrak{f}(z))^2 - 2\mathfrak{f}(z)k = r(z)^2((v_0 r(z))^2 + k).$$

By the proof of Proposition 7.2 and $B_0 = I$, we have $\mathfrak{H}\mathfrak{f}(z) = -A_0 = -S_0$. By the result in [13], we can compute S_t and it is given by

$$
S_t = \begin{cases}
\begin{pmatrix}
\frac{\tau_0(\sin \tau_t - \tau_t \cos \tau_t)}{2 - 2\cos \tau_t - \tau_t \sin \tau_t} & \frac{\tau_0^2(1 - \cos \tau_t)}{2 - 2\cos \tau_t - \tau_t \sin \tau_t} & 0 \\
\frac{\tau_0^2(1 - \cos \tau_t)}{2 - 2\cos \tau_t - \tau_t \sin \tau_t} & \frac{\tau_0^3 \sin \tau_t}{2 - 2\cos \tau_t - \tau_t \sin \tau_t} & 0 \\
0 & 0 & \frac{1}{1-t}
\end{pmatrix} & \text{if } \sigma > 0, \\[3em]
\begin{pmatrix}
\frac{\tau_0(\tau_t \cosh \tau_t - \sinh \tau_t)}{2 - 2\cosh \tau_t + \tau_t \sinh \tau_t} & \frac{\tau_0^2(\cosh \tau_t - 1)}{2 - 2\cosh \tau_t + \tau_t \sinh \tau_t} & 0 \\
\frac{\tau_0^2(\cosh \tau_t - 1)}{2 - 2\cosh \tau_t + \tau_t \sinh \tau_t} & \frac{\tau_0^3 \sinh \tau_t}{2 - 2\cosh \tau_t + \tau_t \sinh \tau_t} & 0 \\
0 & 0 & \frac{1}{1-t}
\end{pmatrix} & \text{if } \sigma < 0, \\[3em]
\begin{pmatrix}
\frac{4}{1-t} & \frac{6}{(1-t)^2} & 0 \\
\frac{6}{(1-t)^2} & \frac{12}{(1-t)^3} & 0 \\
0 & 0 & \frac{1}{1-t}
\end{pmatrix} & \text{if } \sigma = 0,
\end{cases}
$$

where $\tau_t = (1-t)\sqrt{|\sigma|}$.

By setting $t = 0$, we obtain the result. $\qquad\square$

9. Subriemannian Hessian and Laplacian Comparison Theorem

In this section, we prove a Hessian and a Laplacian comparison theorem in our subriemannian setting. Let (M^1, Δ^1, g^1) and (M^2, Δ^2, g^2) be three dimensional contact subriemannian manifolds. Let x_0^i be a point on the manifold M^i, let r_i be the subriemannian distance from the point x_0^i, and let $\mathfrak{f}_i = -\frac{1}{2}r_i^2$. Let v_0^i be the Reeb field in M^i and let $(R_t^{11})^i$ and $(R_t^{22})^i$ be the curvature invariant on M^i introduced in section 2. Finally let

$$
R_t^i = \begin{pmatrix}
(R_t^{11})^i & 0 & 0 \\
0 & (R_t^{22})^i & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

Theorem 9.1. *(Subriemannian Hessian Comparison Theorem I) Let z_1 and z_2 be points on the three dimensional contact subriemannian manifolds M^1 and M^2, respectively, such that $\mathfrak{f}_i$ is twice differentiable at z_i. Assume that $R_t^1\big|_{(d\mathfrak{f}_1)_{z_1}} \leq R_t^2\big|_{(d\mathfrak{f}_2)_{z_2}}$ for all t in the interval $[0,1]$. Then*

$$
\mathfrak{H}\mathfrak{f}_1(z_1) \leq \mathfrak{H}\mathfrak{f}_2(z_2).
$$

Remark 9.2. By the result in [9], $\mathfrak{f}_i$ is twice differentiable Lebesgue almost everywhere.

If we restrict to Sasakian manifolds, then we have the following.

Theorem 9.3. *(Subriemannian Hessian Comparison Theorem II) Let d be the subriemannian distance function of a Sasakian manifold (M, Δ, g) and let $\mathfrak{f}(x) = -\frac{1}{2}d^2(x, x_0)$, where x_0 is a point on M. Assume that the Tanaka-Webster curvature $\mathfrak{W}$ of M satisfies $\mathfrak{W} \geq k$ (resp. $\mathfrak{W} \leq k$). Then the following holds wherever f is twice differentiable.*

$$
\mathfrak{Hf} \geq \begin{cases}
-\begin{pmatrix} \frac{\tau(\sin\tau - \tau\cos\tau)}{2 - 2\cos\tau - \tau\sin\tau} & \frac{\tau^2(1 - \cos\tau)}{2 - 2\cos\tau - \tau\sin\tau} & 0 \\ \frac{\tau^2(1 - \cos\tau)}{2 - 2\cos\tau - \tau\sin\tau} & \frac{\tau^3\sin\tau}{2 - 2\cos\tau - \tau\sin\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma > 0, \\[3em]
-\begin{pmatrix} \frac{\tau(\tau\cosh\tau - \sinh\tau)}{2 - 2\cosh\tau + \tau\sinh\tau} & \frac{\tau^2(\cosh\tau - 1)}{2 - 2\cosh\tau + \tau\sinh\tau} & 0 \\ \frac{\tau^2(\cosh\tau - 1)}{2 - 2\cosh\tau + \tau\sinh\tau} & \frac{\tau^3\sinh\tau}{2 - 2\cosh\tau + \tau\sinh\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma < 0, \\[3em]
-\begin{pmatrix} 4 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma = 0,
\end{cases}
$$

(resp. $\leq$) where $\sigma = (v_0\mathfrak{f})^2 - 2k\mathfrak{f}$ and $\tau = \sqrt{|\sigma|}$.

If we combine Theorem 9.3 and Proposition 7.3, then we have the following sub-Laplacian comparison theorem.

Corollary 9.4. *(Sub-Laplacian Comparison Theorem) Under the notation and assumptions of Theorem 9.3, the following holds wherever r is twice differentiable.*

$$
\Delta_H r \leq \begin{cases}
\frac{\tau(\sin\tau - \tau\cos\tau)}{r(2 - 2\cos\tau - \tau\sin\tau)} & \text{if } \sigma > 0, \\
\frac{\tau(\tau\cosh\tau - \sinh\tau)}{r(2 - 2\cosh\tau + \tau\sinh\tau)} & \text{if } \sigma < 0, \qquad (resp. \ \geq) \\
\frac{4}{r} & \text{if } \sigma = 0,
\end{cases}
$$

where $r(x) = d(x, x_0)$, $\sigma = r(z)^2((v_0 r(z))^2 + k)$, and $\tau = \sqrt{|\sigma|}$.

Proof of Theorem 9.1. Let $\varphi_t^i(x) = \pi(e^{t\vec{H}}((d\mathfrak{f}_i)_x))$ and let

$$
E_t^i = (e_1^i(t), e_2^i(t), e_3^i(t))^T, \quad F_t^i = (f_1^i(t), f_2^i(t), f_3^i(t))^T
$$

be a Darboux frame at $(d\mathfrak{f}_i)_{z_i}$ and let $\iota_j^i = d\pi(f_j(0))$. Let A_t^i and B_t^i be the matrices with jk-th entry $a_{jk}^i(t)$ and $b_{jk}^i(t)$, respectively, defind by

$$
d(d\mathfrak{f}_{z_i})(\iota_j) = \sum_{k=1}^{3}(a_{jk}^i(t)e_k^i(t) + b_{jk}^i(t)f_k^i(t)).
$$

We define the matrix S_t^i by $(B_t^i)^{-1}A_t^i$. As in the proof of Theorem 8.1, we have $\mathfrak{Hf}_i(z_i) = -S_0^i = -A_0^i$ and

$$
\dot{S}_t^i - R_t^i + S_t^i C_1 + C_1^T S_t^i - S_t^i C_2 S_t^i = 0 \quad \lim_{t \to 1}(S_t^i)^{-1} = 0,
$$

where R_t^i here denotes $R_t^i\big|_{df_{z_i}}$.

Therefore, by assumption and the result in [18], we have the following as claimed

$$\mathfrak{H}\mathfrak{f}_2 = -S_t^2 \geq -S_t^1 = \mathfrak{H}\mathfrak{f}_1.$$

$\square$

Proof of Theorem 9.3. Let us first assume that $\mathfrak{W} \geq k$. Here we use the same notation as in proof of Theorem 8.1. The matrix S_t satisfies the equation

$$\dot{S}_t - R_t + S_t C_1 + C_1^T S_t - S_t C_2 S_t = 0, \quad \lim_{t \to 1}(S_t)^{-1} = 0,$$

where

$$R_t = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the result in [18], we have $S_t \leq S_t^k$, where S_t^k is the solution of

$$\dot{S}_t - R^k + S_t C_1 + C_1^T S_t - S_t C_2 S_t = 0, \quad \lim_{t \to 1}(S_t)^{-1} = 0,$$

and

$$R^k = \begin{pmatrix} (v_0\mathfrak{f})^2 - 2k\mathfrak{f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we set $t = 0$, then we get $\mathfrak{H}\mathfrak{f} = -S_0 \geq -S_0^k$. Finally the matrix S_0^k can be computed using the result in [13] which gives the first statement of the theorem. The reverse inequalities under the assumption $\mathfrak{W} \leq k$ are proved in a similar way. $\square$

10. CHEEGER-YAU TYPE THEOREM IN SUBRIEMANNIAN GEOMETRY

In this section, we give a lower bound on the solution of the subriemannian heat equation $\dot{u} = \Delta_H u$ in the spirit of its Riemannian analogue in [11]. More precisely, let ϕ be the function defined by

$$\phi(s) = \begin{cases} \dfrac{\sqrt{k}(\sin(s\sqrt{k}) - s\sqrt{k}\cos(s\sqrt{k}))}{2 - 2\cos(s\sqrt{k}) - s\sqrt{k}\sin(s\sqrt{k})} & \text{if } k > 0 \\ \dfrac{4}{s} & \text{if } k = 0. \end{cases}$$

Let $h = h(t, s) : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be a smooth solution to the following equation

(10.1) $$\dot{h} = h'' + h'\phi.$$

where $\dot{h}$ and h' denotes the derivative with respect to t and s, respectively.

Theorem 10.1. *Let (M, Δ, g) be a three dimension Sasakian manifold with non-negative Tanaka-Webster curvature. Let h be a solution of (10.1) which satisfies the conditions*

$$h'(0, s) \leq 0, \quad \lim_{s \to 0} h'(t, s) \leq 0.$$

Let x_0 be a point on the manifold M and let $r(\cdot) = d(x_0, \cdot)$, where d is the subriemannian distance function. Let Ω be an open set which contains x_0 and have smooth boundary $\partial\Omega$. Let $u = u(t, x)$ be a smooth solution to the subriemannian heat equation $\dot{u} = \Delta_H u$ on $[0, \infty) \times M \backslash \Omega$ which satisfies

$$u(0, \cdot) \geq h(0, r(\cdot)), \quad u(t, y) \geq h(t, r(y)) \quad y \in \partial\Omega.$$

Then we have $u \geq h \circ r$.

Remark 10.2. When $k = 0$, the function

$$h(t, s) = (t + \epsilon)^{-5/2} e^{\frac{-s^2}{4(t+\epsilon)}}$$

is a solution to the equation (10.1) for every $\epsilon > 0$.

Proof. Let r be the subriemannian distance function from the point x_0 (i.e. $r(x) = d(x_0, x)$). By Corollary 9.4 and the chain rule, the following holds η-a.e.

$$\Delta_H r \leq \begin{cases} \frac{\tau(\sin\tau - \tau\cos\tau)}{r(2 - 2\cos\tau - \tau\sin\tau)} & \text{if } \sigma > 0 \\ \frac{4}{r} & \text{if } \sigma = 0 \end{cases}$$

where $\sigma = r^2((v_0 r)^2 + k)$ and $\tau = r\sqrt{(v_0 r)^2 + k}$.

If we differentiate (10.1) with respect to s. Then we get

$$(10.2) \qquad\qquad \dot{h}' = h''' + h''\phi + h'\phi'.$$

By the maximum principle, we see that $h'(t, s) \leq 0$ for all t and for all s since $h'(0, s) \leq 0$ for all s and $h'(t, 0) \leq 0$ for all t by assumptions.

Therefore, the following holds wherever r is twice differentiable.

$$\begin{aligned}
\Delta_H(g(t, r)) &= h''(t, r) + h'(t, r)\Delta_H r \\
(10.3) \qquad &\geq h''(t, r) + h'(t, r)\phi(r) \\
&= \dot{h}(t, r).
\end{aligned}$$

Let (t_0, z) be a local minimum of the function $G(t, x) = u(t, x) - h(t, r(x)) + \delta t$, where δ is a positive constant. Let us assume that $t_0 > 0$ and z in contained in the interior of $M \backslash \Omega$. By the result in [9], r is locally semiconcave on $M \backslash \{x_0\}$. Since g is nondecreasing in

s, $G(t, x)$ is locally semiconcave on $M \setminus \{x_0\}$ as well. Therefore, by [8, Theorem 2.3.2], we can find a sequence of points z_i on the manifold M converging to z and a sequence of numbers ϵ_i converging to 0 such that

$$\Delta_H G(t_0, z_i) \geq -\epsilon_i.$$

Since u is the solution of the subriemannian heat equation, it follows from (10.3) that $\frac{d}{dt} G(t_0, z_i) \geq -\epsilon_i + \delta$. If we let i go to ∞, then we have $0 = \frac{d}{dt} G(t_0, z) \geq \delta$ which is a contradiction.

Since we have the condition $u(t, x) \geq h(t, r(x))$ for all points x on the boundary of Ω and $u(0, \cdot) \geq h(0, r(\cdot))$, it follows that $G \geq 0$. Therefore, if we let δ go to 0, then we have $u \geq h \circ r$ as claimed. $\qquad \square$

References

[1] A. Agrachev, P.W.Y. Lee: Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds, preprint, arXiv: 0903.2550.

[2] F. Baudoin, N. Garofalo: Generalized Bochner formulas and Ricci lower bounds for sub-Riemannian manifolds of rank two, preprint, arXiv:0904.1623.

[3] F. Baudoin, N. Garofalo: Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries, preprint, arXiv:1101.3590.

[4] F. Baudoin, M. Bonnefont, N. Garofalo: A sub-riemannian curvature-dimension inequality, volume doubling property and the Poincaŕe inequality, preprint, arxiv: 1007.1600.

[5] A. Bellaïche, J.-J. Risler: Sub-Riemannian Geometry, Birkhäuser Verlag, Basel, 1996.

[6] R. Bhatia: Matrix analysis. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997.

[7] U. Boscain, F. Rossi: Invariant Carnot-Caratheodory metrics on S^3, SO(3), SL(2), and lens spaces. SIAM J. Control Optim. 47 (2008), no. 4, 1851–1878.

[8] P. Cannarsa, C. Sinestrari: Semiconcave functions, Hamilton-Jacobi equations, and optimal control. Progress in Nonlinear Differential Equations and their Applications, 58. Birkhuser Boston, Inc., Boston, MA, 2004.

[9] P. Cannarsa, L. Rifford: Semiconcavity results for optimal control problems admitting no singular minimizing controls. Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 4, 773–802.

[10] S. Chanillo, P.-C. Yang: Isoperimetric inequalities & volume comparison theorems on CR manifolds. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 2, 279–307.

[11] J. Cheeger, S.T. Yau: A lower bound for the heat kernel. Comm. Pure Appl. Math. 34 (1981), no. 4, 465-480.

[12] N. Juillet: Geometric inequalities and generalized Ricci bounds in the Heisenberg group. Int. Math. Res. Not. IMRN 2009, no. 13, 2347-2373.

[13] J.J. Levin: On the matrix Riccati equation. Proc. Amer. Math. Soc. 10 1959 519–524.

[14] C.B. Li, I. Zelenko: Differential geometry of curves in Lagrange Grassmannians with given Young diagram. Differential Geom. Appl. 27 (2009), no. 6, 723742.

[15] C.B. Li, I. Zelenko: Parametrized curves in Lagrange Grassmannians, C.R. Acad. Sci. Paris, Ser. I, Vol. 345, Issue 11, 647–652.

[16] C.B. Li, I.Zelenko: Jacobi Equations and Comparison Theorems for Corank 1 sub-Riemannian Structures with Symmetries, Journal of Geometry and Physics 61 (2011) 781–807.

[17] R. Montgomery: A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[18] H.L. Royden: Comparison theorems for the matrix Riccati equation, Comm. Pure Appl. Math. 41 (1988), no. 5, 739-746.

[19] S. Tanno: Variational problems on contact Riemannian manifolds. Trans. Amer. Math. Soc. 314 (1989), no. 1, 349-379.

E-mail address: `agrachev@sissa.it`

International School for Advanced Studies, via Bonomea 265, 34136, Trieste, Italy and Steklov Mathematical Institute, ul. Gubkina 8, Moscow, 119991 Russia

E-mail address: `wylee@math.cuhk.edu.hk`

Room 216, Lady Shaw Building, The Chinese University of Hong Kong, Shatin, Hong Kong