QUADRATIC MAPS AND SMOOTH VECTOR-VALUED FUNCTIONS:
EULER CHARACTERISTICS OF LEVEL SETS

A. A. Agrachev and R. V. Gamkrelidze

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Quadratic maps of R^N into R^K are studied. Explicit expressions are obtained for the Euler characteristics of level sets of such maps. The Euler characteristics of level sets of smooth vector-valued functions are also evaluated in terms of their values at critical points.

Any nonsingular quadratic form on $\mathbf{R}^N$ can be reduced by a linear change of variables to the form $x\mapsto\sum_{i=n+1}^{N}x_i^2-\sum_{i=1}^{n}x_i^2$. The number n is the index of the form: it is independent of the specific transformation used to diagonalize the form and in, therefore, the only linear invariant of a nonsingular quadratic form. The index also possesses a simple topological interpretation. Let $\mathcal{P}(\mathbf{R}^N)$ be the space of all quadratic forms on $\mathbf{R}^N$, then the set of all singular forms Π_1 is a hypersurface, which is also an orientable pseudomanifold. Fixing an orientation of Π_1, we can define the intersection number of any curve in $\mathcal{P}(\mathbf{R}^N)$, whose ends lie outside Π_1 with Π_1. This intersection number depends only on the ends of the curve, but not on the path connecting them. The index of a nonsingular form $q\in\mathcal{P}(\mathbf{R}^N)$ is equal to the intersection number of the surface Π_1, suitably oriented, and any curve one of whose ends is q and the other a positive definite form.

Next, a form $q\in\mathcal{P}(\mathbf{R}^N)$ is nonsingular if and only if zero is not a critical value of the restriction of q to the sphere $S^{N-1}\subset\mathbf{R}^N$. In that case $q^{-1}(0)\cap S^{N-1}$ is a smooth submanifold of S^{N-1} and, moreover, submanifolds corresponding to forms in the same connected component of the set $\mathcal{P}(\mathbf{R}^N)\setminus\Pi_1$ are necessarily diffeomorphic. In fact, $q^{-1}(0)\cap S^{N-1}=S^{N-1}\times S^{N-n-1}$, where $n=\operatorname{ind} q$.

Consider now a vector-valued quadratic map $p=(p_1,\ldots,p_k)^T$, $p_i\in\mathcal{P}(\mathbf{R}^N)$, $i=1,\ldots,k$. Already in the case $k=2$, the classification of these maps with respect to linear changes of variables in $\mathbf{R}^N$ and in $\mathbf{R}^k$ involves continuous parameters; and for $k\geq 3$ the problem becomes quite intractable. Nevertheless, it turns out that considerations of a topological nature, similar to those outlined above, work in this situation as well.

A quadratic map p is said to be nonsingular if the zero of $\mathbf{R}^k$ is not a critical value of the restriction of p to S^{N-1}. Under a continuous deformation of p in the class of nonsingular maps, the smooth submanifold $p^{-1}(0)\cap S^{N-1}$ transforms according to some isotopy of the sphere S^{N-1}. Note that $p^{-1}(0)$ is the set of real solutions of a homogeneous system of quadratic equations. If one is interested in systems of quadratic inequalities, one must consider $p^{-1}(C)$, where C is a convex closed cone in $\mathbf{R}^k$. In that event one has the concept of a quadratic map which is nonsingular (or nondegenerate) with respect to C (see Sec. 3).

Any quadratic map p determines a pencil of quadratic forms $\omega p=\sum_{i=1}^{k}\omega_ip_i$, $\omega=(\omega_1,\ldots,\omega_k)\in\mathbf{R}^{k*}$, $|\omega|=1$. This pencil is a sphere of dimension $\leq(k-1)$ embedded in $\mathcal{P}(\mathbf{R}^N)$. It turns out that p is a nonsingular map if and only if this sphere is not tangent to the hypersurface Π_1 at any point.†

†Π_1 has singularities. There are various ways of understanding tangency at a singular point. Here we are thinking of tangency in the strongest possible sense: a sphere is tangent to Π_1 at a singular point if application of a small smooth deformation of the sphere will make it tangent to Π_1 at a nonsingular point (see Secs. 1, 2).

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If a sphere embedded in $\mathcal{P}(\mathbf{R}^N)$ is subjected to a smooth deformation, taking it from one nonsingular position to another, also nonsingular, there will generally be intermediate positions in which the sphere is tangent to Π_1. One can define the algebraic number of points of tangency, which indeed depends only on the initial and final position of the sphere but not on the specific deformation. But one can say even more: beginning with the sphere ωp, $|\omega| = 1$, if the deformation results in a sphere consisting entirely of positive definite forms, then the algebraic number of points of tangency is

$$\frac{(-1)^k}{2}(\chi(p^{-1}(0)\cap S^{N-1})+(-1)^N-1),$$

where $\chi(p^{-1}(0) \cap S^{N-1})$ is the Euler characteristic of the manifold in the parentheses.

The most important invariant of a real quadratic form is its index. For a pencil of forms, however, one must consider not one index but an integer-valued function $\omega \mapsto \operatorname{ind}(\omega p)$, $\omega \in \mathbf{R}^{k*}$, $|\omega| = 1$. This function defines a filtration of the sphere S^{k-1} by the subsets $\Omega_n = \{\omega \in S^{k-1} | \operatorname{ind}(\omega p) \leq n\}$. If $k = 2$, the homotopy type of this filtration is preserved under a smooth deformation of p in the class of nonsingular quadratic maps. This is not so, however, when $k \geq 3$: even the quantity $\min_{|\omega|=1} \operatorname{ind}(\omega p)$ may change under such a deformation. The reason lies in the structure of the singularities of the hypersurface Π_1: a sphere ωp, $\omega \in S^{k-1}$, embedded in $\mathcal{P}(\mathbf{R}^N)$, may leave a domain of given index without having touched Π_1. One consequence of this situation is the existence of quadratic maps with nonconvex images in $\mathbf{R}^k$, $k \geq 3$.

Nevertheless, something is preserved. For example,

$$\chi(p^{-1}(0)\cap S^{N-1})=2\sum_{n=0}^{[\frac{N}{2}]}(\chi(\Omega_{2n+1})-\chi(\Omega_{2n}))+1+(-1)^{N-1}.$$

There is an analogous formula for $\chi(p^{-1}(C) \cap S^{N-1})$, where C is a convex closed cone in $\mathbf{R}^k$ to derive it one has to replace the set Ω_n ny $\Omega_n \cap C^\circ$ (see Theorem 4.1).

Fixed sections of this paper are devoted to establishing the various relationships just described, as well as the necessary preparatory material, examples and generalizations. The topic of the sixth and last section is not quadratic but general smooth vector-valued functions on compact oriented manifolds. Expressions are developed for the Euler characteristics of level sets in terms of the values of vector-valued functions at critical points. The fundamental notion in this context is that of a Morse vector-valued function — a direct generalization of scalar Morse functions.

To each critical point corresponds a quadratic form — the Hessian of the function at that point. The index of the Hessian may vary from point to point and one obtains a partition of the critical set into domains of different indices. Under a favorable confluence of circumstances, one can calculate the Euler characteristics of the preimages of convex cones in terms of those of these domains.

Propositions, theorems, lemmas, and formulas within each section will be numbered separately. In references to other sections we will use double numbers.

1. Spaces of Quadratic Forms

1. Let $\mathcal{P}(\mathbf{R}^N)$ be the space of all real bilinear symmetric forms on $\mathbf{R}^N$, $\dim\mathcal{P}(\mathbf{R}^N) = N \times (N + 1)/2$. If $p\in\mathcal{P}(\mathbf{R}^N)$, $p:\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}^N$, then the quadratic form $x \to p(x, x)$, where $x \in \mathbf{R}^N$, will be denoted by the same symbol p. The subspace $\{x \in \mathbf{R}^N | p(x,y)=0 \ \forall y\in\mathbf{R}^N\}\subset\mathbf{R}^N$ will be called the kernel of the quadratic form p and denoted by $\ker p$. A form p is said to be nonsingular if $\ker p = 0$; otherwise it will be called singular or degenerate. A form p is to be nonnegative (nonpositive) if $\ker p = 0$. It is easy to see that if p is a nonnegative (nonpositive) form then $p(x, x) = 0$ if and only if $x \in \ker p$. Thus, a nonnegative form is positive (a nonpositive form is negative) if and only if it is nonsingular.

For any subspace $V \subset \mathbf{R}^N$, we let $(p|V) \in \mathcal{P}(V)$ denote the quadratic form obtained by restricting the map p to V.

The index of a quadratic form p (denoted by $\operatorname{ind} p$) is the maximum number $k \geq 0$ such that for some subspace $V \subset \mathbf{R}^N$, $\dim V = k$, the form $p|V$ is negative.

The linear changes of coordinates in $\mathbf{R}^N$ define an action of the group $GL(\mathbf{R}^N)$ of linear transformations of $\mathbf{R}^N$ in the space of quadratic forms $\mathscr{P}(\mathbf{R}^N)$. Clearly, the numbers $\dim \ker p$, $\operatorname{ind} p$ are invariant under this action (i.e., independent of the choice of coordinates). It follows from the usual procedure of reducing a quadratic form to a sum of squares that there are no other invariants: quadratic forms with the same indices and kernels of the same dimension can be transformed into one another by a linear change of coordinates.

Let $(\cdot, \cdot)$ be the standard scalar product in $\mathbf{R}^N$. To any form $p\in\mathscr{P}(\mathbf{R}^N)$ there corresponds a symmetric linear operator $P: \mathbf{R}^N\to\mathbf{R}^N$, defined by the identity $p(x, y) = (Px, y)$, $\forall x, y\in\mathbf{R}^N$. The correspondence $p \mapsto P$ is obviously an isomorphism of the linear space $\mathscr{P}(\mathbf{R}^N)$ onto the space of all symmetric linear transformations in $\mathbf{R}^N$. Under this correspondence $\ker p = \ker P$, and $\operatorname{ind} p$ is the number of negative eigenvalues (counting multiplicities) of P.

Define $\mathscr{P}_k(\mathbf{R}^N)=\{p\in\mathscr{P}(\mathbf{R}^N) \mid \operatorname{ind} p \le k\}$. In particular, $\mathscr{P}_0(\mathbf{R}^N)$ – the set of all nonnegative forms – is clearly a convex closed full-dimensional acute-angled cone in $\mathscr{P}(\mathbf{R}^N)$. The cone $\mathscr{P}_0(\mathbf{R}^N)$ defines a partial ordering of the space $\mathscr{P}(\mathbf{R}^N)$: we write $p_1 \le p_2$ if $(p_2 - p_1) \in \mathscr{P}_0(\mathbf{R}^N)$, and $p_1 < p_2$ if in addition the form $(p_2 - p_1)$ is nonsingular. The following useful relationship implies that the map $\operatorname{ind}: \mathscr{P}(\mathbf{R}^N)\to\mathbf{Z}_+$ is monotone nonincreasing.

<u>LEMMA 1.</u> For any $p_1, p_2 \in \mathscr{P}(\mathbf{R}^N)$

$$\operatorname{ind}(p_1+p_2)\leqslant \operatorname{ind} p_1+\operatorname{ind} p_2.$$

<u>Proof.</u> Let V_i be the linear span of the eigenvectors of P_i belonging to nonnegative eigenvalues, $\operatorname{codim} V_i = \operatorname{ind} p_i$, $i = 1, 2$. Then both p_1 and p_2 are nonnegative forms on the space $V_1 \cap V_2$, $\operatorname{codim} V_1 \cap V_2 \le \operatorname{ind} p_1 + \operatorname{ind} p_2$.

Let $P: \mathbf{R}^N\to\mathbf{R}^N$ be a linear symmetric operator and $\lambda_1(P) \le \lambda_2(P) \le \ldots \le \lambda_N(P)$ its eigenvalues, arranged in increasing order; note that $\lambda_1(P)$ is a continuous (but not smooth!) function of P. Let I be the identity map of the space $\mathbf{R}^N$, i.e., the operator corresponding to the quadratic form $\mapsto$, i.e., the operator corresponding to the quadratic form $x \mapsto (x, x)$, $x \in \mathbf{R}^N$. The map $P \mapsto P + (\lambda_{k+1} - \lambda_1)I$ defines a monotone homeomorphism of the space $\mathscr{P}(\mathbf{R}^N)$ onto itself, carrying the subset $\mathscr{P}_k(\mathbf{R}^N)$ onto $\mathscr{P}_0(\mathbf{R}^N)$. Hence it follows, in particular, that $\mathscr{P}(\mathbf{R}^N)$ is homeomorphic to a closed half-space in $\mathbf{R}^{N(N+1)/2}$ for $k = 0, 1, \ldots, N - 1$.

The boundary of the subset $\mathscr{P}_k(\mathbf{R}^N)$ consists of the singular forms and is defined by the equation $\lambda_{k+1}(P) = 0$. Let $\Pi_1(\mathbf{R}^N)$ denote the set of all singular forms. Clearly, $\Pi_1 \times (\mathbf{R}^N) = \{p\in\mathscr{P}(\mathbf{R}^N) \mid P = 0\}$ is an algebraic hypersurface in $\mathscr{P}(\mathbf{R}^N)$. The set of nonsingular forms $\mathscr{P}(\mathbf{R}^N)\setminus\Pi_1(\mathbf{R}^N)$ has $N + 1$ connected components: nonsingular forms p_1, p_2 are in the same component if and only if $\operatorname{ind} p_1 = \operatorname{ind} p_2$. The hypersurface $\Pi_1(\mathbf{R}^N)$, naturally, has singularities; a form $p \in \Pi_1(\mathbf{R}^N)$ is a regular point of the hypersurface if and only if $\dim \ker p = 1$. The local structure of $\Pi_1(\mathbf{R}^N)$ near an arbitrary point – including a singular point – is described by the following assertion.

<u>LEMMA 2.</u> Let $p_0 \in \mathscr{P}(\mathbf{R}^N)$, $V = \ker p_0$. Then there exist a neighborhood $\mathcal{O}_{p_0}$ of p_0 in $\mathscr{P}(\mathbf{R}^N)$ and an analytic map $\Phi: \mathcal{O}_{p_0}\to\mathscr{P}(V)$ such that 1) $\Phi(p_0) = 0$; 2) $\operatorname{ind} p = \operatorname{ind} p_0 + \operatorname{ind}\Phi(p)$; 3) $\dim \ker p = \dim \ker \Phi(p)$; 4) $(D_{p_0}\Phi)p = p|V$, $\forall p$.

<u>Proof.</u> Let γ be some closed contour in the complex plane separating the origin from the nonzero eigenvalues of P_0. Define

$$\pi_p = \frac{1}{2\pi i}\int_\gamma (P - \xi I)^{-1} d\xi$$

for any operator P with no eigenvalues on γ. Clearly, π_p is a symmetric operator, which commutes with P and is analytic with respect to p; moreover, $\pi_p^2 = \pi_p$. Indeed, π_p is the orthogonal projection of $\mathbf{R}^N$ onto the invariant subspace of P corresponding to the eigenvalues in the interior of γ, in particular, $\pi_{p_0}|V = I$. Define $\Phi(p)(v_1, v_2) = p(\pi_p v_1, \pi_p v_2)$, $\forall\, v_1, v_2 \in V$. For p close to p_0, the map $\pi_p|V$ is nonsingular and, so, the form $\Phi(p)$ is equivalent to $p|\operatorname{im}\pi_p$. But the form $p|\operatorname{im}\pi_p^{\perp}$ is nonsingular and has index equal to $\operatorname{ind} p_0$ for p near p_0. Equalities 2 and 3 now follow from the fact that $\operatorname{im}\pi_p$, $\operatorname{im}\pi_p^{\perp}$ are invariant subspaces of P; equalities 1, 4 are verified directly.

Define $\Pi_k(\mathbf{R}^N)=\{p\in\mathscr{P}(\mathbf{R}^N)\,|\,\dim\ker p\geq k\}$, $k = 0, \ldots, N$; then

$$0=\Pi_N(\mathbf{R}^N)\subset\Pi_{N-1}(\mathbf{R}^N)\subset\ldots\subset\Pi_1(\mathbf{R}^N)\subset\Pi_0(\mathbf{R}^N)=\mathscr{P}(\mathbf{R}^N)$$

is a filtration of $\mathscr{P}(\mathbf{R}^N)$ by closed subsets. It follows from Lemma 2, in particular, that for any $k = 0, 1, \ldots, N$ the set $\Pi_k(\mathbf{R}^N)\setminus\Pi_{k+1}(\mathbf{R}^N)$ is an analytic submanifold of codimension $k(k+1)/2$ in $\mathscr{P}(\mathbf{R}^N)$, and, moreover, $\overline{\Pi_k(\mathbf{R}^N)\setminus\Pi_{k+1}(\mathbf{R}^N)}=\Pi_k(\mathbf{R}^N)$. In fact, if $\dim\ker p_0 = k$, then the equation $\Phi(p) = 0$ defines the intersection of $\Pi_k(\mathbf{R}^N)\setminus\Pi_{k+1}(\mathbf{R}^N)$ with a neighborhood of p_0 in $\mathscr{P}(\mathbf{R}^N)$.

It is worth noting that the sets $\Pi_k(\mathbf{R}^N)$ are closed under multiplication by scalars, i.e., they are cones and the manifolds $\Pi_k(\mathbf{R}^N)\setminus\Pi_{k+1}(\mathbf{R}^N)$ have $N+1-k$ connected components: forms $p_1, p_2\in\Pi_k(\mathbf{R}^N)\setminus\Pi_{k+1}(\mathbf{R}^N)$ are in the same component if and only if $\operatorname{ind}p_1 = \operatorname{ind}p_2$.

2. We now consider the algebraic hypersurface $\Pi_1(\mathbf{R}^N)$ in greater detail. Below (both in this subsection and later) we will omit the argument $\mathbf{R}^N$ in the notation $\Pi_k(\mathbf{R}^N)$, wherever no confusion may arise. As already remarked, the set of singular points of Π_1 is just Π_2. Since $\operatorname{codim}\Pi_2 = 3$ and $\operatorname{codim}\Pi_1 = 1$, it follows that Π_1 is a pseudomanifold. Similarly, the set of singular points of the algebraic manifold Π_1 is just Π_3, and so on: for any k the set of singular points of Π_k is just Π_{k+1} — this follows from Lemma 2.

The space of quadratic forms $\mathscr{P}(\mathbf{R}^N)$ is, by definition, the dual space of the symmetric products of $\mathbf{R}^N$ by itself, i.e., $\mathscr{P}(\mathbf{R}^N)=(\mathbf{R}^N\odot\mathbf{R}^N)^*$, where the symbol "$\odot$" denotes the symmetric product. Consequently, $\mathscr{P}(\mathbf{R}^N)^*=\mathbf{R}^N\odot\mathbf{R}^N$. In particular, for any $x, y\in\mathbf{R}^N$ we have $\langle p, x\odot y\rangle = p(x, y)$.

Let $\xi\in\mathbf{R}^N\odot\mathbf{R}^N$, in view of the canonical isomorphisms

$$\mathscr{P}(\mathbf{R}^{N*})=(\mathbf{R}^{N*}\odot\mathbf{R}^{N*})^*=\mathbf{R}^N\odot\mathbf{R}^N=\mathscr{P}(\mathbf{R}^N)^*$$

we can consider ξ as a quadratic form on $\mathbf{R}^{N*}$. Let $\mathbf{R}^{N*}\supset\ker\xi$ be the kernel of this form and denote $L(\xi) = (\ker\xi)^\perp\subset\mathbf{R}^N$, $\operatorname{rank}\xi = \dim L(\xi)$.

An arbitrary element $\xi\in\mathbf{R}^N\odot\mathbf{R}^N$ can be represented (but not uniquely) as $\xi = \sum_{i=1}^k \alpha_i x_i\odot x_i$, $\alpha_i\neq 0$; a direct check verifies the identity

$$L\left(\sum_{i=1}^k\alpha_i x_i\odot x_i\right)=\operatorname{span}\{x_i,\ i=1,\ldots,k\},\quad\text{if}\quad\alpha_i\neq 0,\ i=1,\ldots,k.$$

We note, moreover, that $\operatorname{rank}\xi$ is the minimum possible number of terms in the representation $\xi = \sum_{i=1}^k\alpha_i x_i\odot x_i$.

Let $p\in\Pi_k/\Pi_{k+1}$. It is easy to see that an element $x\odot x\in\mathbf{R}^N\odot\mathbf{R}^N=\mathscr{P}(\mathbf{R}^N)^*$ is normal to Π_k at the point p if and only if $x\in\ker p$. Clearly,

$$\operatorname{span}\{x\odot x\,|\,x\in\ker p\}=\{\xi\in\mathscr{P}(\mathbf{R}^N)^*\,|\,L(\xi)\subset\ker p\}.$$

At the same time, the canonical isomorphism $\mathscr{P}(\mathbf{R}^{N*})=\mathscr{P}(\mathbf{R}^N)^*$ induces an isomorphism

$$\{\xi\in\mathscr{P}(\mathbf{R}^N)^*\,|\,L(\xi)\subset\ker p\}=\mathscr{P}(\ker p^*).$$

Since $\dim\mathscr{P}(\ker p^*) = [k(k+1)/2] = \operatorname{codim}\Pi_k$, we see that the normal subspace to the submanifold $\Pi_k\setminus\Pi_{k+1}$ coincides with $\mathscr{P}(\ker p^*) = \operatorname{span}\{x\odot x|\ \in\ker p\}$.

If $k = 1$, $p\in\Pi_1\setminus\Pi_2$, then there exists a normal $x\odot x$, $x\in\ker p$ to the hypersurface Π_1 at p, which is unique up to a scalar factor. We see that the normals to the points of $\Pi_1\setminus\Pi_2\subset\mathscr{P}(\mathbf{R}^N)$ are the elements of rank 1 in $\mathscr{P}(\mathbf{R}^N)^*=\mathscr{P}(\mathbf{R}^{N*})$. We now assume that p is a singular point of Π_1, $\dim\ker p > 1$.

A normal to Π_1 at the point p is an arbitrary element of the form $\alpha x \circledcirc x \in \mathscr{P}(\mathbf{R}^N)^*$, where $x \in \ker p$, $\alpha \in \mathbf{R}$, or equivalently an arbitrary element of rank 1 in $\mathscr{P}(\ker p^*) \subset \mathscr{P}(\mathbf{R}^{N*})$. Let $\mathscr{N}_p$ denote the set of all normals to Π_1 at p. Our definition of normals at singular points is justified by the following easily verified equality:

$$\mathscr{N}_p = \bigcap_{O_p \subset \Pi_1} \bigcup_{q \in O_p} \mathscr{N}_q$$ (the intersection is taken over all neighborhoods O_0 of p in Π_1).

Normals of the form $x \circledcirc x (=(-x) \circledcirc (-x))$, where $x \neq 0$, will be called positive. If $p \in \Pi_1 \setminus \Pi_2$, there is exactly one positive normal at p, up to a positive factor and, therefore, the positive normals define an orientation of the manifold $\Pi_1 \setminus \Pi_2$. The set of all positive normals to Π_1 at an arbitrary point $p \in \Pi_1$ will be denoted by $\mathscr{N}_p^+$. It turns out that the set of all positive normals to Π_1 of unit length, relative to the corresponding points of Π_1, form a manifold and, so, resolve the singularities of the algebraic manifold Π.

More precisely, let us consider the following subset of the space $\mathscr{P}(\mathbf{R}^N) \oplus \mathscr{P}(\mathbf{R}^N)^*$

$$\tilde{\Pi}_1 = \{(p,\ x \odot x) \mid x \in \ker p,\ |x| = 1\}.$$

It is easy to see that $\tilde{\Pi}_1$ is a smooth submanifold of $\mathscr{P}(\mathbf{R}^N) \oplus \mathscr{P}(\mathbf{R}^N)^*$. Indeed, this follows from the fact that for any $x \neq 0$ the map $p \to Px$ from $\mathscr{P}(\mathbf{R})$ to $\mathbf{R}^N$ is of rank N.

Since normals are defined at every point of Π_1, we can also speak of tangent vectors. Specifically: a form $p' \in \mathscr{P}(\mathbf{R}^N)$ is a tangent vector to Π_1 at a point $p \in \Pi_1$ if and only if $p'(x, x) = 0$ for some $x \in \ker p \setminus \{0\}$. More generally: for any $k = 1, \ldots, [N(N+1)/2] - 1$, let $\mathscr{L}^+(k, N)$ denote the manifold of all oriented k-dimensional planes in $\mathscr{P}(\mathbf{R}^N)$ [clearly, $\mathscr{L}^+(k, N) \simeq G^+(k, (N(N+1)/2) - k)$, where, as usual, $G^+(k, m)$ is the Grassmann manifold of k-dimensional oriented planes in $\mathbf{R}^{k+m}$).

A plane $H \in \mathscr{L}^+(k, N)$ is said to be tangent to Π_1 at a point p if there exists $x \in \ker p \setminus \{0\}$ such that $p'(x, x) = 0$ $\forall p' \in H$. We let $G_k^+(p; \Pi_1) \subset \mathscr{L}^+(k, N)$ denote the set of all k-dimensional oriented planes tangent to Π_1 at p. Finally, we define

$$G_k^+(\Pi_1) = \{(p, H) \mid p \in \Pi_1,\ H \in G_k^+(p; \Pi_1)\} \subset \mathscr{P}(\mathbf{R}^N) \times \mathscr{L}^+(k, N).$$

It is easy to see that $G_k^+(\Pi_1)$ is an algebraic subset of $\mathscr{P}(\mathbf{R}^N) \times \mathscr{L}^+(k, N)$, with $\operatorname{codim} G_k^+(\Pi_1) = k + 1$. It turns out that the set of singular points of the algebraic manifold $G_k^+(\Pi_1)$ has codimension $2(k+1)$ in $\mathscr{P}(\mathbf{R}^N) \times \mathscr{L}^+(k, N)$ and, so, $G_k^+(\Pi_1)$ is a smooth pseudomanifold. In order to verify this, we consider the set

$$\overset{\frown}{G_k^+}(\Pi_1) = \{(p,\ x \odot x,\ H) \mid p \in \Pi_1,\ x \in \ker p,\ |x| = 1,\ H \in \mathscr{L}^+(k, N),$$
$$H(x, x) = \{0\}\} \subset \mathscr{P}(\mathbf{R}^N) \times \mathscr{P}(\mathbf{R}^N)^* \times \mathscr{L}^+(k, N).$$

The set $\overset{\frown}{G_k^+}(\Pi_1)$ is a smooth manifold, for the same reason as $\tilde{\Pi}_1$; the map $(p, x \circledcirc x, H) \mapsto (p, H)$ from $\overset{\frown}{G_k^+}(\Pi_1)$ to $G_k^+(\Pi_1)$ "resolves the singularities" of the algebraic manifold $G_k^+(\Pi_1)$. The desired result follows from

LEMMA 3. The smooth map $(p, x \circledcirc x, H) \mapsto (p, H)$ from $\overset{\frown}{G_k^+}(\Pi_1)$ to $\mathscr{P}(\mathbf{R}^N) \times \mathscr{L}^+(k, N)$ has maximum rank at all points except for a certain subset of codimension $k + 1$.

Proof. We must single out those points in $\overset{\frown}{G_k^+}(\Pi_1)$ at which the differential of the map $(p, x \circledcirc x, H) \mapsto (p, H)$ has nonzero kernel. The kernel is obviously the set of all vectors $(0, x \circledcirc y, 0)$ which lie in the tangent space $T_{(p, x \circledcirc x, H)} \overset{\frown}{G_k^+}(\Pi_1)$. At the same time, a vector $(0, x \circledcirc y, 0)$ is tangent to the manifold $\overset{\frown}{G_k^+}(\Pi_1)$ at a point $(p, x \circledcirc x, H)$ if and only if $(x, y) = 0$, $y \in \ker p$, $p'(x, y) = 0$ $p' \in H$. An elementary argument now shows that the condition that there exist such a nonzero vector y defines an algebraic subset of codimension $k + 1$ in $\overset{\frown}{G_k^+}(\Pi_1)$.

Let us dwell in more detail on the choice of orientations of the pseudomanifolds Π_1 and $G_k^+(\Pi_1)$. An orientation of a pseudomanifold is by definition any orientation of its manifold of nonsingular points. The manifold of nonsingular points for Π_1 is $\Pi_1 \setminus \Pi_2$. As mentioned above, the hypersurface $\Pi_1 \setminus \Pi_2$ is orientable: at every point $p \in \Pi_1 \setminus \Pi_2$ there is a unique (up to a positive factor) positive normal ν_p. Moreover, $\Pi_1 \setminus \Pi_2$ consists of N components, and there are therefore 2^N different orientations on it: let $\tau:\{0, 1, \ldots, N-1\}$ $\{0, 1\}$ be an arbitrary sequence of zeros and ones; then an arbitrary orientation of $\Pi_1 \setminus \Pi_2$ is defined by the normals $(-1)^{\mathrm{ind}\,p}\nu_p$.

Proceeding to consideration of the pseudomanifold $\widetilde{G_k^+}(\Pi_1)$, we note that in this case the manifold of nonsingular points is connected — this follows from Lemma 3, since $\widetilde{G_k^+}(\Pi_1)$ is connected and $k + 1 \geq 2$. Thus, there are at most two orientations on $G_k^+(\Pi_1)$. And the two in fact exist. First, any orientation of $\Pi_1 \setminus \Pi_2$ determines an orientation of $G_k^+(\Pi_1 \setminus \Pi_2)$ in the standard manner. The orientation on $G_k^+(\Pi_1 \setminus \Pi_2)$ corresponding to an orientation ν_p on $\Pi_1 \setminus \Pi_2$ will be denoted by $G_k^+(\nu_p)$; and an arbitrary orientation on $G_k^+(\Pi_1 \setminus \Pi_2)$ will have the form $(-1)^{\nu(\mathrm{ind}\,p)} G_k^+(\nu_p)$. We put $\mathrm{ind}(p, H) \overset{\mathrm{def}}{=} \mathrm{ind}\,p$; and then the manifold $G_k^+(\Pi_1 \setminus \Pi_2)$ is precisely the set of points of continuity of the integer-valued lower semicontinuous function ind on $G_k^+(\Pi_1)$.

Remark. Suppose that M is a triangulable manifold and α is a lower semicontinuous integer-valued function on this manifold, such that the subset $\alpha^{-1}((-\infty, n])$ is a submanifold of M with connected boundary, for any $n \in Z$. It can be shown that if the set of continuity points of α (which is open and dense in M) is an orientable manifold, then M is also an orientable manifold.

It follows from the remark that $G_k^+(\Pi_1)$ is an orientable manifold. It remains to choose, of the 2^N orientations on $G_k^+(\Pi_1 \setminus \Pi_2)$, one which extends to $G_k^+(\Pi_1)$. A direct, though cumbersome, calculation shows that the required orientation is $(-1)^{\mathrm{ind}\,p} G_k^+(\nu_p)$.

Remark. It is not hard to show that the zero-dimensional cocycle $p \mapsto \mathrm{ind}\,p$ on $\mathscr{P}(\mathbf{R}^N) \setminus \Pi_1$ is the Alexander–Pontryagin dual of the pseudomanifold Π_1, oriented by the normals ν_p, $p \in \Pi_1 \setminus \Pi_2$; while the cocycle $p \mapsto 1 + (-1)^{\mathrm{ind}\,p}$ on $\mathscr{P}(\mathbf{R}^N) \setminus \Pi_1$ is the dual of Π_1 oriented by the normals $(-1)^{\mathrm{ind}\,p}\nu_p$, $p \in p \in \mathscr{P}(\mathbf{R}^N) \setminus \Pi_1$.

2. Quadratic Maps and Pencils of Quadratic Forms

1. Let $k > 0$ and consider the space $\mathscr{P}(\mathbf{R}^N)^k$ of symmetric bilinear maps $p\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$. If $p \in \mathscr{P}(\mathbf{R}^N)^k$, then the quadratic map $x \to p(x, x)$ will be denoted by the same symbol p. Throughout the sequel we will employ the abbreviated notation $\mathscr{P}(\mathbf{R}^N)^k = \mathscr{P}(N, k)$ in order to avoid cumbersome formulas.

A quadratic map $p \in \mathscr{P}(N, k)$ is said to be singular or degenerate if zero is a critical value of the map $p|(\mathbf{R}^N \setminus \{0\})$. This means that for some $x \in \mathbf{R}^N \setminus \{0\}$, $\omega \in \mathbf{R}^{k*} \setminus \{0\}$ it is true that $p(x, x) = 0$, $\omega p x = 0$. But if these equalities are not both true for any nonzero x, ω, we say that p is a nonsingular map. For quadratic forms ($k = 1$) the definition of singular and nonsingular maps obviously reduces to the standar done (see the previous section). If a quadratic map $p \in \mathscr{P}(N, k)$ is nonsingular, then $p^{-1}(0) \cap S^{N-1}$ is a (possibly empty) smooth closed submanifold of S^{N-1}.

LEMMA 1. The set of all singular maps is a proper algebraic subset of $\mathscr{P}(N, k)$.

Proof. That the set of singular maps is algebraic follows from the fact that by definition it is the image under the projection $(p, x, \omega) \mapsto p$ of some algebraic subset of $\mathscr{P}(N, k) \times \mathbf{R}^N \times \mathbf{R}^k$. In order to prove that the set of singular maps is proper, it will suffice to exhibit at least one nonsingular map; one example is the map

$$x \mapsto (x, x)\,l, \text{ where } l \in \mathbf{R}^k \setminus \{0\},\ x \in \mathbf{R}^N.$$

It follows from Lemma 4, in particular, that the nonsingular maps fill out an open dense subset of $\mathscr{P}(N, k)$.

An arbitrary linear map from $\mathbf{R}^{k+1*}$ to $\mathscr{P}(\mathbf{R}^N)$ is called a k-dimensional (linear) pencil of N-dimensional quadratic forms. To every quadratic map $p \in \mathscr{P}(N, k)$ one can associate, in an obvious way, a $(k-1)$-dimensional pencil of quadratic forms, $p^*\colon \omega \mapsto \omega p$, $\omega \in \mathbf{R}^{k*}$. The correspondence $p \mapsto p^*$ determines a canonical isomorphism of $\mathscr{P}(N, k)$ onto the space of $(k-1)$-dimensional pencils of quadratic forms: $\mathscr{P}(N, k) = \mathrm{Hom}(\mathbf{R}^{k*}, \mathscr{P}(\mathbf{R}^N))$.

In many cases it is more convenient to investigate quadratic maps in the language of pencils of quadratic forms. A pencil of forms is simply a subspace of $\mathcal{P}(R^N)$ and we can directly use all the information about $\mathcal{P}(R^N)$, established in the previous section.

Let M be a smooth manifold and $f: M \to \mathcal{P}(R^N)$ be a smooth map. We will say that f is tangent to the hypersurface $\Pi_1 \subset \mathcal{P}(R^N)$ at a point $\mu \in M$ if $f(\mu) \in \Pi_1$ and $D_\mu f(T_\mu M) \perp \nu$ for some normal $\nu \in \mathcal{N}_{f(\mu)} \setminus \{0\}$.

Definition. Let $p \in \mathcal{P}(N, k)$. Then the pencil $p^*: R^{k*} \to \mathcal{P}(R^N)$ is said to be singular if p^* is tangent to Π_1 at some point $\omega \in R^{k*} \setminus \{0\}$. Otherwise p^* is said to be nonsingular.

It is easy to see that a pencil p^* is singular if and only if the map p is singular – we have simply rephrased the degeneracy condition for p in geometric language. Later this rephrasing device will furnish us with some important generalizations.

Recall that the nonsingular quadratic maps form an open dense subset of $\mathcal{P}(N, k)$; we denote this subset by (N, k).

Definition. Nonsingular $p_1, p_2 \in \mathring{\mathcal{P}}(N, k)$ are said to be rigidly isotopic if they are in the same (connected) component of $\mathring{\mathcal{P}}(N, k)$.†

It follows from Thom's standard Isotopy Lemma [3, 6] that, if $p_1, p_2 \in \mathring{\mathcal{P}}(N, k)$ are rigidly isotopic, then there exists a smooth isotopy F_t of the sphere S^{N-1}, $t \in [0, 1]$, mapping $p_1^{-1}(0) \cap S^{N-1}$ into $p_2^{-1}(0) \cap S^{N-1}$, i.e.,

$$F_0 = \mathrm{id}, \quad F_1(p_1^{-1}(0) \cap S^{N-1}) = p_2^{-1}(0) \cap S^{N-1}.$$

If k = 1, two nonsingular quadratic forms are, of course, rigidly isotopic if and only if they have the same index. In the case k = 2, classical results about one-dimensional pencils of quadratic forms can be used to derive a complete classification of two-dimensional quadratic maps with respect to rigid isotopy. It is simpler, however, to carry out this classification directly. Let us assume that the quadratic maps take values in the complex plane $C = R^2$ and begin with the simplest nontrivial case, N = 2.

Proposition 1. Any quadratic map in $\mathring{\mathcal{P}}(2, 2)$ is rigidly isotopic to one of the following three:

$$I_2(x, x) = x_1^2 + x_2^2; \quad \Gamma(x, x) = x_1^2 - x_2^2 + 2ix_1x_2;$$

$$\bar{\Gamma}(x, x) = x_1^2 - x_2^2 - 2ix_1x_2,$$

where $i = \sqrt{-1} \in C$, $x = (x_1, x_2) \in R^2$, these three maps are not rigidly isotopic to one another.

Remark. Putting $z = x_1 + ix_2$, we obtain $I_2(z, z) = \bar{z}z$, $\Gamma(x, z) = z^2$, $\bar{\Gamma}(z, z) = \bar{z}^2$.

Proof. Let $p \in \mathring{\mathcal{P}}(2, 2)$. Consider the quadratic equation

$$\det(\omega P) = 0, \quad \omega \in R^{2*} \setminus \{0\}. \tag{1}$$

There are three possible cases.

1) The discriminant of Eq. (1) is positive. Then there is a pair of noncollinear covectors $\omega, \hat{\omega} \in R^{2*}$ such that $\omega P x = \hat{\omega} P \hat{x} = 0$ for some $x, \hat{x} \in R^2 \setminus \{0\}$. Since $\omega p(x, \hat{x}) = \hat{\omega} p(x, \hat{x}) = 0$, it is obvious that $p(x, \hat{x}) = 0$. Clearly, $x \neq \hat{x}$, because p is nonsingular. Consequently, p must have the form $p(y, y) = (a, y)^2 e^{i\theta_1} + (\hat{a}, y)^2 e^{i\theta_2}$, where $(a, x) = (\hat{a}, \hat{x}) = 0$, $\theta_1 \neq \pm\theta_2$.

The image of R^2 under the quadratic map p is the convex cone spanned by $e^{i\theta_1}$ and $e^{i\theta_2}$. An obvious homotopy brings this map to the form $(a, y)^2 + (\hat{a}, y)^2$ (the cone is flattened out into a half-line) and, then, a change of variables with positive determinant brings it to the form (y, y).

2) The discriminant of Eq. (1) is negative. Then the map $p|\, R^2 \setminus \{0\}$ is regular. Consequently, for every $\omega \in R^{2*} \setminus \{0\}$, $\operatorname{ind} \omega p = 1$, $\ker \omega p = 0$. A linear change of variables with positive determinant now brings p to the form

$$p((x_1, x_2), (x_1, x_2)) = x_1^2 - x_2^2 + i(\alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2).$$

†This notion corresponds to the accepted terminology in real algebraic geometry.

The condition on the discriminant is equivalent to the inequality $|\alpha+\beta| < |\gamma|$. An obvious homotopy brings p to the form $x_1^2 - x_2^2 + 2ix_1x_2$, if $\gamma > 0$, or to the form $x_1^2 - x_2^2 - 2ix_1x_2$, if $\gamma < 0$.

3) The discriminant of Eq. (1) vanishes. The maps p satisfying this condition form a proper algebraic subset of $\mathscr{P}(2, 2)$ and, so, by slightly perturbing p, we can reduce everything to the situation of one of cases 1 or 2.

If $p \in \mathring{\mathscr{P}}(2, 2)$, then obviously $p(x, x) \neq 0$, $\forall x \neq 0$. Let $\deg p$ denote the degree of the map $p_1 \oplus p_2$ relative to zero. Clearly, $\deg p$ is invariant under a nonsingular homotopy. We have $\deg I_2 = 0$, $\deg \Gamma = 2$, $\deg \overline{\Gamma} = -2$. Consequently, the maps I_2, Γ, $\overline{\Gamma}$ are not homotopic to one another.

Let $p_i \in \mathscr{P}(N_i, k)$, $i = 1, 2$. We define the direct sum $p_1 \oplus p_2$ of quadratic maps p_1 and p_2 as the map $p \in \mathscr{P}(N_1 + N_2, k)$ defined by

$$p((x_1, x_2), (x_1, x_2)) = p_1(x_1, x_1) + p_2(x_2, x_2), \quad \forall x_i \in \mathbf{R}^{N_i}, \quad i = 1, 2.$$

<u>Proposition 2.</u> Any $p \in \mathring{\mathscr{P}}(N, 2)$ is rigidly isotopic to a quadratic map of the form $\overset{n}{\underset{i=1}{\oplus}} p_i \oplus \left(\overset{m}{\underset{j=1}{\oplus}} q_j\right)$, where $p_i \in \mathring{\mathscr{P}}(1, 2)$, $q_j \in \mathring{\mathscr{P}}(2, 2)$, $n + 2m = N$.

<u>Proof.</u> We may assume without loss of generality that the discriminant of the map

$$\det(\omega P) = 0, \quad \omega \neq 0, \tag{1}$$

does not vanish.

The symmetric bilinear map p has a unique extension as a symmetric $\mathbf{C}$-bilinear map $p_{\mathbf{C}}\colon \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^2$, where $\mathbf{C}^N = \mathbf{C} \otimes \mathbf{R}^N$, $\mathbf{C}^2 = \mathbf{C} \otimes \mathbf{R}^2$. Denote the quadratic map $X \mapsto p_{\mathbf{C}}(X, X)$, $X \in \mathbf{C}^N$, by the same symbol $p_{\mathbf{C}}$. The extension of Eq. (1) to the complex domain is

$$\deg(\Omega P_{\mathrm{C}}) = 0, \quad \Omega \in \mathbf{C}^{2*} \setminus \{0\}. \tag{1$_{\mathrm{C}}$}$$

Equation (1) has N pairwise noncollinear roots and, moreover, if Ω is a root, then so is $\overline{\Omega}$ (where the bar "¯" denotes complex conjugation). Let Ω_1, Ω_2 be two distinct roots of Eq. (1_{C}) such that $\Omega_2 \neq \Omega_1$; for some $X_1, X_2 \in \mathbf{C}^N \setminus \{0\}$, we have $\Omega_1 P_{\mathrm{C}} X_1 = 0$, $\Omega_2 P_{\mathrm{C}} X_2 = 0$. Since

$$\Omega_1 p_{\mathrm{C}}(X_1, X_2) = \Omega_2 p_{\mathrm{C}}(X_1, X_2) = \overline{\Omega}_1 p_{\mathrm{C}}(\overline{X}_1, X_2) = \overline{\Omega}_2 p_{\mathrm{C}}(\overline{X}_1, X_2) = 0,$$

it follows that $p_{\mathrm{C}}(X_1, X_2) = p_{\mathrm{C}}(\overline{X}_1, X_2) = 0$. Let x_i and y_i be, respectively, the real and imaginary parts of the vector $X_i \in \mathbf{C}^N$, $i = 1, 2$. It follows from the last equalities that $p(x_1, x_2) = p(x_1, y_2) = p(y_1, x_2) = p(y_1, y_2) = 0$. Consequently, $p|\mathrm{span}\{x_i, y_i, i = 1, 2\} = p|\mathrm{span}\{x_1, y_1\} \oplus p|\mathrm{span}\{x_2, y_2\}$. Let $\Omega_1, \ldots, \Omega_N$ be the pairwise noncollinear roots of Eq. (1_{C}) and let $\Omega_i P_{\mathrm{C}} X_i = 0$, where $X_i \neq 0$, $i = 1, \ldots, N$. To finish the proof, it remains to be shown that the vectors $X_1, \ldots, X_N$ form a basis of $\mathbf{C}^N$. Fix a covector $\Omega_0 \in \mathbf{C}^{2*}$, which is not a root of Eq. (1_{C}). Then $\Omega_i = \Omega_1 + \alpha_i \Omega_0$, $i = 1, \ldots, N$, where $\alpha_i \in \mathbf{C}$, $\alpha_i \neq \alpha_j$, for $i \neq j$. It follows from the equations $\Omega_1 P_{\mathrm{C}} X_i + \alpha_i \Omega_0 P_{\mathrm{C}} X_i = 0$ that $-\alpha_1$ are eigenvalues and X_i eigenvectors of the operator

$$(\Omega_0 P_{\mathrm{C}})^{-1} \Omega_1 P_{\mathrm{C}}, \quad i = 1, \ldots, N. \ \blacktriangleright$$

For any $n > 0$, let I_n denote the quadratic form $x \mapsto (x, x)$, where $x \in \mathbf{R}^n$. Clearly, $I_n = \underbrace{I_1 \oplus \ldots \oplus I_1}$. We also define $\Gamma_n = \underbrace{\Gamma \oplus \ldots \oplus \Gamma}$; $\Gamma_n \in \mathscr{P}(2n, 2)$. We also put $\Gamma_0 = 0$.

<u>THEOREM 1.</u> If $N \geq 3$, any $p \in \mathring{\mathscr{P}}(N, 2)$ is rigidly isotopic to one of the following maps:

$$q(n_1, \ldots, n_{2k-1}; m) = \overset{2k-1}{\underset{j=1}{\oplus}} I_{n_j} e^{\frac{2\pi i}{2k-1}} + \Gamma_m, \quad k, m \geqslant 0, \tag{2}$$

$$\sum_{j=1}^{2k-1} n_j + 2m = N.$$

The map $q(n_1, \ldots, n_{2k-1}; m)$ is rigidly isotopic to a map $q(n_1', \ldots, n_{2k'-1}'; m')$ if and only if $k' = k$, $m' = m$, and the sequence of numbers $n_1', \ldots, n_{2k-1}'$ is a cyclic permutation of $n_1, \ldots, n_{2k-1}$.

<u>Proof.</u> By Propositions 1, 2, p is rigidly isotopic to some map of the form $\bigoplus_{j=1}^{n} I_1 a_j \oplus \Gamma_\ell \oplus \overline{\Gamma_{\bar\ell}}$, where $a_j \in \mathbf{C} \setminus 0$, $n + 2\ell + 2\bar\ell = N$. Changing variables $x_j \mapsto |a_j|^{-1/2} x_j$, if necessary, we may assume that $|a_j| = 1$, i.e., $a_j = e^{\theta_j i}$. Let $\mathbf{R}^N = \mathbf{R}^m \oplus \mathbf{C}^m$, where $m = \ell + \bar\ell$; without loss of generality, we may assume that

$$p(x_1, \ldots, x_n, z_1, \ldots, z_m) = \sum_{j=1}^{n} x_j^2 e^{\theta_j i} + \sum_{j=1}^{l} z_j^2 + \sum_{j=l+1}^{m} \bar z_j^2. \tag{3}$$

We first get rid of terms of the type $\bar z_j^2$. If $n \neq 0$, a change of variables in $\mathbf{R}^N$ with positive determinant, replacing x_1 by $(-1)^{\bar\ell} x_1$, the coordinates z_j with $j = \ell + 1, \ldots, m$ by $\bar z_j$ and leaving the other coordinates unaffected, will bring (3) to the form of

$$\sum_{j=1}^{n} x_j^2 e^{\theta_j i} + \sum_{j=1}^{m} z_j^2. \tag{4}$$

Since any linear transformation with positive determinant is homotopic to the identity, it follows that the map (4) is rigidly isotopic to (3). A similar argument goes through in the case that $n = 0$ and $\bar\ell$ is even. To eliminate the terms $\bar z_j^2$ in the case that $n = 0$ and $\bar\ell$ is odd, it will suffice to show that the quadratic map $z_1^2 + \bar z_2^2$ from $\mathbf{C}^2$ to $\mathbf{C}$ is rigidly isotopic to the map $z_1^2 + z_2^2$.

Let $\mathcal{R}$ denote the set of all singular quadratic maps $r \in \mathcal{P}(4, 2)$ such that the equation $\det(\omega R) = 0$ has exactly one (necessarily multiple) root in $\mathbf{RP}^1 = (\mathbf{R}^{2*} \setminus 0)/(\omega \sim \alpha\omega, \alpha \neq 0)$, where $\omega r(x, x) = (\omega R x, x)$, $x \in \mathbf{R}^4$.

It is easy to see that $\mathcal{R}$ is a semi-algebraic subset of $\mathcal{P}(4, 2)$. The "general point" r_0 of the hypersurface $\mathcal{R}$ has the following properties:

a) The multiplicity of the single root of the equation $\det(\omega R_0) = 0$ in $\mathbf{RP}^1$ is two;

b) For some neighborhood O_{r_0} of r_0 in $\mathcal{P}(4, 2)$ the set $O_{r_0} \setminus (O_{r_0} \cap \mathcal{R}) = O_{r_0} \cap \mathring{\mathcal{P}}(4, 2)$ has at most two arcwise-connected components.

Let $\det(\omega_0 R_0) = 0$, $\omega_0 \in \mathbf{R}^{2*} \setminus 0$. By assumption, the symmetric linear operator $\omega_0 R_0$ has a two-dimensional kernel $H_0 \subset \mathbf{R}^4$. The equation $\det(\omega R_0) = 0$ has another pair of nonzero complex-conjugate roots in $\mathbf{CP}^1$. Reasoning as in the proof of Proposition 2, we see that $r_0 = r_0|H_0 \oplus r_0|H_1$, where H_1 is a two-dimensional subspace of $\mathbf{R}^4$, transverse to H_0. Denote $r_0|H_0 = r_0^0$, $r_0|H_1 = r_0^1$; then the equation $\det(\omega r_0^1) = 0$ has no nonzero real roots and the equation $\det(\omega r_0^0) = 0$ has a double root ω_0. Since $r_0^0 \in \mathcal{P}(2, 2)$, it follows that $\omega_0 r_0^0 = 0$. Thus the values of the map r_0^0 lie on a straight line orthogonal to ω_0. We may assume without loss of generality that this is the real line $\mathbf{C} = \mathbf{R}^2$. Then $r_0^0(x, x) = (Qx, x)$, where $Q: \mathbf{R}^2 \to \mathbf{R}^2$ is a symmetric operator. This operator Q is nonsingular [otherwise it would be true that $\det(\omega r_0^0) \equiv 0$]. Recall, finally, that the map r_0 is singular. This is possible only if that is true of the map r_0^0 (considered as a quadratic map from $\mathbf{R}^2$ to $\mathbf{R}^2$). This in turn is true if and only if $(Qx_0, x_0) = 0$ for some $x_0 \neq 0$. Consequently, (Qx, x) is a nondefinite quadratic form and we may assume without loss of generality that $r_0^0(x, x) = x_1^2 - x_2^2$.

Finally, consider the quadratic maps defined by

$$r_\varepsilon^0 = x_1^2 - x_2^2 + i\varepsilon x_1 x_2, \quad \hat r_\varepsilon^0 = x_1^2 - x_2^2 + i\varepsilon(x_1^2 + x_2^2)$$

(here $i = \sqrt{-1} \in \mathbf{C} = \mathbf{R}^2$), as well as $r_\varepsilon = r_\varepsilon^0 \oplus r_0^1$, $\hat r_\varepsilon = \hat r_\varepsilon^0 \oplus r_0^1$. If ε is sufficiently near zero, then r_ε and $\hat r_\varepsilon$ lie in O_{r_0}; in addition, as long as $\varepsilon \neq 0$, both maps r_ε and $\hat r_\varepsilon$ are nonsingular. We note that the equation $\det(\omega R_\varepsilon) = 0$ has no nonzero roots in $\mathbf{R}^{2*}$ if $\varepsilon \neq 0$, while the equation $\det(\omega \hat R_\varepsilon) = 0$ has two noncollinear roots. It is readily seen that the sum of multiplicities of roots in $\mathbf{RP}^1$ is invariant under rigid isotopy. Consequently, the map r_{ε_1} is not rigidly isotopic to $\hat r_{\varepsilon_2}$ for any $\varepsilon_1, \varepsilon_2 \neq 0$. Since $O_{r_0} \cap \mathring{\mathcal{P}}(4, 2)$ has at most two arcwise-connected components, this implies that for all $\varepsilon \neq 0$ sufficiently close to zero the maps r_ε and $r_{-\varepsilon}$ are rigidly isotopic. At the same time, $r_\varepsilon{}^0$ is rigidly isotopic to Γ if $\varepsilon > 0$ and to $\overline{\Gamma}$ if $\varepsilon < 0$. Since $r_0{}^1$ is also rigidly isotopic to either Γ or $\overline{\Gamma}$, and $\overline{\Gamma} \bullet \overline{\Gamma}$ is rigidly isotopic to $\Gamma \bullet \Gamma$, it follows that $\Gamma \bullet \Gamma$ is rigidly isotopic to $\Gamma \bullet \overline{\Gamma}$.

We have thus shown that any map in $\mathscr{P}(N, 2)$ is rigidly isotopic to some map of the form (4). All our further manipulations will take place within the class of all such maps. It is directly verifiable that a map (4) is nonsingular if and only if $e^{\theta_{j_1} i} + e^{\theta_{j_2} i} \neq 0$ $\forall j_1, j_2$.

Let us divide the set $\{e^{\theta_1 i}, \ldots, e^{\theta_n i}\}$ into several subsets, according to the following rule: put elements $e^{\theta_{j_1} i}$, $e^{\theta_{j_2} i}$ into the same subset if one of the arcs connecting the points $e^{\theta_{j_1} i}$ and $e^{\theta_{j_2} i}$ on the circle $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ contains no points of the form $(-e^{\theta_j i})$, $j = 1, \ldots, n$. It is not difficult to show that if $n \neq 0$ the number of subsets is odd, say $2k - 1$. Via a rigid isotopy, it is easy to merge all the points in one subset into a single point and, then, to arrange these $2k - 1$ points at the vertices of a regular $(2k - 1)$-gon. Thus, any nonsingular quadratic map from $\mathbf{R}^N$ to $\mathbf{R}^2 = \mathbf{C}$ is indeed rigidly isotopic to one of the maps (2).

It remains to be shown that the rigid isotopy class determines the sequence of numbers $n_1, \ldots, n_{2k-1}$ up to cyclic permutations. First, the number $\sum_{j=1}^{2k-1} n_j = N - 2m$ is the sum of multiplicities of the roots of the equation $\det(\omega P) = 0$ in $\mathbf{RP}^1$ and is, therefore, invariant to a rigid isotopy.

Let $p \in \mathring{\mathscr{P}}(N, 2)$ and let $\omega_1, \ldots, \omega_n \in \mathbf{R}^{2*} \setminus 0$ be representative of the roots of the equation $\det(\omega O) = 0$ in $\mathbf{RP}^1 = (\mathbf{R}^{2*} \setminus 0)/(\omega \sim \alpha\omega, \alpha \neq 0)$. For each $j = 1, \ldots, n$, the quadratic map $p|\ker \omega_j p$ takes values on the straight line $\omega_j{}^\perp$. Moreover, since p is nonsingular, the quadratic form $p|\ker \omega_j p$ is sign-definite. In particular, $\forall x \in \ker \omega p \setminus 0$ the vector $p(x, x)$ does not vanish and the vector $(1/|p(x, x)|) \cdot p(x, x)$ is independent of $x \in \ker \omega p \setminus 0$. Define $(1/|p(x, x)|) \cdot p(x, x) = e^{\theta_j i}$, $x \in \ker \omega_j p \setminus 0$. Now the roots of the equation $\det(\omega P) = 0$ are continuous functions of $p \in \mathscr{P}(N, 2)$, provided that roots are counted with their multiplicities. The same is true of the vectors $e^{\theta_j i}$, provided that their multiplicities are assumed equal to those of the corresponding roots. The fact that p is a nonsingular map implies that $e^{\theta_{j_1} i} + e^{\theta_{j_2} i} \neq 0$ $\forall j_1, j_2$. Divide the set $\{e^{\theta_1 i}, \ldots, e^{\theta_n i}\}$ into subsets according to the familiar rule: put points in the same subset if they can be connected in S^1 by an arc not containing any points $(-e^{\theta_j i})$, $j = 1, \ldots, n$. This gives an odd number of sets, say $T_1, \ldots, T_{2k-1}$. To each T_j associate the unique arc $\Delta_j \subset S^1$ of length less than π satisfying the conditions $\partial\Delta_j \subset T_j \subset \Delta_j$ (where $\partial\Delta_j$ is the boundary, i.e., the endpoints of Δ_j; in the case $\#\Delta_j = 1$ the arc Δ_j degenerates to a point). It is easy to see that $\Delta_{j_1} \cap \Delta_{j_2} = \emptyset$ if $j_1 \neq j_2$. To each arc Δ_j there corresponds a multiplicity n_j, equal to the sum of multiplicities of the points occurring in the set T_j. A rigid isotopy may change the length of the arc, but it preserves multiplicity and the relative positions of arcs on S^1. To allow for these positions we will assume that the arcs Δ_j are numbered in such a way that when S^1 is described in the positive sense, beginning from Δ_1, one encounters the arcs $\Delta_2, \Delta_3, \ldots, \Delta_{2k-1}$ in succession. This rule determines the numbering up to cyclic permutations. ▸

Unfortunately, the elementary techniques used to classify quadratic maps with values in $\mathbf{R}^2$, are no longer applicable to maps with values in $\mathbf{R}^3$. The problem is that a quadratic map in general position in $\mathscr{P}(N, 3)$ cannot be expressed as $p_1 \bullet p_2$ with nonzero p_1, p_2.

Indeed, to each $p \in \mathscr{P}(N, 3)$ we can associate an algebraic curve of degree N in $\mathbf{RP}^2 = (\mathbf{R}^{3*} \setminus 0)/(\omega \sim \alpha\omega, \alpha \neq 0)$, defined by an equation $\det(\omega P) = 0$, $\omega \in \mathbf{R}^{3*}$. If $p = p_1 \bullet p_2$, then $\det(\omega P) = \det(\omega P_1)\det(\omega P_2)$, and the curve is reducible. At the same time, the general curve of degree N in $\mathbf{RP}^2$ is irreducible. Let us compare the dimensions of the space of quadratic maps and the projective space of curves. In so doing we must allow for the fact that the group of linear changes of variables acting on the space $\mathscr{P}(N, 3)$ is $GL(N)$, and a change of variables in p does not affect the curve $\det(\omega P) = 0$. The dimension of the quotient space of $\mathcal{P}$ (N, 3) modulo the action of GL(N) is $[(3N(N + 1)/2] - N^2 = [(N^2 + 3N)/2]$. The dimension of the space of curves of degree N in RP^2 is $C_{N+2}{}^2 - 1 = \{[(N + 2)(N + 1)/2 - 1] = [(N^2 + 3N)/2]$.

Thus, the dimensions are the same. It can be shown that the map carrying each $p \in \mathcal{P}$ (N, 3) into the curve $\det(\omega P) = 0$ has regular points and so its image contains irreducible curves. A study of this map is beyond the scope of this paper. For a detailed investigation in the complex situation we refer the reader to [4].

2. We now resume our study of pencils of quadratic forms p* corresponding to quadratic homotopic if and only if p_0 and p_1 are rigidly isotopic.

Definition. Let $p_0, p_1 \in \mathcal{P}(N, k)$. The pencils $p_0{}^*$ and $p_1{}^*$ are said to be nonsingularly homotopic if and only if p_0 and p_1 are rigidly isotopic.

Thus, pencils $p_0{}^*$ and $p_1{}^*$ are nonsingularly homotopic if and only if they can be embedded in some continuous family of pencils $p_t{}^*$, $t \in [0, 1]$ such that the maps $\omega \to \omega p_t$, $\omega \neq 0$, are not tangent to the hypersurface Π_1 $\forall t \in [0, 1]$. At the same time, however, the maps $\omega \to \omega p_t$ may be tangent (i.e., not transverse) to one or other of the submanifolds $\Pi_n \backslash \Pi_{n+1} \subset \Pi_1$, $n > 1$.

In fact, let $p \in \mathcal{P}(N, k)$, $\omega_0 \in R^{k*}$, and $\omega_0 p \in \Pi_n \backslash \Pi_{n+1}$. The map $\omega \mapsto \omega p$ is tangent to Π_1 at ω_0 if and only if $p(x, x) = 0$ for some $x \in \ker \omega_0 p \backslash \{0\}$. Let $x_1, \ldots, x_n$ be a basis of the space $\ker \omega_0 p$. The map $\omega \mapsto \omega p$ will be tangent (i.e., not transverse) to the submanifold $\Pi_n \backslash \Pi_{n+1}$ at $\omega_0 p$ if and only if

$$\sum_{i=1}^{n} \alpha_i p(x_i, x_i) = 0,$$

for some $\alpha_i \in R$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \alpha_i{}^2 \neq 0$. This follows from an equality established in Sec. 1:

$$(T_{\omega_0 p}\Pi_n)^{\perp} = \operatorname{span} \mathcal{N}_p = \operatorname{span}\{x \odot x \mid x \in \ker \omega_0 p\}.$$

Henceforth we will refer to a pencil of quadratic forms $\omega \mapsto \omega p$ which is transverse to all the submanifolds $\Pi_n \backslash \Pi_{n+1}$, $n \geq 1$, where $\omega \neq 0$, simply as a transverse pencil. It follows from the standard Transversality Theorem that the quadratic maps corresponding to transverse pencils fill out an open dense subset of $\mathcal{P}$ (N, k). We denote this subset by $\mathcal{P}^t$ (N, k). It is clear that $\mathcal{P}^t(N, k) \subset \mathring{\mathcal{P}}$ (N, k).

Recall that $\mathcal{P}_n$ denotes the set $\{p \in \mathcal{P}(R^N) \mid \operatorname{ind} p \leq n\}$.

Let $p \in \mathring{\mathcal{P}}$ (N, k); if a nonsingular homotopy of pencils preserves the submanifold $p^{-1} \times (0) \cap S^{N-1} \subset S^{N-1}$ (up to isotopy), then a transverse homotopy also preserves the subsets $p^{*-1}(\mathcal{P}_n) \cap S^{k-1}$, n = 0, 1, 2, ... (up to isotopy). This will follow from the following assertion.

LEMMA 1. Let M be a smooth closed manifold, $f_t: M \to$ $t \in [0, 1]$ be a smooth homotopy of smooth maps. If the maps f_t are transverse to the submanifolds $\Pi_n \backslash \Pi_{n+1}$, n = 1, ..., N, for any $t \in [0, 1]$, then there exists a smooth isotopy $F_t: M \to M$, $t \in [0, 1]$, $F_0 = \mathrm{id}$, of M satisfying the condition

$$F_t(f_0^{-1}(\mathcal{P}_n)) = f_t^{-1}(\mathcal{P}_n), \quad n = 0, 1, \ldots, N.$$

Proof. If the boundary of the set $\mathcal{P}_n$ were a smooth manifold, everything would be reduced to Thom's Isotopy Lemma. However, the method usually used to prove the Isotopy Lemma [3] works here too — with the help of Lemma 1.2 (local structure of the algebraic manifold Π_1). We must find a family of diffeomorphisms $F_t: M \to M$, $F_0 = \mathrm{id}$, such that for an $x \in M$

$$f_0(x) \in \mathcal{P}_n \Leftrightarrow f_t \circ F_t(x) \in \mathcal{P}_n, \quad \forall t \in [0, 1]. \tag{2}$$

Let us try to find F_t as the general solution of the ordinary differential equation

$$\frac{\partial}{\partial t} F_t(x) = X_t(F_t(x)), \ t \in [0, 1]$$

where X_t is a nonstationary vector field on M.

Differentiation of (2) shows that it will suffice to prove the existence of a smooth vector field X with the following properties if $y \in M$ and $f_t(y) \in \Pi_n \setminus \Pi_{n+1}$, then $[(\partial f/\partial t)\times(y) + f_{ty}'X_t(y)] \in T_{f_t(y)}\Pi_n$, where $f_{ty}': T_yM \to \mathcal{P}(R^N)$ is the differential of the map f_t at y. Furthermore, it will suffice to construct the field X_t locally, in the neighborhood of a fixed point $y_0 \in M$, subsequently using a partition of unity in the standard way. If $f_t \times (y_0) \in \Pi_n \setminus \Pi_{n+1}$, it follows at once from the transversality condition that there exists such a field for $y \in f_t^{-1}(\Pi_n \setminus \Pi_{n+1})$. The fine point here is that $f_t(y_0) \in \Pi_i$ for $i = 1, \ldots, n-1$. But Lemma 1.2 implies the existence of local coordinates Ψ in the neighborhood of $f(y_0)$ on $\mathcal{P}(R^N)$, such that $\Psi_* T_y \Pi_i \supset \Psi_* T_{y_0} \Pi_n$ for all y close to y_0, $y \in \Pi_i$, $i = 1, \ldots, n$. Hence it quickly follows that the field X_t may be extended regularly to the entire neighborhood of y_0 in M.

Let $p \in \mathcal{P}(N, k)$ be a quadratic map such that $\omega p \notin \Pi_2$ for any $\omega \neq 0$. In this case, if the pencil p^* is nonsingular, it is also transverse. The algebraic set Π_2 has codimension 3 in $\mathcal{P}(R^N)$ and, therefore, for typical families p_t^*, $t \in [0, 1]$ of zero- and one-dimensional pencils, the condition $\omega p_t \notin \Pi_2$ $\forall\, \omega \neq 0$, $t \in [0, 1]$ is fulfilled. We have thus proved

<u>Proposition 3.</u> Let $p_1, p_2 \in \mathcal{P}(N, k)$, where $k \leq 2$, and let the pencils be transverse. If p_1^* and p_2^* are nonsingularly homotopic, then they are transversally homotopic.

In the case $k \geq 3$, transversal homotopy is a much stronger condition than nonsingular homotopy. The difference between these two types of homotopy in the language of quadratic maps is brought out by the following

<u>LEMMA 2.</u> Let $p_1, p_2 \in \mathring{\mathcal{P}}(N, k)$, where $k \geq 2$. 1) If $p_1(R^N) \neq R^k$ and $p_2(R^N) \neq R^k$, then P_1^* and P_2^* are nonsingularly homotopic.

2) If p_1^* and p_2^* are transversally homotopic and $p_1(R^N)$ lies in some half-space in R^k, then $p_2(R^N)$ is also in a half-space in R^k.

<u>Proof.</u> 1) Let $\ell_i \in R^k \setminus p_i(R^N)$, $i = 1, 2$; clearly, $\alpha \ell_i \in R^k \setminus p_i(R^N)$ $\forall\, \alpha > 0$. Let $I_N \ell_i$ denote the quadratic map $x \mapsto (x, x)\ell_i$. The family $t \mapsto (1-t)p_i - tI_N\ell_i$ defines a rigid isotopy between the quadratic maps p_i and $-I_N\ell_i$. That the maps $-I_N\ell_1$ and $-I_N\ell_2$ are rigidly isotopic is obvious.

2) The set $p(R^N)$ is a subset of a half-space if and only if $\operatorname{ind} \omega p = 0$ for some $\omega \neq 0$; in other words, if $p^{*-1}(\mathcal{P}_0) \cap S^{k-1} \neq \emptyset$. But we know that the set $p^{*-1}(\mathcal{P}_0) \cap S^{k-1}$ is preserved (up to isotopy) under transversal homotopy.

<u>Remark.</u> It follows form Lemma 2 and Proposition 3, in particular, that the image of an arbitrary $p \in \mathcal{P}(N, 2)$ is either in a half-space or is all of R^2. But it is readily proved directly that $p(R^N)$ is a convex cone in R^2.

Let $k \geq 3$. In order to construct an example of a pair of nonsingularly homotopic pencils which are not transversally homotopic, it will suffice to find some $p \in \mathcal{P}(N, k)$ which satisfies the conditions: $\overline{p(R^N)} \neq R^k$, $\operatorname{conv} p(R^N) = R^k$ (the map p does not even have to be nonsingular, since a suitable small perturbation will make p nonsingular without affecting its having these same properties). The simplest example of this kind is the map $q_3: (x_1, x_2, x_3)^\top \mapsto (x_1x_2, x_2x_3, x_1x_2)^\top$ in $\mathcal{P}(3, 3)$.

Indeed, $\overline{q_3(R^3)} = \{(y_1, y_2, y_3)^\top \in R^3 \mid y_1y_2y_3 \geq 0\}$.

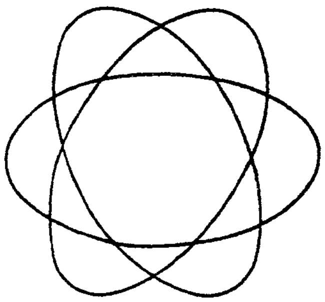
A map $p \in \mathcal{P}(3, 3)$ is nonsingular if and only if $p(x, x) \neq 0$ $\forall x \in R^3 \setminus 0$. Let $RP^2 = S^2/(x \sim (-x))$, where S^2 is the unit sphere in R^3. Since $p(x, x) = p(-x, -x)$ for any $p \in \mathring{\mathcal{P}}(3, 3)$, the map $\bar{p}: RP^2 \to S^2$ carrying any point $\{x, -x\} \in RP^2$ into $(1/|p(x, x)|)\cdot p(x, x)$ is well defined. Let $\deg \bar{p}$ be the degree of this map (since RP^2 is not orientable, the degree is defined only modulo 2). It turns out that for typical $p \in \mathring{\mathcal{P}}(3, 3)$ the condition $\deg \bar{p} = 0$ implies $\bar{p}(RP^2) \neq S^2$.

We outline the proof.

For any point y in general position on S^2, the sets $\bar{p}^{-1}(y)$ and $\bar{p}^{-1}(-y)$ must consist of an even number of points. At the same time, the set $\bar{p}^{-1}(y) \cup \bar{p}^{-1}(-y)$ is a transverse intersection of two quadratics in $\mathbb{R}P^2$ and, so, by Bezout's Theorem, it is either empty or consists of two or four points. If at least one of the sets $\bar{p}^{-1}(y)$, $\bar{p}^{-1}(-y)$ is empty, there is nothing more to prove. There remains the possibility $\#\bar{p}^{-1}(y) = \#\bar{p}^{-1}(-y) = 2$, i.e., for any point $y \in S^2$ in general position the set $\bar{p}^{-1}(y)$ consists of exactly two points. But since the map $\bar{p}: \mathbb{R}P^2 \to S^2$ cannot be a cover, it must have folds and the preimage of a point on one side of the fold contains two points more than that of a point on the other side. This contradiction completes the proof.

As a corollary, using Lemma 2, we deduce that all the maps $p \in \mathring{\mathcal{P}}(3, 3)$ such that deg $\bar{p} = 0$ are rigidly isotopic to one another. Using this fact one can construct numerous examples of pencils which are nonsingularly but not transversally homotopic. The following beautiful example was pointed out to us by Arnol'd.

Let $p = (p_1, p_2, p_3)^T \in \mathcal{P}(3, 3)$, where the equations $p_i(x, x) = 0$, $i = 1, 2, 3$, define ellipses in $\mathbb{R}P^2$, arranged as shown in the drawing:



(each pair of ellipses intersects at four points, two of which lie inside the third ellipse and two outside).

Each of the coordinate forms p_i takes values of opposite signs at points lying inside and outside "its" ellipse. Hence we conclude that $p(\mathbb{R}^3)$ intersects all the octants into which $\mathbb{R}^3$ is divided by the coordinate planes. Consequently, $p(\mathbb{R}^3)$ is not contained in any half-space in $\mathbb{R}^3$ and, so, by part 2 of Lemma 2 p^* is not transversally homotopic to any pencil containing a sign-definite form. On the other hand, ind $\bar{p} = 0$; the poles of the sphere S^2 are regular values of the map $p: \mathbb{R}P^2 \to S^2$ and the image of each pole consists of just two points. Thus p^* is nonsingularly homotopic to pencils containing sign-definite forms.

3. Complexes of Quadratic Forms

1. Let $K^\circ \subset \mathbb{R}^k$ be a convex acute-angled polyhedral cone in $\mathbb{R}^k$ with its apex at zero, $K = \{w \in \mathbb{R}^k | \omega l \leqslant 0 \ \forall \ l \in K^\circ\}$ be the dual cone.

In Sec. 2 we examined the nonsingular quadratic maps, i.e., those $p \in \mathcal{P}(N, k)$ for which $p|(\mathbb{R}^N \setminus \{0\})$ is transverse to the point 0 in $\mathbb{R}^k$. In this subsection we are going to generalize the situation to some extent, considering maps transverse to the cone K°.

<u>Definition.</u> Let M be a smooth manifold, $f: M \to \mathbb{R}^k$ be a smooth map and f_m' be its differential at a point m. We will say that the map f is tangent to the cone K^0 at a point $m \in M$ if $f(m) \in K^\circ$ and $f_m'(T_mM) + K^\circ + \mathbb{R}f(m) \neq \mathbb{R}^k$ [or, equivalently, $f_m'(T_mM) \perp \omega$ and $f(m) \perp \omega$ for some $\omega \in K \setminus 0$]. The map f is said to be transverse to K° if it is not tangent to K° at any point.

Now let $p \in \mathcal{P}(N, k)$. A quadratic map $p: \mathbb{R}^N \to \mathbb{R}^k$ is tangent to K° at a point $x \neq 0$ if and only if $\omega_0 Px = 0$ for some $\omega_0 \in K \setminus 0$ and $\omega p(x, x) \leq 0$ for any $\omega \in K$. We see that the tangency condition completely determines the values of the pencil p^* on K. An arbitrary linear map from K to $\mathcal{P}(\mathbb{R}^N)$ will be called a pencil of quadratic forms on K and the linear space of all these pencils will be denoted by $\mathcal{P}(N; K)$.

If $p \in \mathcal{P}(N, k)$, we will denote the pencil $p^*|_K \in \mathcal{P}(N; K)$ by p^K.

<u>Definition.</u> A pencil $p^K \in \mathcal{P}(N; K)$ is said to be singular if there exists $\omega_0 \in K \setminus 0$ such that $\omega_0 p \in \Pi_1$ and, for some normal $\nu \in \mathcal{N}^+_{\omega_0 p} \setminus 0$, it is true that $\langle \nu, \omega p \rangle \leq 0 \ \forall \omega \in K$ [in other words, ν is in the cone $p^K(K)^\circ$ dual to $p^K(K)$]. Otherwise we will call p^K nonsingular.

It is easy to see that a pencil p^K is singular if and only if the map p is tangent to K° at some point $x \neq 0$. The set of all nonsingular pencils in $\mathscr{P}(N; K)$ will be denoted by $\mathring{\mathscr{P}}(N; K)$.

LEMMA 1. $\mathring{\mathscr{P}}(N; K)$ is an open and dense subset of $\mathscr{P}(N; K)$.

Proof. If $x \neq 0$, the map $p \mapsto p(x, x)$ from $\mathscr{P}(N; K)$ to $\mathbf{R}^k$ is of rank k. Therefore, an open dense subset of $\mathscr{P}(N; K)$ consists of quadratic maps p such that $p|(\mathbf{R}^N\setminus 0)$ is transverse to all subsets span Γ, where Γ is an arbitrary face of the cone K° (this is a corollary of the standard Transversality Theorem). The corresponding pencils p^K lie in $\mathring{\mathscr{P}}(N; K)$.

Let $K_i \subset \mathbf{R}^{k_i *}$ be convex polyhedral cones and $A: K_1 \to K_2$ be a linear map. To each pencil $p^{K_2} \in \mathscr{P}(N; K_2)$ we associate the pencil $A^*p^{K_2} \in \mathscr{P}(N; K_1)$, where $A^*p^{K_2}(A\omega)$. Clearly, $A^*p^{K_2} = (A^*p)^{K_1}$, where $A^*p \in \mathscr{P}(N, k_1)$ is the composition of the quadratic map $p \in \mathscr{P}(N; k_2)$ and the linear map $A^*: \mathbf{R}^{k_2} \to \mathbf{R}^{k_1}$. If A is surjective, then obviously

$$(A^*p)^{-1}(K_1^\circ) = \bigcap_{\omega \in K_1} \{x \in \mathbf{R}^N \mid \omega p(x, x) \leq 0\} = p^{-1}(K_2^\circ).$$

The following assertions are also obvious.

LEMMA 2. If a linear map $A: K_1 \to K_2$ is surjective and never vanishes on $K_1\setminus 0$, then for any pencil $p^{K_2} \in \mathscr{P}(N; K_2)$:

$$p^{K_2} \in \mathring{\mathscr{P}}(N; K_2) \Leftrightarrow A^*p^{K_2} \in \mathring{\mathscr{P}}(N; K_1).$$

LEMMA 3. Define $\mathbf{R}^n_+ = \{(y_1, \ldots, y_n) \in \mathbf{R}^n \mid y_i \geq 0,\ i = 1, \ldots, n\}$. For any convex polyhedral cone K there exist $n_1, n_2 \geq 0$ and a surjective map $A: \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}_+ \to K$ such that A does not vanish on $(\mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}_+)\setminus 0$. (Here $n_1 = \dim(K \cap (-K))$ and $n_2 \geq$ the number of generators of the acute-angled cone $K/(K \cap (-K))$.

LEMMA 4. Let $p_t{}^K \in \mathring{\mathscr{P}}(N; K)$ be smoothly dependent on $t \in [0, 1]$. Then there exists a Lipschitzian isotopy $F_t: S^{N-1} \to S^{N-1}$, $t \in [0, 1]$, $F_0 = \mathrm{id}$, such that $F_1(p_0{}^{-1}(K^\circ) \cap S^{N-1} = p_1{}^{-1}(K^\circ) \cap S^{N-1}$.

Proof. The quadratic maps $p_t|S^{N-1}$ are transverse to the cone K°, $t \in [0, 1]$. Define $f_t = p_t|S^{N-1}$.

Choose an arbitrary point y_0 in the relative interior of K°; the Minkowski function on K° is defined by the formula

$$\mu(y) = \inf\{\mu \geq 0 \mid \mu K^\circ \ni p\}\ \ \forall p \in \operatorname{span} K^\circ.$$

It is clear that $\mu^{-1}(1) = \partial K^\circ$ is the relative boundary of K. Approximate the convex positively homogeneous function μ by a smooth convex positively homogeneous function μ_ε such that $\mu_\varepsilon - \varepsilon \leq \mu \leq \mu_\varepsilon$; then $K_\varepsilon{}^\circ = \{y \in \mathbf{R}^k \mid \mu_\varepsilon \leq 1\}$ is a convex set with smooth boundary $\partial K_\varepsilon{}^\circ$ contained in K°.

The transversality condition implies the existence of a smooth vector field X(x) on S^{N-1} such that for any $x \in f_t{}^{-1}(K^\circ)$ the vector $f_t{}'xX(x)$ applied at the point f(x) looks into the relative interior of K° (the local existence of a field satisfying this condition follows directly from the transversality condition; the global field is constructed using a partition of unity). Consider the differential equation

$$\frac{dx}{ds} = X(x(s)) \text{ on } S^{N-1}.$$

It is readily shown that the function $s \mapsto \mu \circ f_t(x(s))$ decreases monotonically at a nonzero rate for all x(s) in a fixed neighborhood O of the set $f_t{}^{-1}(\partial K^\circ)$. Choosing ε small enough, we can ensure that $f_t{}^{-1}(\partial K_\varepsilon{}^\circ) \subset O$ and, in addition, $(d/ds)\cdot\mu_\varepsilon \circ f_t(x(s)) \leq 0$ for $x(s) \in O$ (this follows from standard results of convex analysis concerning the behavior of the subdifferentials of a convergent sequence of convex functions, see [2]). Moving along trajectories of the system $(dx/ds) = X(x(s))$ from $f_t{}^{-1}(\partial K^\circ) = (\mu \circ f_t)^{-1}(1)$ to $f_t{}^{-1}(\partial K_\varepsilon) = (\mu_\varepsilon \circ f_t{}^{-1})(1)$, we obtain a Lipschitzian isotopy of S^{N-1} taking the set $f_t{}^{-1}(K^\circ)$ into the set $f_t{}^{-1}(K_\varepsilon{}^\circ)$.

Lemma 4 now follows from the following assertion:

Let $V \subset \mathbf{R}^k$ be a smooth manifold with boundary and $f_t: S^{N-1} \to \mathbf{R}^k$, $t \in [0, 1]$, be a smooth homotopy of smooth maps transverse to V. Then there exists a smooth isotopy $F_t: S^{N-1} \to S^{N-1}$, $t \in [0, 1]$, $F_0 = \mathrm{id}$, such that $F_t(f_0^{-1}(V)) = f_t^{-1}(V)$, $\forall\, t \in [0, 1]$. This slightly strengthened version of Thom's Isotopy Lemma (instead of a manifold without boundary — a manifold with boundary) can be proved in the same way as Lemma 2.1, with the appropriate simplifications.

Arguments similar to those used in the proof of Lemma 4 furnish a proof of the following

LEMMA 5. Let C be a closed convex subset of $\mathbf{R}^k$ and $f: M \to \mathbf{R}^k$ a smooth map of a smooth closed manifold M into $\mathbf{R}^k$. Assume that f is transverse to C [i.e., for no $x \in M$ is im $\times$ $f'(x)$ a supporting plane of C at the point $f(x)$]. Then for any convex closed set C_p sufficiently close to C, there is an isotopy $F_t: M \to M$, $t \in [0, 1]$, $F_0 = \mathrm{id}$, such that $F_1(f^{-1} \times (C)) = f^{-1}(C_p)$.

Lemmas 2 and 3 yield a natural way to generalize the concept of nonsingular homotopy to pencils defined on different cones.

Definition. Nonsingular pencils $p_i^{K_i} \in \mathscr{P}(N; K_i)$, $i = 1, 2$, are said to be nonsingularly homotopic if there exist a cone K and linear surjective maps $A_i: K \to K_i$, $i = 1, 2$, not vanishing on $K\backslash 0$, such that the pencils $A_i^* p_i^{K_i} \in \mathring{\mathscr{P}}(N; K)$, $i = 1, 2$, lie in the same connected component of the set $\mathring{\mathscr{P}}(N; K)$.

Remark. The new concept of nonsingular homotopy is wider than that defined in Sec. 2 even if $K_1 = K_2$ is a linear space; nevertheless, for nonsingularly homotopic $p_i^{K_i}$ the sets $p_i^{-1}(K_i^0) \cap S^{N-1}$ are isotopic in S^{N-1}.

The fact that a pencil $p_1^{K_1}$ is nonsingularly homotopic to a pencil $p_2^{K_2}$ will be written thus: $p_1^{K_1} \sim p_2^{K_2}$.

In Sec. 2, in connection with pencils defined on linear spaces, we considered, apart from nonsingular pencils, also transverse pencils. There is an analogous concept (which, as we will see later, is very useful) for pencils defined on convex cones. Recall that a pencil $p^K \in \mathscr{P}(N; K)$ is said to be nonsingular if for any $\omega_0 \in K\backslash\, 0$ we have $p^K(K)^\circ \cap \mathscr{N}^{+}_{\omega, p} = 0$.

We call a pencil $p^K \in \mathscr{P}(N; K)$ transverse if, for any $\omega_0 \in K\backslash 0$,

$$p^K(K) \cap \mathrm{conv}\, \mathscr{N}^{+}_{\omega_0 p} = 0.$$

Since $\mathscr{N}^{+}_{\omega_0 p} = \{x \otimes x \mid x \in \ker \omega_0 p\}$, we obtain, after separating the cones $p^K(K)^\circ$ and conv $\times$ $\mathscr{N}^{+}_{\omega_0} p$ by a hyperplane:

A pencil p^K is transverse if and only if, for any $\omega_0 \in K\backslash 0$, there exists $\omega \in K$ such that $\omega p|\ker \omega_0 p > 0$. The set of all transverse pencils in $\mathscr{P}(N; K)$ is denoted by $\mathscr{P}^t(N; K)$; clearly, $\mathscr{P}^t(N; K) \subset \mathring{\mathscr{P}}(N; K)$.

LEMMA 6. $\mathscr{P}^t(N; K)$ is an open dense subset of $\mathscr{P}(N; K)$.

Proof. That $\mathscr{P}^t(N; K)$ is an open set is obvious. Now assume that $\mathscr{P}(N; K)$ is not a transverse pencil; let $\omega_0 \in K\backslash 0$ be a point at which the transversality fails and let Γ_{ω_0} be the open fact of K which contains ω_0. Then the pencil $p^*|\mathrm{span}\,\Gamma_{\omega_0}$ defined on a subspace is also not transverse. Since K has only finitely many faces, the statement of the lemma follows from the standard Transversality Theorem.

Definition. Pencils $p_i^{K_i} \in \mathscr{P}^t(N; K_i)$ are said to be tranversally homotopic if there exist a cone K and linear surjective maps $A_i: K \to K_i$, $i = 1, 2$, not vanishing on $K\backslash 0$, such that $A_i^* p_i^{K_i} \in \mathscr{P}^t(N; K)$, $i = 1, 2$, lie in the same connected component of the set $\mathscr{P}^t(N; K)$.

The fact that pencils $p_1^{K_1}$ and $p_2^{K_2}$ are transversally homotopic will be written $p_1^{K_1} \overset{t}{\sim} p_2^{K_2}$. The next assertion is obvious:

Proposition 1. Let K_1, i = 1, 2, be acute-angled cones and $p_i^{K_i} \in \mathcal{P}(N; K_i)$ be pencils such that the forms ωp_i are nonsingular $\forall \omega \in K_i\backslash 0$, i = 1, 2. Then $p_1^{K_1}$ is tranversally homotopic to $p_2^{K_2}$ if and only if $\operatorname{ind}\omega_1 p_1 = \operatorname{ind}\omega_2 p_2$ for some $\omega_i \in K_i\backslash 0$, i = 1, 2.

Let $p_0 \in \mathcal{P}(R^N)$, with $\ker p_0 = 0$, $\operatorname{ind} p_0 = n$. Let n denote the class of all pencils which are nonsingularly homotopic to a pencil $\alpha \mapsto \alpha p_0$, $\alpha \in R_+$ in $\mathcal{P}(N; R_+)$, and n_t the class of all pencils which are transversally homotopic to it. Each of the pencils described in Proposition 1 is in some class n_t, where n = 0, 1, ..., N.

The next proposition shows that "small" transverse pencils belong to one of the classes n_t even if they do not meet the conditions of Proposition 1.

Proposition 2. Let $p^K \in \mathcal{P}(N; K)$, let the pencil p^K be transverse at ω_0 and $\operatorname{ind}\omega_0 p = n$. Then for any sufficiently small conical neighborhood U_{ω_0} of ω_0 in K the pencil $p^{U_{\omega_0}} = p^K|U_{\omega_0}$ belongs to the class n_t.

Proof. Let $V = \ker \omega_0 p$, $\dim V = r$; the transversality condition implies the existence of $\omega_1 \in K$ such that $\omega_1 p|_V > 0$.

Let $\omega \in K$ and assume that corresponding to ωp we have a symmetric operator $\omega P: R^N \to R^N$ with eigenvalues $\lambda_1(\omega P) \le \lambda_2(\omega P) \le \dots \lambda_N(\omega P)$. Let $V(\omega)$ denote the linear span of the eigenvalues of ωP belonging to the eigenvalues $\lambda_{k+1}(\omega P), \dots, \lambda_{k+r}(\omega P)$. We have $V(\omega_0) = V$ and for ω close to ω_0 the subspace $V(\omega)$ is close to V. Choose a neighborhood U such that for $\forall \omega \in U$ and some $\varepsilon > 0$

$$\lambda_k(\omega p) < -\varepsilon(\omega, \omega_0) < \lambda_{k+1}(\omega p),\ \lambda_{k+r}(\omega p) > 0;$$

$$(\omega_1 p - \omega_0 p)\,|_{V(\omega)} > 0.$$

In addition, we will assume that U is contained in the union of the faces of K adjacent to ω_0.

For any $\tau \in [0, \varepsilon]$ and $\omega \in U$ we define $\omega p_\tau = \omega p + \tau(\omega, \omega_0)I$, where $I(x, x) \overset{\text{def}}{=} (x, x)\ \forall x \in R^N$, and the scalar product in the space span K is so chosen that $(\omega_1 - \omega_0, \omega_0) > 0$. Clearly, $\operatorname{ind}\omega p_\varepsilon = n$, $\ker \omega p_\varepsilon = 0\ \forall \omega \in U$. Consequently, the pencil $p_\varepsilon^U \in \mathcal{P}(N; U)$ belongs to class n_t . It remains to be shown that the pencils p_τ^U are transverse for all $\tau \in [0, \varepsilon]$. It is easy to see that $\ker \omega p_\tau \subset V(\omega)$, $\forall \omega \in U$, $\tau \in [0, \varepsilon]$. At the same time, for all sufficiently small $\alpha > 0$ we have $(\alpha(\omega_1 - \omega_0) + \omega) \in U$ and

$$(\alpha\omega_1 - \alpha\omega_0 + \omega)\, p_\tau \,|\, \ker \omega p_\tau = \alpha(\omega_1 - \omega_0)\, p_\tau \,|\, \ker \omega p_\tau \geqslant$$
$$\geqslant \alpha(\omega_1 - \omega_0)\, p \,|\, \ker \omega p_\tau > 0. \ \blacktriangleright$$

2. In this subsection we again consider a significant expansion of the class of pencils of quadratic forms under examination. Up to now we have confined attention to linear maps in the space of quadratic forms; we will now admit arbitrary piecewise smooth maps. We begin with a rigorous definition of "piecewise smooth."

Definition. Let M be a C^∞ manifold. A closed subset $\Omega \subset M$ will be called a manifold with angles if any point $\omega \in \Omega$ has a neighborhood O_ω in M and local coordinates $\varphi: O_\omega \to R^k$, $(\omega) = 0$, such that $\varphi(O_\omega \cap \Omega)$ is a convex polyhedral cone in R^k.

A vector $\xi \in T_\omega M$ is said to be tangent to Ω at a point $\omega \in \Omega$ if there exists a smooth curve $\gamma: [0, \varepsilon] \to \Omega$ such that $\gamma(0) = \omega$, $\gamma'(0) = \xi$. The set of all vectors tangent at a point ω to a manifold with angles Ω is a convex polyhedral cone in $T_\omega M$, which we denote by $T_\omega\Omega$. It follows from the definition of a manifold with angles that there exists a diffeomorphism $\Phi: O_\omega \to T_\omega M$ of a neighborhood of ω in M onto $T_\omega M$ such that

$$\Phi(\Omega \cap O_\omega) = T_\omega\Omega.$$

Definition. A connected smooth submanifold $\Gamma \subset M$ is called an open face of a manifold with angles $\Omega \subset M$ if $\Gamma \subset \Omega$ and Γ is not a proper subset of any connected smooth submanifold of M which is also a subset of Ω; the closure $\overline{\Gamma}$ of an open face Γ is called a closed face.

LEMMA 7. Let Ω be a manifold with angles and $\omega \in \Omega$. Then: i) ω lies in a unique open face Γ_ω of Ω; ii) Any closed face $\overline{\Gamma} \subset \Omega$ is itself a manifold with angles and the correspondence $\bar{\Gamma} \to T_\omega\bar{\Gamma}$, where $\omega \in \bar{\Gamma}$, is a one-to-one correspondence between the closed faces containing ω and the closed faces of the cone $T_{\omega\Omega}$; moreover, $T_\omega\Gamma_\omega = T_\omega\Omega \cap (-T_\omega\Omega)$.

Proof. It is readily seen that the sets $\{\omega \in \Omega | \dim(T_\omega\Omega \cap (-T_\omega\Omega)) = i\}$ are i-dimensional smooth submanifolds of M. Consequently, the connected components of these sets, and they alone, are the i-dimensional faces of Ω. This implies part (i) of the lemma and the second part is now obvious.

The inclusion relation "$\subset$" defines a partial ordering of the set of all closed faces of a manifold with angles Ω. It also generates a partial order relation on the set of all open faces: a face Γ_1 is "less than" a face Γ_2 if $\Gamma_1 \subset \bar{\Gamma}_2$. It is not hard to see that any closed face $\bar{\Gamma}$ is a topological manifold with boundary $\partial\bar{\Gamma} = \bar{\Gamma}\setminus\Gamma$ (the boundary need not be smooth) and the maximal closed faces are precisely the connected components of Ω.

Let $\tilde{M}$ be another C^∞ manifold. A map $f:\Omega \to \tilde{M}$ is said to be smooth if it is the restriction to Ω of a smooth map $F:M \to \tilde{M}$. The differential $f_\omega':T_\omega\Omega \to T_{f(\omega)}\tilde{M}$ is defined as the restriction to $T_\omega\Omega$ of the differential $F_\omega':T_\omega M \to T_{f(\omega)}\tilde{M}$. This restriction obviously depends only on f, but not on the choice of the smooth extension F. A map $\varphi:\Omega \to \Omega$ is called a diffeomorphism if φ is the restriction to Ω of a diffeomorphism $\Phi:M \to M$. Note that a diffeomorphism φ need not be an "onto" map.

Definition. Let $f:\Omega \to \tilde{M}$ be a smooth map and $\tilde{M} \supset \tilde{\Omega}$ be a manifold with angles. We will say that f is transverse to $\tilde{\Omega}$ at a point $\omega \in \Omega$ if, whenever $f(\omega) \in \tilde{\Omega}$, we have $(-\operatorname{im} f_\omega') + T_{f(\omega)}\tilde{\Omega} = T_{f(\omega)}\tilde{M}$. The map f is said to be transverse to $\tilde{\Omega}$ if it is transverse to $\tilde{\Omega}$ at every point.

Definition. Let $\mathcal{K} = \{\Omega_1, \ldots, \Omega_n\}$ be a finite set of compact manifolds with angles, $\Omega_i \subset M$, $i = 1, \ldots, n$. We will call $\mathcal{K}$ a piecewise smooth complex if $\forall$ i, j the intersection $\Omega_i \cap \Omega_j$ is a closed face of Ω_i and a closed face of Ω_j and, in addition, $\Omega_i \cap \Omega_j \in \mathcal{K}$. The support of a piecewise smooth complex $\mathcal{K}$ is the set $\bar{\mathcal{K}} = \bigcup_{i=1}^{n} \Omega_i \subset M$.

Definition. Let $M \supset \Omega$ be a manifold with angles and $f:\Omega \to \mathcal{R}(R^N)$ be a smooth map. Then f is said to be nonsingular if, for any $\omega \in \Omega$, whenever $f(\omega) \in \Pi_1$, we have $f_\omega'(T_\omega\Omega)^\circ \cap \mathcal{N}^+_{f\omega} = 0$; it is said to be transverse if, under the same assumptions,

$$f'_\omega(T\omega\Omega)^\circ \cap \operatorname{conv} \mathcal{N}^+_{f(\omega)} = 0.$$

The last definition may be interpreted along the same lines as done in subsection 1 for pencils of quadratic forms: a smooth map $f:\Omega \to \mathcal{P}(R^N)$ is nonsingular if and only if, for any $\omega \in \Omega$, $x \in \ker f(\omega)\setminus 0$, there exists $\xi \in T_\omega\Omega$ such that $(f_\omega'\xi)(x, x) > 0$; it is transverse if and only if, for any $\omega \in \Omega$, there exists $\xi \in T_\omega\Omega$ such that $(f_\omega'\xi)|_{\ker f(\omega)} > 0$.

Let K be a polyhedral cone in R^{k*} and S^{k-1} a sphere in R^{k*}. Clearly, $K \cap S^{k-1}$ is a manifold with angles. It is not hard to see that a pencil $p^K \in \mathcal{P}(N; k)$ is nonsingular (transverse) if and only if the map $p^K|_{K \cap S^{N-1}}$ is nonsingular (transverse).

Definition. Let $\mathcal{K}$ be a piecewise smooth complex and $\bar{\mathcal{K}} \subset M$ be its support. We define a complex of quadratic forms to be an arbitrary continuous map $f: \bar{\mathcal{K}} \to \mathcal{P}(R^N)$, which is smooth on every manifold with angles $\Omega \in \mathcal{K}$, $\Omega \subset \bar{\mathcal{K}}$. A complex of quadratic forms f is said to be nonsingular (transverse) if, for any $\Omega \in \mathcal{K}$, the map $f|_\Omega:\Omega \to \mathcal{P}(R^N)$ is nonsingular (transverse).

An arbitrary homotopy $f_t: \bar{\mathcal{K}} \to \mathcal{P}(R^N)$, $t \in [0, 1]$, of complexes of quadratic forms is said to be nonsingular (transverse) if f_t is nonsingular (transverse) $\forall t \in [0, 1]$.

The space of all smooth maps of a manifold with angles $\Omega \subset M$ into $\mathcal{P}(R^N)$ will be denoted by $C^\infty(\Omega, \mathcal{P}(R^N))$, and the space of all complexes of quadratic forms $f: \bar{\mathcal{K}} \to \mathcal{P}(R^N)$ by $C^\infty(\mathcal{K}, \mathcal{P}(R^N))$. The Whitney topology of $C^\infty(M, \mathcal{P}(R^N))$ induces a Fréchet space structure in $C^\infty(\Omega, \mathcal{P}(R^N))$. We will say that a sequence of complexes $f_n \in C^\infty(\mathcal{K}, \mathcal{P}(R^N))$, $n = 1, 2, \ldots$, converges to a complex $f \in C^\infty(\mathcal{K}, \mathcal{P}(R^N))$ if $\forall \Omega \in \mathcal{K}$ the maps $f_n|\Omega$ converge to $f|\Omega$ in the Fréchet space $C^\infty(\Omega, \mathcal{P}(R^N))$. This convergence concept clearly defines a Fréchet space structure in $C^\infty(\mathcal{K}, \mathcal{P}(R^N))$.

The following assertion (similar to Lemma 6) follows easily from the standard Transversality Theorem.

LEMMA 8. Let $\mathcal{K}$ be a piecewise smooth complex; the transverse quadratic complexes form an open dense subset of $C^\infty(\mathcal{K}, \mathcal{P}(\mathbb{R}^N))$.

LEMMA 9. Let $f_t: \overline{\mathcal{K}} \to \mathcal{P}(\mathbb{R}^N)$, $t \in [0, 1]$, be a smooth transversal homotopy of complexes of quadratic forms. Then there exists a smooth family of continuous one-to-one maps $F_t: \overline{\mathcal{K}} \to \overline{\mathcal{K}}$, $F_0 = \mathrm{id}$, such that

$$F_t(f_0^{-1}(\mathcal{P}_n)) \subset f_t^{-1}(\mathcal{P}_n), \quad n = 0, 1, \ldots, N, \quad t \in [0, 1],$$

where, if $\mathcal{K}$ consists of a single manifold with angles Ω, then $F_t\Omega \to \Omega$ is a diffeomorphism.

Proof. Lemma 9 is a parallel to Lemma 2.1, although it by no means includes the latter as a special case, since the equality of Lemma 2.1 is replaced here by an inclusion. The method of proof of Lemma 8 is the same as that of Lemma 2.1 (or of any other variant of Thom's Isotopy Lemma).

Let X be a continuous vector field on M. We will call the restriction of X to $\overline{\mathcal{K}}$ a vector field on $\mathcal{K}$, if. for any y, Ω, $y \in \Omega \in \mathcal{K}$, it is true that $X(y) \in T_y\Omega$. A vector field on $\mathcal{K}$ is said to be smooth (Lipschitzian) if $X|_\Omega$ is smooth (Lipschitzian) for any $\Omega \in \mathcal{K}$.

To prove Lemma 9 it will suffice to construct a nonstationary Lipschitzian (or, if $\mathcal{K} = \{\Omega\}$, smooth) field X_t on $\mathcal{K}$ with the following properties: if $y \in \Omega \in \mathcal{K}$, then

$$\left(\frac{\partial f}{\partial t}(y) + f'_{ty} X_t(y)\right)\Big|_{\ker f_t(y)} > 0. \tag{1}$$

This field need only be constructed locally, near a fixed point $y_0 \in \Omega_0 \in \mathcal{K}$, the rest follows by using a partition of unity.

Let $\mathcal{P}_+(\ker f_t(y))$ be the convex closed cone of all quadratic forms on $\mathbb{R}^N$, which are nonnegative on the subspace $\ker f_t(y)$. We know from the results of Sec. 1.2 that

$$\mathcal{P}_+(\ker f_t(y))^\circ = -\operatorname{conv} \mathcal{N}^+_{f_t(y)}. \tag{2}$$

Going over to dual cones in the transversality condition

$$\operatorname{conv} \mathcal{N}^+_{f_t(y_0)} \cap f'_{ty_0}(T_{y_0}\Omega_0) = 0,$$

we get

$$f'_{ty_0}(T_{y_0}\Omega_0) + \mathcal{P}_+(\ker f_t(y_0)) = \mathcal{P}(\mathbb{R}^N).$$

This equality will obviously be maintained if we replace the cone $\mathcal{P}_+(\ker f_t(y_0))$ by its own interior, i.e., the set of forms which are positive on $\ker f_t(y_0)$. Consequently, there exists a vector $X_t(y_0) \in T_{y_0}\Omega_0$ such that

$$\left(\frac{\partial f}{\partial t}(y_0) + f'_{ty_0} X_t(y_0)\right)\Big|_{\ker f_t(y_0)} > 0.$$

Clearly, any continuous vector field X_t on M taking the value $X_t(y_0)$ at the point y_0 will satisfy inequality (1) for any y close to y_0.

It remains to be proved the existence of at least one Lipschitzian (or, if $\mathcal{K} = \{\Omega\}$, smooth) field on $\mathcal{K}$, whose value at y_0 is $X_t(y_0)$. The construction of a smooth field in Ω_0 is trivial: it is a field mapped onto a constant by the diffeomorphism $\Phi: O_{y_0} \cap \Omega_0 \to T_{y_0}\Omega_0$.

Extension of this vector field on Ω_0 to a Lipschitzian field on any $\Omega \in \mathcal{K}$ is also not a difficult task. The extension can be accomplished by induction on the dimensions of faces, in which case the problem easily reduces to the following one: given a Lipschitzian vector field on the boundary of a convex acute-angled polyhedral cone, extend it to a Lipschitzian field on the entire cone. This is done by picking any point in the interior of the cone, and considering the ray through this point and the apex of the cone; define the field at the point as equal to its value at the apex. Then, joining the points of the ray to the boundary of the cone by segments in a hyperplane perpendicular to the ray, define the fields along each segment by taking convex combinations of the values of the field at the ends of the segment.

COROLLARY. Under the assumptions of Lemma 8, the homotopy type of a pair $(\bar{\mathcal{K}}, f_t^{-1}(\mathcal{P}_n))$ is independent of $t \in [0, 1]$ for $n = 0, 1, \ldots, N$. In particular, the cohomology groups $H^i(\bar{\mathcal{K}}, f^{-1}(\mathcal{P}_n))$ are transversal homotopy invariants.

Let $f:\bar{\mathcal{K}} \to \mathcal{P}(\mathbf{R}^N)$ be a nonsingular complex of quadratic forms such that $f(\bar{\mathcal{K}}) \cap \Pi_2 = \emptyset$. Then f is a transverse complex. Since Π_2 has codimension 3 in $\mathcal{P}(\mathbf{R}^N)$, we obtain the following obvious generalization of Proposition 2.3:

Proposition 3. Let $\mathcal{K}$ be a zero- or one-dimensional complex and $f_i: \bar{\mathcal{K}} \to \mathcal{P}(\mathbf{R}^N)$ be transverse complexes of quadratic forms, i = 1, 2. If f_1 and f_2 are nonsingularly homotopic, they are also transversally homotopic.

In the situation considered in Proposition 3, the homotopy type of the pairs $(\mathcal{K}, f_i^{-1} \times (\mathcal{P}_n))$, i = 1, ..., N, is a nonsingular homotopy invariant. In particular, this is the case for the complexes

$$\omega \mapsto \omega q(n_1, \ldots, n_{2k-1};\ m),\ \omega \in S^1 \subset \mathbf{R}^{2*} \tag{2}$$

[for the definition of the quadratic maps $q(n_1, \ldots, n_{2k-1};\ m)$, see the text of Theorem 2.1].

Recall that $f^{-1}(\mathcal{P}_n) = \{y \in \bar{\mathcal{K}}\ |\operatorname{ind} f(y) \le n\}$, so description of the sets $f^{-1}(\mathcal{P}_n)$ reduces to calculation of the function $y \to \operatorname{ind} f(y)$. This may be done as follows for the complex (2).

Let $\Delta_0, \ldots, \Delta_{2k-3}$ be the open arcs marked off on the circle S^1 by the $(4k-2)$-th roots of unity, numbered so that the indices increase monotonically as the circle is described in the positive sense from Δ_0.

Put $n_j = n_{j-2k+1}$ for $2k \le j \le 3k-2$. Then

$$\operatorname{ind}(\omega q(n_1, \ldots, n_{2k-1}: m)) = \begin{cases} m + \sum_{j=\alpha+1}^{\alpha+k} n_j, & \omega \in \Delta_{2\alpha} \\ m + \sum_{j=\alpha+2}^{\alpha+k} n_j, & \omega \in \bar{\Delta}_{2\alpha+1} \end{cases}, \quad \alpha = 0, 1, \ldots, 2k-2. \tag{3}$$

Hence one readily infers that a complex (2) can be nonsingularly homotopic to a complex $\omega \mapsto \omega q(n_1', \ldots, n_{2k'-1}';\ m')$, $\omega \in S^1$, only if $m = m'$, $k = k'$, and the sequence $n_1', \ldots, n_{2k-1}'$ is a cyclic permutation of $n_1, \ldots, n_{2k-1}$. We emphasize that this fact does not follow from Theorem 2.1, where we were concerned with homotopy in the class of nonsingular pencils of quadratic forms, i.e., "linear" nonsingular maps of S_1 into $\mathcal{P}(\mathbf{R}^N)$, whereas here we are dealing with homotopy in the class of arbitrary smooth nonsingular maps.

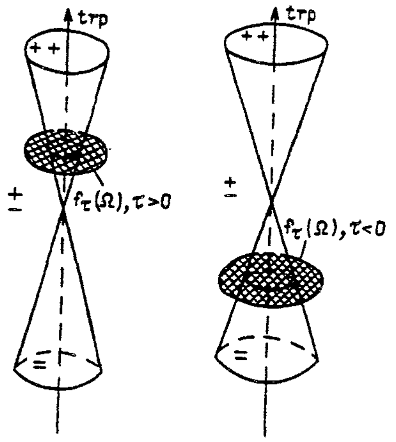
For complexes of quadratic forms of dimension greater than one, nonsingular homotopy, as might be expected, by no means implies transversal homotopy. Indeed, even for smooth maps $f: B^2 \to \mathcal{P}(\mathbf{R}^2)$ of the two-dimensional disk into the space of quadratic forms of two variables, the cohomology groups $H^i(B^2, f^{-1}(\mathcal{P}_n))$ are not nonsingular homotopy invariants.

We present a "model" example.

Let $\Omega = \{p \in \mathcal{P}(\mathbf{R}^2)\ |\operatorname{tr} P = 0,\ \operatorname{tr} P^2 \le 1\}$ be a disk in the three-dimensional space $\mathcal{P}(\mathbf{R}^2)$. Define a map $f_\tau: \Omega \to \mathcal{P}(\mathbf{R}^2)$ by the formula $f_\tau(p) = p + \tau I$, $p \in \Omega$, $\tau \in (-1, 1)$. The map f_0 is nonsingular, but not transvsers; the maps f_τ, where $\tau \in (-1, 1)\setminus\{0\}$, are transverse. We have $H^i(\Omega, f_\tau^{-1}(\mathcal{P}_n)) = 0\ \forall\, i, n$ for $\tau > 0$; but $H^0(\Omega, f_\tau^{-1}(\mathcal{P}_0)) = H^2(\Omega, f_\tau^{-1}(\mathcal{P}_1)) = \mathbf{Z}$ for $-1 < \tau < 0$. (See picture at the top of the next page.)

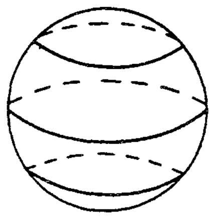
Let q_0 be a nonsingular quadratic form in $\mathcal{P}(\mathbf{R}^N)$. Replacing f_τ by $f_\tau \oplus q_0: \Omega \to \mathcal{P}(\mathbf{R}^{N+2})$, we obtain an analogous picture for any $N > 0$.

The examples considered at the end of Sec. 2, of quadratic maps of $\mathbf{R}^3$ into $\mathbf{R}^3$, enable one to demonstrate the same effect for pencils of quadratic forms. Let $p \in \mathcal{P}(3, 3)$. The equation $\det(\omega P) = 0$ defines a cubic curve in the projective plane $\mathbf{RP}^2$. For a map p in general position, this curve is nonsingular. In general, a nonsingular cubic curve may

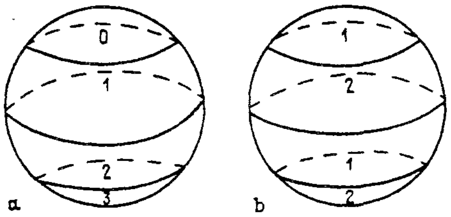


have either one or two connected components. It turns out that it has two components if and only if the pencil p* is nonsingularly homotopic to a pencil containing a sign-definite form. Upon passage from $\mathbf{RP}^2$ to a twofold cover – the sphere $S^2 \subset \mathbf{R}^{3*}$ – one of the components (which is contractible in $\mathbf{RP}^2$) is doubled, but the other (incontractible) one is not.

The following picture is obtained: the curve divides S^2 into four regions.



Now, for nonsingular forms ωp it is true that $\operatorname{ind}(-\omega p) = 3 - \operatorname{ind}\omega p$, while the points of adjacent regions must correspond to adjacent indices. Hence there are only two possible waves of dividing the sphere into domains with different indices:



In case (a) the pencil p* contains sign-definite forms, but in case (b) it does not. As shown in Sec. 2, both situations may actually occur.

Let $f: \mathcal{K} \to \mathcal{P}(\mathbf{R}^N)$ be a complex of quadratic forms and $p_0 \in \mathcal{P}_0(\mathbf{R}^{N'})$ be a fixed positive form. Then the map $\omega \mapsto f(\omega) \oplus p_0$ defines a complex of quadratic forms $f \oplus p_0: \mathcal{K} \to \mathcal{P}(\mathbf{R}^{N+N'})$. If f is a nonsingular complex, then obviously $f \oplus p_0$ is also nonsingular; if f is transverse, then so is $f \oplus p_0$. In addition,

$$f^{-1}(\mathcal{P}_n) = (f \oplus p_0)^{-1}(\mathcal{P}_n), \quad \forall n \text{ and } f \oplus p_0 \overset{t}{\sim} f \oplus p_1$$

for any positive forms $p_0, p_1 \in \mathcal{P}_0(\mathbf{R}^{N'})$. The operation associating to any complex f the complex $f \oplus p_0$ enables us to extend the concepts of nonsingular and transversal homotopy to complexes of forms of different numbers of variables.

<u>Definition.</u> We will say that complexes $f_i: \mathcal{K} \to \mathcal{P}(\mathbf{R}^{N_i})$, i = 1, 2, are stable nonsingularly (transversally) homotopic if there are positive forms $p_i \in \mathcal{P}(\mathbf{R}^{N_i'})$, $N_1 + N_1' = N_2 + N_2'$, such that the complexes $f_i \oplus p_i$, i = 1, 2, are nonsingularly (transversally) homotopic.

4. EULER CHARACTERISTIC

Let $\mathcal{T}$ be a topological space with the homotopy type of a finite cell complex. The symbol $\chi(\mathcal{T})$ will denote the Euler characteristic of $\mathcal{T}$, $\chi(\mathcal{T}) = \sum \operatorname{rank} H^i(\mathcal{T})(-1)^i$. The Euler characteristic is, of course, additive: if C_1, C_2 are subcomplexes of a finite cell complex, then

$$\chi(C/C_1)=\chi(C)-\chi(C_1),\quad \chi(C_1\cup C_2)+\chi(C_1\cap C_2)=\chi(C_1)+\chi(C_2).$$

Definition. Let $f:\overline{\mathscr{K}}\to\mathscr{P}(\mathbf{R}^N)$ be a nonsingular complex of quadratic forms. We put

$$\chi(f)=\sum_{n=0}^{N}(-1)^n\chi(\overline{\mathscr{K}}/f^{-1}(\mathscr{P}_n))=\sum_{i=0}^{[\frac{N}{2}]}\chi(f^{-1}(\mathscr{P}_{2i+1})/f^{-1}(\mathscr{P}_{2i}))$$

and call $\chi(f)$ the Euler characteristic of the complex f.

Let $\mathscr{K}$, $\mathscr{L}$ be piecewise smooth complexes, $\overline{\mathscr{K}}, \overline{\mathscr{L}}\subset M$, such that $\overline{\mathscr{K}\cap\mathscr{L}}=\overline{\mathscr{K}}\cap\overline{\mathscr{L}}$. Then $\mathscr{K}\cap\mathscr{L}$, $\mathscr{K}\cup\mathscr{L}$ are also piecewise smooth complexes and for any complex of quadratic forms $f:\overline{\mathscr{L}\cup\mathscr{K}}\to\mathscr{P}(\mathbf{R}^N)$ there is an obvious definition of the complexes of quadratic forms $f|\mathscr{K}$, $f|\mathscr{L}$, $f|\mathscr{K}\cap\mathscr{L}$. The Euler characteristic is obviously additive in the sense that

$$\chi(f)+\chi(f|\mathscr{K}\cap\mathscr{L})=\chi(f|\mathscr{K})+\chi(f|\mathscr{L}).$$

It can be shown that the Euler characteristic $\chi(f)$ [unlike its terms $\chi(\overline{\mathscr{K}}/f^{-1}(\mathscr{P}_n))$] is not changed under nonsingular homotopy. In the case of pencils this follows from the next theorem and Lemma 3.4.

THEOREM 1. Let $p\in\mathscr{P}(N, k)$ be a quadratic map from $\mathbf{R}^N$ to $\mathbf{R}^k$, K be a convex closed cone in $\mathbf{R}^{k*}$, such that pencil $p^K:K\to\mathscr{P}(\mathbf{R}^N)$ is transverse. Define $K_n=(p^K)^{-1}(\mathscr{P}_n)\backslash 0$, $n\geq 0$. Then

$$\frac{1}{2}\chi(p^{-1}(K^\circ)\cap S^{N-1})=\sum_{i=0}^{[\frac{N}{2}]}\chi(K_{2i+1}/K_{2i})+\frac{\varepsilon}{2}(1+(-1)^{N-1})=\chi(p^K|S^{k-1})+\frac{\varepsilon}{2}\chi(S^{N-1}), \tag{1}$$

where $\varepsilon=\begin{cases}(-1)^{\dim K}, & K=-K\\ 0, & K\neq -K.\end{cases}$

Proof. The proof will be divided into several steps.

1) Assume that K is a half-line, $K=\{\alpha\omega_0|\alpha\geq 0\}$. Then $p^{-1}(K^\circ)=\{x\in\mathbf{R}^N|\omega_0p(x, x)\leq 0\}$, the set $p^{-1}(K^\circ)\cap S^{N-1}$ has the homotopy type of a sphere of dimension $(\operatorname{ind}\omega_0p-1)$; consequently, $\chi(p^{-1}(K^\circ)\cap S^{N-1})=1+(-1)^{\operatorname{ind}\omega_0p-1}$. At the same time,

$$\chi(K_n/K_{n-1})=\begin{cases}1, & n=\operatorname{ind}\omega_0p,\\ 0, & n\neq\operatorname{ind}\omega_0p.\end{cases}$$

So (1) is true when K is a half-line.

2) Let K be an arbitrary polyhedral cone and $\omega_0\in K\backslash 0$. Then (1) is true for the pencil $p^{U_{\omega_0}}$, where U_{ω_0} is an arbitrary but sufficiently small conical neighborhood of ω_0 in K. Indeed, by Lemma 3.4 and the corollary to Lemma 3.9, the left- and right-hand sides of (1) remain unchanged under a transversal homotopy, while Proposition 3.2 states that the pencil $p^{U_{\omega_0}}$ is transversally homotopic to a pencil defined on a half-line; we are thus in the situation of the first step of the proof.

Remark. We are assuming that $\dim\ker\omega p\leq r$, $\forall\omega\in K\backslash 0$. Close attention to the proof of Proposition 3.2 will show that in this case the dimensions of the neighborhood U_{ω_0} in Step 2 may be chosen equal for all ω_0, so that $\omega_0\in\Pi_r$.

In order to move on, we need a simple property of convex cones.

Proposition 1. Let K_1, K_2 be convex closed cones in $\mathbf{R}^k$. Then

$$K_1\cup K_2=K_1+K_2\Leftrightarrow K_1^\circ\cup K_2^\circ=K_1^\circ+K_2^\circ.$$

Since we have not been able to locate a proof of this proposition in the textbook literature, we will present one here. For any convex cones it is true that $K_1^\circ+K_2^\circ=(K_1\cap K_2)^\circ$. Suppose that $\omega\in(K_1\cap K_2)^\circ\backslash(K_1^\circ\cup K_2^\circ)$. Then $\langle\omega, x\rangle\leq 0$ $\forall x\in K_1\cap K_2$, but there exist $x_1\in K_1$ and $x_2\in K_2$ such that $\langle\omega, x_1\rangle>0$, $\langle\omega, x_2\rangle>0$. For any $\alpha\in[0, 1]$ we have $\langle\omega,$

$\alpha x_1 + (1 - \alpha)x_2\rangle > 0$ and, so, the whole of the segment conv$\{x_1, x_2\}$ lies outside the set $K_1 \cap K_2$. But the sets $K_i \cap$ conv$\{x_1, x_2\}$ are closed and nonempty. Consequently, conv$\{x_1, x_2\}$ contains points lying outside $K_1 \cup K_2$ and, so, the set $K_1 \cup K_2$ is not convex. ▸

3) Let K, L be convex polyhedral cones such that $K \cup L$ is convex and $K \cap L \neq 0$. If (1) is true for pencils p^K, p^L, $p^{K\cap L}$, then it is also true for the pencil $p^{K\cup L}$.

Indeed, $(K \cup L)^\circ = (K^\circ \cap L^\circ)$ and, by Proposition 1, $(K \cap L)^\circ = K^\circ \cup L^\circ$. Thus

$$\chi(p^{-1}((K\cup L)^\circ)\cap S^{N-1}) = \chi(p^{-1}(K^\circ)\cap p^{-1}(L^\circ)\cap S^{N-1}) =$$
$$= \chi(p^{-1}(K^\circ)\cap S^{N-1}) + \chi(p^{-1}(L^\circ)\cap S^{N-1}) - \chi(p^{-1}(K^\circ)\cup p^{-1}(L^\circ)\cap S^{N-1}) =$$
$$= \chi(p^{-1}(K^\circ)\cap S^{N-1}) + \chi(p^{-1}(L^\circ)\cap S^{N-1}) - \chi(p^{-1}((K\cap L)^\circ\cap S^{N-1}).)$$

It remains to use the assumption and the additivity of the Euler characteristic of quadratic forms.

4) Equality (1) is true for arbitrary convex polyhedral cones K.

We will use double induction, with respect to dim K and max(dim ker ωp). Cutting K repeatedly by hyperplanes transverse to it and using the result of Step 3, we can reduce everything to the situation of Step 2. We note that if K is a linear space, one step of this reduction will involve representing a straight line in $\mathbf{R}^{k*}$ as a union of two rays. The intersection of these rays is the zero cone 0. Since $0^\circ = \mathbf{R}^k$, it follows that $\chi(p^{-1} \times (0^\circ) \cap S^{N-1}) = \chi(S^{N-1})$, so there appears in formula (1) a term with coefficient ε.

The truth of (1) for an arbitrary convex closed cone now follows from the Approximation Lemma 3.5. ▸

Formula (1) already gives meaningful results in the case k = 2, K = 0. In that case all possible situations are exhausted by the quadratic maps $q(n_1, \ldots, n_{2k-1}; m)$ (see Theorem 2.1). Define

$$V(n_1, \ldots, n_{2k-1}; m) = \{x \in S^{N-1} \mid q(n_1, \ldots, n_{2k-1}; m)(x, x) = 0\}$$

– the complete intersection of two real quadratics of $\mathbf{R}^N$, cut by the sphere S^{N-1}, $N = \Sigma n_j + 2m$. Theorem 1, in combination with equality (3.3), implies the following device for evaluating the Euler characteristic of this manifold.

Let I denote the subset of $\{1, \ldots, 2k-1\}$ consisting of all j such that n_j is odd and I' the set $I \cup \{j + 2k - 1 \mid j \in I\}$; define $\theta(j) = \#(I' \cap \{j, j+1, \ldots, j+2k-1\}$. The manifold $V(n_1, \ldots, n_{2k-1}; m)$ is of dimension N – 3; if N is even this is an odd-dimensional manifold and its Euler characteristic is zero; but if N is odd then

$$\frac{1}{2}\chi(V(n_1, \ldots, n_{2k-1}; m)) = (-1)^m \sum_{j\in I} (-1)^{\theta(j)} + 1.$$

For fixed odd N, the Euler characteristic of maximum absolute value is obtained in the case $k = [(N + 1)/2]$, m = 0 and, then, $(1/2)\cdot\chi(V(1, \ldots, 1; 0)) = (-1)^{[(N+1)/2]}\cdot N + 1$.

Now let us consider the quadratic maps $p \in \mathscr{P}(N, 3)$. The equation $\det(\omega P) = 0$ defines an algebraic curve of degree N in $\mathbf{RP}^2 = (\mathbf{R}^{3*} \setminus 0)/(\omega \sim \alpha\omega, \alpha \neq 0)$. Let us assume that this is a nonsingular curve (this assumption holds for maps p in general position and guarantees transversality of p^*). If N is odd, then $p^{-1}(0) \cap S^{N-1}$ is an odd-dimensional manifold and its Euler characteristic is zero; if N is even equality (1) is easily rewritten as

$$\frac{1}{4}\chi(p^{-1}(0)\cap S^{N-1}) = \chi(\{\bar\omega \in \mathbf{RP}^2 \mid \det(\omega p) \leqslant 0\}.$$

It now follows from the classical Petrowski inequalities [7] that

$$|\chi(p^{-1}(0)\cap S^{N-1})| \leqslant \frac{3}{2}N(N-2) + 4.$$

Remark. Petrowski's inequalities for fourth-degree curves are best possible. It can be shown that the curves for which they become equalities are represented by equations of the form $\det(\omega p) = 0$ for some $p \in \mathscr{P}(3, 3)$. Consequently, the above inequality is also best possible.

$\alpha x_1 + (1 - \alpha)x_2\rangle > 0$ and, so, the whole of the segment conv $\{x_1, x_2\}$ lies outside the set $K_1 \cap K_2$. But the sets $K_i \cap$ conv $\{x_1, x_2\}$ are closed and nonempty. Consequently, conv $\{x_1, x_2\}$ contains points lying outside $K_1 \cup K_2$ and, so, the set $K_1 \cup K_2$ is not convex. ▸

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<u>Remark.</u> Petrowski's inequalities for fourth-degree curves are best possible. It can be shown that the curves for which they become equalities are represented by equations of the form $\det(\omega p) = 0$ for some $p \in \mathcal{P}(3, 3)$. Consequently, the above inequality is also best possible.

2. In this subsection we will introduce some expressions for $\chi(f)$ on the assumption that $f: M^k \to \mathscr{P}(\mathbf{R}^N)$ is defined on a smooth manifold M^k.

Let $F(m_0): \mathbf{R}^N \to \mathbf{R}^N$ be a linear symmetric operator corresponding to a quadratic form $f(m_0)$ and $\lambda_n(m_0)$ the n-th eigenvalue of the operator $F(m)$, $m \in M$, $n = 1, \ldots, N$ (counted in increasing order). If $\lambda_n(m_0)$ is a multiple eigenvalue, the function $m \mapsto \lambda_n(m)$ is smooth near m_0. In that case, we let $\lambda_n'(m_0)$ denote the differential of the function $m \mapsto \lambda_n(m)$ at m_0 and $\lambda_n''(m_0)$ its Hessian (recall that the Hessian of a scalar function is defined only when the differential vanishes). We also define $F_\lambda(m) = (F(m) - \lambda I)|\ker(F(m) - \lambda I)^\perp$ and let $f_\lambda(m)$ be the quadratic form on $\ker(F(m) - \lambda I)^\perp$ corresponding to the operator $F_\lambda(m)$.

Let x be an eigenvector belonging to a simple eigenvalue $\lambda_n(m)$ of $F(m)$, $|x| = 1$ and $\mu \in T_n M^k$. Then it is not hard to show that $\langle \lambda'(m), \mu \rangle = (\partial/\partial\mu)\cdot f(m)(x, x)$,

$$\lambda''(m)(\mu, \mu) = \frac{\partial^2}{\partial\mu^2} f(m)(x, x) - 2 f_{\lambda_n(m)}\left(F^{-1}_{\lambda_n(m)} \frac{\partial F(m)}{\partial\mu} x, F^{-1}_{\lambda_n(m)} \frac{\partial F(m)}{\partial\mu} x\right).$$

<u>LEMMA 1.</u> A typical nonsingular map $f: M^k \to \mathscr{P}(\mathbf{R}^N)$ of class C^∞ has the following properties: if $m_0 \in M^k$, $\lambda \in \mathbf{R}$, $x \in \mathbf{R}^N \setminus 0$ are such that $F(m_0)x = \lambda x$ and m_0 is a critical point of the function $m \mapsto f(m)(x, x)$, then i) $\dim\ker f(m_0) = 1$; ii) the quadratic form $\lambda''(m_0)$ is nonsingular on $T_{m_0}M^k$.

This assertion can be derived from the standard Transversality Theorem. We will not give a detailed proof, only pointing how to evaluate the parameters in such situations. For example, in order to prove part (i), we have to consider a system of equations $F(m)x = \lambda x$, $F(m)y = \lambda y$, $(\partial/\partial m)\cdot f(m)(x, y) = 0$, $(x, y) = 0$ in the variables $m \in M^k$, $\lambda \in \mathbf{R}$, $x, y \in S^{N-1}$. This system contains $2N + k$ independent equations [but not $2N + k + 1$, because the equality $(F(m)y, x) = \lambda(y, x)$ follows from $F(m)x = \lambda x$] and $2N + k - 1$ unknowns and, so, for a typical f the equations need not be solvable.

<u>Proposition 2.</u> If the map $f: M^k \to \mathscr{P}(\mathbf{R}^N)$ satisfies the assumptions of Lemma 1, then

$$\chi(f) = \sum_{n=1}^{N} (-1)^{n+k-1} \sum_{\{m \mid \lambda_n(m) < 0, \lambda_n'(m) = 0\}} (-1)^{\operatorname{Ind}\lambda_n''(m)}. \tag{2}$$

<u>Proof.</u> If $t > 0$ is sufficiently large, then $(f + tI)(M) \subset \mathscr{P}_0$; in particular, $\chi(f + tI) = 0$. If the map $f + tI$ is nonsingular for $t_1 \le t \le t_2$, then $\chi(f + t_1 I) = \chi(f + t_2 I)$. Consequently, in order to evaluate $\chi(f)$ we must locate the points $t_i > 0$ for which $f + f_i I$ is singular and determine the change in $\chi(f + t_i I)$ when the parameter t goes through the value t_i.

The map $f + t_0 I$ is singular if and only if, for some $m_0 \in M$, $x_0 \in S^{N-1}$, it is true that $F(m_0)x_0 = -t_0 x_0$, $(\partial/\partial m)\cdot f(m_0)(x_0, x_0) = 0$. It follows from the assumptions of Proposition 2 that such triples (t_0, m_0, x_0) are isolated points of $(0, +\infty) \times M^k \times S^{N-1}$ and, so, there are only a finite number of them (t_0 close to zero and t_0 too large are both inadmissible).

Thus, if $f + t_0 I$ is nonsingular then $-t_0 = \lambda_n(m)$ for some n, where m_0 is a critical point of the function $\lambda_n(m)$.

Let $m_{01}, \ldots, m_{0\ell}$ be all the critical points of $\lambda_n(m)$ corresponding to the critical value $-t_0$. It follows from the assumptions of this proposition that all the critical points are nonsingular. Note that $(f + tI)^{-1}(\mathscr{P}_{n-1}) = \lambda_n^{-1}([-t, +\infty))$. It follows from Morse theory that

$$\chi((f + (t_0 + \varepsilon) I)^{-1}(\mathscr{P}_{k-1})) - \chi((f + (t_0 - \varepsilon) I)^{-1}) = (-1)^k \sum_{i=1}^{l} (-1)^{\operatorname{Ind}\lambda_n''(m_{0i})}$$

for all sufficiently small $\varepsilon > 0$.

Summing over all $t_0 > 0$, we get

$$\begin{array}{c}\chi(M^k) - \chi(f^{-1}(\mathscr{P}_{n-1})) \\ \| \\ \chi(M^k / f^{-1}(\mathscr{P}_{n-1}))\end{array} = (-1)^k \sum_{\{m \mid \lambda_n(m) < 0, \lambda_n'(m) = 0\}} (-1)^{\operatorname{ind}\lambda_n''(m)}.$$

Finally, summing over n with the appropriate signs, we obtain (2).

Remark. An equality analogous to (2) is true in the case that f is defined on an arbitrary manifold with angles. The proof involves a modification of Morse theory for manifolds with angles.

We now want to establish an expression for $\chi(f)$ as the linking coefficient of two cycles. Since the choice of sign in the coefficients of the linking coefficient is a matter of convention, while the sign of the Euler characteristic is rigidly defined, we must first agree on some sign convention in order to avoid ambiguities. Let U^n be a smooth oriented manifold, $\varphi_1, \ldots, \varphi_k \in C^\infty(U^n)$ and $V^{n-k} = \{v \in U^n | \varphi_1(v) = \varphi_2(v) = \ldots = \varphi_k(v) = 0\}$, such that $d_v\varphi_1, \ldots, d_v\varphi_k$ are linearly independent at every point $v \in V^{n-k}$. The functions $\varphi_1, \ldots, \varphi_k$ determine the orientation of the smooth manifold V^{n-k} as follows: an exterior $(n-k)$-form ω on T_vV^{n-k} defines the positive orientation on T_vV^{n-k} if and only if the n-form $d_v\varphi_1 \wedge \ldots \wedge d_v\varphi_k \wedge \omega$ defines the positive orientation on T_vU^n (the order of the factors is significant!)

Now let M be a smooth oriented manifold of arbitrary dimension, $\varphi \in C^\infty(M)$, $W = \varphi^{-1}(0)$, such that $d_w \varphi \neq 0$ for $w \in W$. Let $G_k^+(M)$ denote the manifold of all k-dimensional oriented planes tangent to M (this is the locally trivial bundle with base space M and fiber $G^+(k, \dim M - k)$) and $G_k^+(W)$ the manifold of all k-dimensional oriented planes tangent to W. It is readily seen that $G_k^+(W)$ is a submanifold of codimension $k + 1$ in $G_k^+(M)$. The manifolds $G_k^+(M)$ and $G_k^+(W)$ are clearly orientable. the choice of orientation on $G_k^+(M)$ is immaterial for us, but on the other hand we must be able to define the orientation on $G_k^+(W)$ uniquely, given an orientation on $G_k^+(M)$; to do this, we define $G_k^+(W)$ in $G_k^+(M)$ by a set of equations. Let $\mu_1, \ldots, \mu_k$ be a positively oriented basis of an oriented plane $H \subset T_mM$, $H \in G_k^+(M)$. The plane H will be in $G_k^+(W)$ if and only if $\varphi(m) = \frac{\partial\varphi}{\partial\mu_1} = \ldots = \frac{\partial\varphi}{\partial\mu_k} = 0$. The independent functions $\varphi(m), \frac{\partial\varphi}{\partial\mu_1}, \ldots, \frac{\partial\varphi}{\partial\mu_k}$ (the order is significant!) define an orientation in $G_k^+(W)$ in accordance with the above prescription.*

Consider the smooth hypersurface $\Pi_1 \setminus \Pi_2$ in the manifold $\mathscr{P}(R^N) \setminus \Pi_2$,

$$\Pi_1 \setminus \Pi_2 = \{p \in \mathscr{P}(R^N) \setminus \Pi_2 \,|\, \lambda_{\operatorname{ind} p+1}(p) = 0\}.$$

Using the notation of Sec. 1.2, we see that $\nu_p = d_p\lambda_{\operatorname{ind}p+1}$ and the orientation $G_k^+(\nu_p)$ of $G_k^+(\Pi_1 \setminus \Pi_2)$ is now automatically defined by the orientation of the manifold

$$G_k^+(\mathscr{P}(R^N) \setminus \Pi_2) = (\mathscr{P}(R^N) \setminus \Pi_2) \times G^+\left(k, \frac{N(N+1)}{2} - k\right).$$

It was observed in Sec. 1.2 that the orientation $(-1)^{\operatorname{ind}p}G_k^+(\nu_p)$ extends to a well-defined orientation of the pseudomanifold $G_k^+(\Pi_1)$, $G_k^+(\Pi_1) \subset \mathscr{P}(R^N) \times \mathscr{L}^+(k, N)$ (in the notation of Sec. 1.2).

The one-point compactification $G_k^+(\Pi_1) \cup \{\infty\}$ of the pseudomanifold $G_k^+(\Pi_1)$ with orientation $(-1)^{\operatorname{ind}p}\nu_p$ defines an integral singular cycle of codimension $k + 1$ in $\mathscr{P}(R^N) \times \mathscr{L}^+(k, N) \cup \{\infty\}$. With some abuse of notation, we will denote this cycle by the same symbol $G_k^+ \times (\Pi_1)$. It is readily seen that the cycle $G_k^+(\Pi_1)$ is homologous to zero in $\mathscr{P}(R^N) \times \mathscr{L}^+(k, N) \cup \{\infty\}$. Indeed, as the image of the pseudomanifold $G_k^+(\Pi_1)$ under the projection $\mathscr{P}(R^N) \times \mathscr{L}^+ \times (k, N) \to \mathscr{P}(R^N)$ is not $\mathscr{P}(R^N)$, our cycle can be mapped by a homotopy onto the point at infinity ∞.

We now recall the definition of the linking coefficient, let γ_1, γ_2 be two cycles, homologous to zero, with disjoint supports, in the one-point compactification $U^n \cup \{\infty\}$ of some smooth oriented manifold U^n, where $\dim\gamma_1 + \dim\gamma_2 = n - 1$. Then the linking coefficient $\ell(\gamma_1, \gamma_2)$ is defined to be the intersection number of the chains Γ_1 and γ_2, where $\partial\Gamma_1 = \gamma_1$; that γ_2 is homologous to zero ensures that the intersection number is independent of the choice of Γ_1; and moreover $\ell(\gamma_2, \gamma_1) = (-1)^{\dim\gamma_1 \dim\gamma_2 + n}\ell(\gamma_1, \gamma_2)$.

*Strictly speaking, the functions $\partial\varphi/\partial\mu_1$ are defined not on planes but on bases; but by transforming, e.g., to orthogonal bases in some Riemannian metric one readily verifies that the definitions are all legitimate.

THEOREM 2. Let $f:M^k \to \mathscr{P}(R^N)$ be a nonsingular map of a smooth oriented manifold M^k into $\mathscr{P}(R^N)$. In addition, let f be an immersion (i.e., $\operatorname{rank} f_m' = k \ \forall m \in M^k$) and let Tf be the singular k-dimensional cycle in $\mathscr{P}(R^N) \times \mathscr{L}^+(k, N)$ defined by the map $m \mapsto (f(m), \operatorname{im} \times f_m')$, where the orientation of the plane in f_m' is defined by that of the space $T_m M^k$. Then the cycle Tf is homologous to the cycle $T(f + tI)$ and, if t is sufficiently large,

$$\chi(f) = (-1)^{k} I(T(f+tI) - Tf, G_k^+(\Pi_1)) \tag{3}$$

Proof. That the cycles Tf and $T(f + tI)$ are homologous (where t is the homotopy parameter) is obvious. In addition, we may assume that f satisfies the assumptions of Lemma 1.

We must calculate the intersection number of the oriented singular chain defined by the map

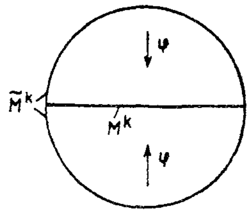
$$(\tau, m) \mapsto (f(m) + \tau I, \operatorname{im} f_m')$$

with the cycle $G_k^+(\Pi_1)$. The point $f(m_0) + \tau_0 I$, $\operatorname{im} f'm_0$ is an intersection point of these two chains if, for some n, we have $\lambda_n(m_0) + \tau_0 = 0$, $\lambda_n'(m_0) = 0$. It follows from our assumptions that the chains intersect transversely; the definition of the orientation of the cycle $G_k^+(\Pi_1)$ implies that the intersection number at the point $(f(m_0) + \tau I, \operatorname{im} f_m')$ is $(-1)^{n-1}$ times the sign of the Jacobian of the map $(\tau, m) \mapsto (\lambda_n(m) + \tau, \lambda_n'(m))$ at the point (τ_0, m_0). In view of the condition $\lambda_n'(m_0) = 0$, we see that this intersection number is $(-1)^{n-1} \operatorname{sgn} \det \lambda_n''(m_0) = (-1)^{n-1+\operatorname{ind}\lambda_n''(m_0)}$. The assertion of the theorem now follows from (2).

Formula (3) is meaningful only if f is an immersion, but this condition is not overly restrictive. It follows from the classical theorem of Whitney that if $2k \leq [(N(N + 1)/2]$ a typical map $f:M^k \to \mathscr{P}(R^N)$ of class C^∞ is an immersion.

Apart from its geometrical transparency, formula (3) is convenient in that its right-hand side is, "by definition," an invariant of nonsingular homotopy: an immersion $f:M^k \to \mathscr{P}(R^N)$ is nonsingular if and only if the chains Tf and $G_k^+(\Pi_1)$ have disjoint supports. A formula similar to (3) is also valid when M^k is an oriented manifold with boundary; we offer a brief description of the construction, without proofs.

Let M^k be a manifold with boundary ∂M^k. Gluing together two copies of M^k along the boundary, we obtain a smooth orientable manifold without boundary $\tilde{M}^k$. The smooth structure in $\tilde{M}^k$ is defined by means of a tubular neighborhood of the boundary in M^k (see [5]). This tubular neighborhood makes it possible to define a smooth "gluing" map $\varphi:\tilde{M}^k \to M^k$, which is singular on $\varphi^{-1}(\partial M^k)$.



Let $f:M^k \to \mathscr{P}(R^N)$ be a nonsingular map. Then the map $f \circ \varphi:\tilde{M}^k \to \mathscr{P}(R^N)$ is also nonsingular and it is true that $\chi(f) = (1/2)\cdot(\chi(f \circ \varphi) + \chi(f|\partial M^k))$. If $2k \leq [(N(N + 1)/2]$, a small deformation will transform the map $f \circ \varphi$ an immersion and we are back in the situation in which the identity (3) is applicable.

5. Complexes of Hermitian Forms

1. Let C^N be N-dimensional complex space, $C^N \approx R^{2N}$. An Hermitian form on C^N is defined as an arbitrary symmetric bilinear form $p \in \mathscr{P}(R^{2N})$, which is preserved under multiplication of its arguments by the imaginary unit: a form $p \in \mathscr{P}(R^{2N})$ is said to be Hermitian if $p(iz, iw) = p(z, w)$, $\forall z, w \in C^N$. The space of all Hermitian forms on C^N is denoted by $\mathscr{P}(C^N)$ and the corresponding quadratic forms $z \to p(z, z)$ will also be called hermitian forms (in the more widespread terminology Hermitian forms are symmetric sesquilinear complex-valued forms; the forms we are calling Hermitian here are the real parts of sesquilinear forms).

A form $p \in \mathcal{P}(\mathbf{R}^{2N})$ is Hermitian if and only if the corresponding symmetric operator $P: \mathbf{R}^{2N} \to \mathbf{R}^{2N}$ commutes with multiplication by i; in other words, if and only if P is an Hermitian operator in $\mathbf{C}^N$, $P^T = \bar{P}$. Consequently, $\dim \mathcal{P}(\mathbf{C}^N) = N^2$; if $p \in \mathcal{P}(\mathbf{C}^N)$, then ind p, dim ker p are even numbers, and Hermitian forms with the same index and dimension of kernel can be transformed into one another by coordinate transformations in $GL(\mathbf{C}^N)$.

Define $\mathcal{P}_k(\mathbf{C}^N) = \mathcal{P}_k(\mathbf{R}^{2N}) \cap \mathcal{P}(\mathbf{C}^N)$, $\Pi_k(\mathbf{C}^N) = \Pi_k(\mathbf{R}^{2N}) \cap \mathcal{P}(\mathbf{C}^N)$; clearly,

$$\mathcal{P}_{2k+1}(\mathbf{C}^N) = \mathcal{P}_{2k}(\mathbf{C}^N),\ \Pi_{2k-1}(\mathbf{C}^N) = \Pi_{2k}(\mathbf{C}^N).$$

Let $p_0 \in \mathcal{P}(\mathbf{C}^N)$; $V = \ker p_0$. Then V is a complex subspace of $\mathbf{C}^N$ and, as is readily seen, the map Φ constructed in Lemma 1.2 transforms Hermitian forms into Hermitian forms. Consequently, Φ describes the local structure of the hypersurface $\Pi_1(\mathbf{C}^N)$ in $\mathcal{P}(\mathbf{C}^N)$ near the point $p_0 \in \mathcal{P}(\mathbf{C}^N)$. As a corollary, we deduce that $\Pi_{2k}(\mathbf{C}^N) \setminus \Pi_{2(k+1)}(\mathbf{C}^N)$ is an analytic submanifold of codimension k^2 in $\mathcal{P}(\mathbf{C}^N)$ for any k = 1, 2, ..., N.

Proceeding now to a description of the space $\mathcal{P}(\mathbf{C}^N)^*$ dual to $\mathcal{P}(\mathbf{C}^N)$, we emphasize that the functors "*" and "$\odot$" are being consider only over the real field: the complex structure in the space $\mathbf{C}^{N*}$ is defined by the operator adjoint to multiplication by i in $\mathbf{C}^N$. It follows almost immediately from the definition of Hermitian forms that

$$\mathcal{P}(\mathbf{C}^N)^* = \left\{ \left(\sum_\alpha z_\alpha \odot u_\alpha \right) \in \mathbf{C}^N \odot \mathbf{C}^N \,\middle|\, \sum_\alpha iz_\alpha \odot iu_\alpha = \sum_\alpha z_\alpha \odot u_\alpha \right\} \subset \mathcal{P}(\mathbf{R}^{2N})^*.$$

If the vectors $e_1, \ldots, e_N$ form a basis in $\mathbf{C}^N$, then the vectors

$$e_\alpha \odot e_\beta + ie_\alpha \odot ie_\beta,\ 1 \leq \alpha \leq \beta \leq N, \text{ и } e_\alpha \odot ie_\beta - ie_\alpha \odot e_\beta,\ 1 \leq \alpha < \beta \leq N,$$

form a basis in $\mathcal{P}(\mathbf{C}^N)^*$. If the spaces $\mathcal{P}(\mathbf{R}^{2N})^*$ and $\mathcal{P}(\mathbf{R}^{2N*})$ are identified (see Sec. 1.2), the subspace $\mathcal{P}(\mathbf{C}^N)^*$ is identified with $\mathcal{P}(\mathbf{C}^{N*})$ – the space of Hermitian forms on $\mathbf{C}^{N*}$. Therefore, if $\eta \in \mathcal{P}(\mathbf{C}^N)^* = \mathcal{P}(\mathbf{C}^{N*})$, then $L(\eta) = \ker \eta^{\perp}$ is a complex subspace of $\mathbf{C}^N$; in particular, rank η is an even number.

Let $p \in \Pi_{2k}(\mathbf{C}^N) \setminus \Pi_{2(k+1)}(\mathbf{C}^N)$. It is easy to see that an element $\eta \in \mathcal{P}(\mathbf{C}^N)^*$ is normal to $\Pi_k(\mathbf{C}^N)$ at p if and only if $L(\eta) \subset \ker p$. When this is the case, the identification $\mathcal{P}(\mathbf{C}^N)^* = \mathcal{P}(\mathbf{C}^{N*})$ induces an isomorphism of the space $T_p\Pi_{2k}(\mathbf{C}^N)^{\perp} \cap \mathcal{P}(\mathbf{C}^N)$ of Hermitian normals to $\Pi_{2k}(\mathbf{C}^N)$ at p and the space $\mathcal{P}(\ker p^*)$ of Hermitian forms on $\ker p^*$.

If k = 1, $p \in \Pi_2(\mathbf{C}^N) \setminus \Pi_4(\mathbf{C}^N)$, then there is a unique (up to a real factor) Hermitian normal $z \odot z + iz \odot iz$, $z \in \ker p$ to the hypersurface $\Pi_1(\mathbf{C}^N)$ at p. We see that the only possible Hermitian normals to points of the hypersurface $\Pi_2(\mathbf{C}^N) \setminus \Pi_4(\mathbf{C}^N)$ are the elements of rank 2 in $\mathcal{P}(\mathbf{C}^N)^* = \mathcal{P}(\mathbf{C}^{N*})$. Let us assume now that p is a singular point of $\Pi_2(\mathbf{C}^N)$, dim ker p > 2. Define a Hermitian normal to $\Pi_2(\mathbf{C}^N)$ at p to be any element of rank 2 in $\mathcal{P}(\ker p^*) \subset \mathcal{P}(\mathbf{C}^{N*})$. Call a Hermitian normal positive if the corresponding Hermitian form on $\ker p^*$ is positive definite. Let $\mathcal{N}_p(\mathbf{C})$ denote the set of all hermitian normals to $\Pi_2(\mathbf{C})$ at p and $\mathcal{N}_p^+(\mathbf{C})$ the set of all positive Hermitian normals. As in the real situation (cf. Sec. 1.2), we have equalities

$$\mathcal{N}_p(\mathbf{C}) = \bigcap_{O_p \subset \Pi_1(\mathbf{C}^N)} \bigcup_{q \in O_p} \mathcal{N}_q(\mathbf{C});\ \mathcal{N}_p^+(\mathbf{C}) = \bigcap_{O_p \subset \Pi_1(\mathbf{C}^N)} \bigcup_{q \in O_p} \mathcal{N}_q^+(\mathbf{C})$$

(the intersection extends over all neighborhoods of p in $\Pi_1(\mathbf{C}^N)$).

2. Let $\mathcal{P}_\mathbf{C}(N, k) = \mathcal{P}(\mathbf{C}^N)^k$ denote the space of Hermitian maps from $\mathbf{C}^N$ to $\mathbf{R}^k$; clearly, $\mathcal{P}_\mathbf{C}(N, k) \subset \mathcal{P}(2N, k)$.

Let K be a convex closed cone in $\mathbf{R}^{k*}$. Define a pencil of Hermitian forms on K to be an arbitrary linear map from K to $\mathcal{P}(\mathbf{C}^N)$, and denote the space of all such pencils by $\mathcal{P}_\mathbf{C}(N; K)$.

Definition. A pencil $p^K \in \mathcal{P}_\mathbf{C}(N; K)$ is said to be singular at a point $\omega_0 \in K \setminus 0$ if $p^K \times (K)^\circ \cap \mathcal{N}^+_{\omega_0 p}(\mathbf{C}) \neq 0$. The pencil p^K is said to be nonsingular if it is not singular at any point.

It is not hard to show that a pencil $p^K \in \mathcal{P}_\mathbf{C}(N; K)$ is nonsingular if and only if the corresponding map $p \in \mathcal{P}_\mathbf{C}(N, k)$ is transverse to the cone K°. In general, all the results and definitions of Sec. 3, from Lemma 3.1 through the corollary to Lemma 3.9, can be rephrased in an obvious way for the Hermitian case. The Hermitian analog of Proposition 3.3 is valid not only for zero- and one-dimensional but even for two-dimensional piecewise smooth complexes, since $\Pi_2(\mathbf{C}^N)$ has codimension 4 in $\mathcal{P}(\mathbf{C}^N)$.

To each quadratic form $p \in \mathscr{P}(R^N)$ we can associate a canonical Hermitian form $p_C \in \mathscr{P}(C^N)$, where $C^N = C \otimes R^N$, putting $p_C(i \bullet x, y) = 0$, $p_C(x, y) = p(x, y)$, $\forall x, y \in R^N$ (in this case $p_C i \bullet x = i \bullet p_C x$, $\forall x \in R^N$). Thus, to any complex of quadratic forms $f:\overline{\mathscr{H}} \to \mathscr{P}(R^N)$ there corresponds a complex $f_C:\overline{\mathscr{H}} \to \mathscr{P}(C^N)$ of Hermitian forms.

It is readily shown that the complex f_C is transverse if and only if f is transverse. If f_C is nonsingular, then f is also nonsingular, but the converse is false! This is quite evident from the "model" example considered in Sec. 3.2. Let

$$\Omega = \{p \in \mathscr{P}(R^2) \mid \mathrm{tr}\, p = 0, \ \mathrm{tr}\, p^2 = 1\} \text{ is a disk in } \mathscr{P}(R^2)$$

and let $f:\Omega \to \mathscr{P}(R^2)$ be the identity map, $f(p) = p$, $p \in \Omega$. A trivial check verifies that f is nonsingular, but f_C is singular at the point $0 \in \Omega$.

Thus, the nonsingular homotopy invariants of a complex of Hermitian forms f_C need not necessarily be nonsingular homotopy invariants of f.

Definition. Let $g:\overline{\mathscr{H}} \to \mathscr{P}(C^N)$ be a nonsingular complex of Hermitian forms. Put

$$\chi_C(g) = \sum_{n=0}^{N} \chi(\overline{\mathscr{H}}) g^{-1}(\mathscr{P}_{2k}(C^N)).$$

It can be shown that $\chi_C(g)$ remains unchanged under a nonsingular homotopy of complexes of Hermitian forms. In the case of pencils this follows from the Hermitian analog of Theorem 4.1, which we now state.

Let S^{2N-1} be the unit sphere in C^N and $CP^{N-1} = S^{2N-1}/(z \sim e^{i\theta}z, \theta \in R)$ complex projective space. If $p \in \mathscr{P}_C(N, k)$, then $p(e^{i\theta}z, e^{i\theta}z) = p(z, z)$. Let $\bar{p}:CP^{N-1} \to R$ denote the map induced by the quadratic map $z \to p(z, z)$, $z \in S^{2n-1}$.

THEOREM 1. Let $p \in \mathscr{P}_C(N, k)$ be a Hermitian quadratic map from C^N to R^k, K be a convex closed cone in R^{k*}, such that the pencil $p^K:K \to \mathscr{P}(C^N)$ is transverse. Define $K_n = (p^K)^{-1}(\mathscr{P}^\circ(C\ )) \cap S^{k-1}$. Then

$$\chi(\bar{p}^{-1}(K^\circ)) = \sum_{n=0}^{N} \chi(K_{2N}/K_{2N}) + \varepsilon\chi(CR^{N-1}) = \chi(p^K \mid K_{2N}) + \varepsilon N \tag{I}$$

where

$$\varepsilon = \begin{cases} (-1)^{\dim_R K}, & K = -K, \\ 0, & K \neq -K. \end{cases}$$

Proof. Suppose that K is a half-line, $K = \{\alpha\omega_0 \mid \alpha \geq 0\}$. Then $p^{-1}(K^\circ) = \{z \in C^N \mid \omega_0 p \times (z, z) \leq 0\}$, the set $\bar{p}^{-1}(K^\circ)$ has the homotopy type of the space $CP^{\mathrm{ind}\,\omega_0 p/2-1}$, and, consequently, $\chi(\bar{p}^{-1}(K^\circ)) = (1/2)\cdot \mathrm{ind}\,\omega_0 p$, so that in this case (I) is true. The rest of the proof is exactly the same as in the proof of Theorem 4.1.

In Sec. 4.2 we considered complexes of quadratic forms defined on a smooth manifold and derived a representation of the Euler characteristic of any such complex as the linking coefficient of two cycles (Theorem 4.2). Naturally, there is an analogous representation for complexes of Hermitian forms. We leave it to the reader to introduce the necessary modifications in the arguments of Sec. 4.2.

Remark. Consider N-dimensional space over the quaternion division ring $H^N \approx C^{2N} \approx R^{4N}$. Put

$$\mathscr{P}(H^N) = \left\{p \in \mathscr{P}(R^{4N}) \mid p(ix, iy) = p(jx, jy) = p(x, y) \ \ \forall x, y \in H^N \approx R^{4N}\right\} \subset \mathscr{P}(C^{2N}).$$

All the concepts and results for pencils and complexes of Hermitian forms, including Theorem 1, have obvious analogs for pencils and complexes with values in $\mathscr{P}(H^N)$.

6. Critical Sets of Smooth Vector-Valued Functions

In this section we will define Morse functions in the vector-valued case — an immediate generalization of scalar Morse functions (i.e., functions with nondegenerate critical points). Some rather simple arguments will demonstrate the remarkable duality between lev sets of a vector-valued function and its critical points. To each critical point there corresponds a quadratic form — the Hessian — and we will obtain an important example of a nonlinear complex of quadratic forms.

1. Let M be a smooth orientable compact n-dimensional manifold and $f:M \to R^k$ be a smoo map, i.e., $f \in C_\infty{}^k(M)$. Consider the set

$$C_f=\{(\psi, x)\in R^{k*}\times M \mid d_x(\psi f)=0,\ |\psi|=1\}\subset S^{k-1}\times M$$

and call C_f the critical set of the vector-valued function f.

<u>Definition.</u> Let $F \in C_\infty{}^k(M)$. We call f a Morse (vector-valued) function if the map $(\psi, x) \to d_x(\psi f)$ from $S^{k-1} \times M$ to T^*M is transverse to the zero section in T^*M.

<u>Proposition 1.</u> The Morse functions constitute an open dense subset of $C_\infty{}^k(M)$. This is a direct corollary of Thom's Transversality Theorem.

If f is a Morse function, the critical set C_f is obviously a smooth orientable $(k - 1)$ dimensional manifold.

A scalar function $a \in C_\infty(M)$ is usually called a Morse function if the Hessian of a at each critical point is a nonsingular quadratic form. It is not difficult to show that this is equivalent to our definition. But this is not all:

<u>LEMMA 1.</u> A map $f \in C_\infty{}^k(M)$ is a Morse function if and only if $\forall (\psi, x) \in C_f$

$$\ker \operatorname{ges}_x(\psi f)\cap \ker D_x f=0.$$

The <u>proof</u> amounts to deciphering the transversality condition in the definition of a Morse function.

For an arbitrary Morse function $f \in C_\infty{}^k(M)$ we let $f_C:C_f \to R^k$ denote the map $(\psi, x) \to f(x)$ and $\psi_C:C_f \to S^{k-1}$ the map $(\psi, x) \to \psi^T$, $(\psi, x) \in C_f$.

From this point on, we will always assume that $n \geq k$. Let $x \in M$ and denote $f'(x) = D_x f:T_xM \to R^k$. Then (ψ, x) is in C_f if and only if the rank of the linear map $f'(x)$ is less than k and $\psi \in (\operatorname{im} f'(x))^\perp$.

Thus, at points of the critical set the rank of the map $f'(x)$ is at most $k - 1$. It turns out that in the typical situation the subset of C_f consisting of the points (ψ, x) such that $\operatorname{rank} f'(x) \leq k - 2$ is of codimension $(n - k) + 2 \geq 2$.

<u>Proposition 2.</u> For any f in some open dense subset of $C_\infty{}^k(M)$, the set of all points $(\psi, x) \in C_f$ such that $\operatorname{rank} f'(x) \leq k - 2$ is the union of finitely many submanifolds of codimension at least $n - k + 2$ in C_f.

<u>Proof.</u> The map $x \to f'(x)$ is a section of the vector bundle $T^*M \otimes R^k$ over M. The fiber of this bundle consists of $n \times k$ matrices.

As we know, the matrices of rank $r \leq k$ form a smooth submanifold of codimension $(n - r)(k - r)$ in the space of matrices. If the matrix $f'(x)$ for some given $x \in M$ is of rank r, then the set $\{\psi \in S^{k-1} | (\psi, x) \in C_f\}$ is a $(k - r - 1)$-dimensional sphere. It remains to apply the Transversality Theorem to the family of maps $x \to f'(x)$, $f \in C_\infty{}^k(M)$, and to calculate the number of parameters.

<u>Remark.</u> Throughout the sequel we will repeatedly encounter properties of smooth maps valid for arbitrary f in some open dense subset of $C_\infty{}^k(M)$. In such cases we will say that a typical map $f \in C_\infty{}^k(M)$ has the property.

Let $x \in M$ and $\operatorname{rank} f'(x) < k$. Then $\operatorname{ges}_x f:\ker f'(x) \times \ker f'(x) \to \operatorname{coker} f'(x)$ is defined and is a symmetric bilinear map; in that case $\forall \psi \in (\operatorname{im} f'(x))^\perp$ we have $\psi \operatorname{ges}_x f = \operatorname{ges}_x \psi \times f|_{\ker f'(x)\times\ker f'(x)}$. The quadratic map $\xi \to \operatorname{ges}_x f(\xi, \xi)$, $\xi \in \ker f'(x)$ will be denoted by $f''(x)$.

<u>LEMMA 2.</u> Let $f \in C_\infty{}^k(M)$ be a Morse function and $(\psi, x) \in C_f$. The map $f_C:C_f \to R^k$ is an immersion at the point (ψ, x) if and only if $\operatorname{rank} f'(x) = k - 1$ and $\operatorname{ges}_x f$ is a nonsing-

ular bilinear form. The map $\varphi_c:(\psi, x) \mapsto \psi^T$ from C_f to S^{k-1} is an immersion (and a submersion) at (ψ, x) if and only if $ges_x\psi f$ is a nonsingular bilinear form.

The proof is by direct calculation.

The critical set C_f of a (vector-valued) Morse function is an orientable manifold. Generally speaking it is a disconnected manifold and can be oriented in more than one way.

There is one special orientation, however, to which we will refer as "canonical." It is constructed as follows.

Let $\Phi:(\psi, x) \mapsto \psi f'(x)$ be a smooth map from $S^{k-1} \times M_\pi$ to T^*M. The C_f is the complete preimage of the zero section of the cotangent bundle $T^*M \to M$ under the map Φ. Fix a point $(\psi, x) \in C_f$ and let $u_1, \ldots, u_{k-1}$ be a basis of the space $T_{(\psi,x)}C_f$; complete it by vectors $v_1, \ldots, v_n$ to a basis of the space $T_{(\psi,x)}S^{k-1} \times M \supset T_{(\psi,x)}C_f$, and let $w_1, \ldots, w_{k-1}$ be a positively oriented basis of the space $T_\psi S^{k-1} \subset T_{(\psi,x)}S^{k-1} \times M = T_\psi S^{k-1} \oplus T_xM$.

We will say that the basis $(u_1, \ldots, u_{k-1})$ defines the positive orientation on $T_{(\psi,x)} \times C_f$, if the bases $(u_1, \ldots, u_{k-1}, v_1, \ldots, v_n)$ and $(w_1, \ldots, w_{k-1}, (id - \pi_*\Phi_*)v_1, \ldots, (id-\pi_*\Phi_*)v_r)$ define the same orientation on $T_{(\psi,x)}S^{k-1} \times M = T_\psi S^{k-1} \oplus T_xM$.

It is easy to see that this construction yields a well-defined canonical orientation on C_f. The canonically oriented critical set C_f will henceforth be called the critical manifold of the Morse function f.

If (ψ, x) is an injectivity point of the map f_c, i.e., $\operatorname{rank}(D_{(\psi,x)}f_c) = k - 1$, there is a more explicit description of the canonical orientation on $T_{(\psi,x)}f_c$. Let ν_ψ be a $(k - 1)$-form on $\mathbb{R}^k$, such that the k-form $\psi \wedge \nu_\psi$ defines the positive orientation on $\mathbb{R}^k$. Then the form $\nu_\psi|\operatorname{im} f'(x)$ defines a certain orientation on the subspace $\operatorname{im} f'(x) = \{\psi\}^\perp \subset \mathbb{R}^k$, and the form $f_c{}^*\nu_\psi$ defines an orientation on $T_{(\psi,x)}C_f$.

LEMMA 3. Let $(\psi,x) \in C_f$ be an injectivity point of f_c. Then the form $(-1)^{\operatorname{ind}\psi f''(x)+k-1}f_c{}^*\nu_\psi$ defines the canonical orientation on $T_{(\psi,x)}C_f$.

The proof is again by a direct calculation, but it is more involved than our previous calculation, and we will present it here. Fixing bases in the spaces T_xM and $T_\psi S^{k-1} = \{\psi\}^\perp \subset \mathbb{R}^k$, we represent the map $f'(x):T_xM \to \{\psi\}^\perp$ by a $(k - 1) \times n$ matrix and the bilinear form $ges_x\psi f$ by a symmetric $n \times n$ matrix. Choose bases $\{u_1, \ldots, u_{k-1}\} \subset \{\psi\}^\perp$ and $\{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{n-k+1}\} \subset T_xM$ so that

1) $f'(x)u_i = w_i$, $i = 1, \ldots, k - 1$;

2) $ges_x\psi f(u_i, \ker f'(x)) = 0$, $i = 1, \ldots, k - 1$;

3) $v_j \in \ker f'(x)$, $j = 1, \ldots, n - k + 1$;

4) $\langle \nu_\psi, w_1 \wedge \ldots \wedge w_{k-1}\rangle > 0$.

In this basis our matrices have the following form: $f'(x) = (E, 0)$, where E is the $(k - 1) \times (k - 1)$ identity matrix:

$$ges_x \psi f = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_0 \end{pmatrix},$$

where $Q_0 = ges_x\psi f|_{\ker f'(x)} = \psi ges_x f$ is a symmetric $(n - k + 1) \times (n - k + 1)$ matrix. In this situation the vectors $(-Q_1w_i) \oplus u_i \in \{\psi^\perp\} \oplus T_xM = T_{(\psi,x)}(S^{k-1} \times M)$ form a basis of the space $T_{(\psi,x)}C_f$, $i = 1, \ldots, k - 1$. Since $f'(x)u_i = w_i$, the basis $(-Q_1w_1) \oplus u_1, \ldots, (-Qu_{k-1}) \oplus u_{k-1}$ defines the same orientation on $T_{(\psi,x)}C_f$ as $f^*\nu_\psi$. In order to ascertain the sign of this orientation relative to the canonical orientation, we augment the $(n + k - 1) \times (k - 1)$ matrix

$$\begin{pmatrix} -Q_1 \\ E \\ 0 \end{pmatrix},$$

which is composed of the elements of our basis of $T_{(\psi,x)}C_f$, by adding on the $(n + k - 1) \times n$ matrix

$$\begin{pmatrix} f'(x) \\ \mathrm{ges}_x \psi f \end{pmatrix} = \begin{pmatrix} E, & 0 \\ Q_1, & 0 \\ 0, & Q_0 \end{pmatrix}$$

and then find the sign of the determinant. This gives

$$\det Q_0 (-1)^{k-1} \det \begin{pmatrix} E, & -Q_1 \\ Q_1, & E \end{pmatrix} = (-1)^{k-1} \det Q_0 \det (Q_1^2 + E).$$

Since $\det(Q_1^2 + E) > 0$, the orientation on $T_{(\psi,x)}C_f$ defined by the basis $(-Q_1 w_1) \bullet u_i$, $i = 1, \ldots, k - 1$, has the sign $(-1)^{k-1} \mathrm{sgn} \det Q_0 = (-1)^{k-1+\mathrm{ind} Q_0}$ relative to the canonical orientation.

Remark. If (ψ, x) is an injectivity point of the map $\varphi_c: (\psi, x) \mapsto \psi^\perp$ from C_f to S^{k-1}, then the form $(-1)^{\mathrm{ind\,ges}_x \psi f} \varphi_c^* \nu_\psi$ defines the canonical orientation on $T_{(\psi,x)}C_f$; we will not present the calculation here.

Our next theorem is a direct many-dimensional generalization of the following trivial fact, which is true for smooth functions of one real argument that have only simple zeros: if the function takes values with different signs at two consecutive zeros of its derivative, the function itself must have a single zero between these points; if the function has the same sign at consecutive zeros of its derivative, it has no zero between these two points.

Let $g: N \to R^k$ be a continuous map of some oriented $(k - 1)$-dimensional manifold N into R^k and let $\alpha \in R^k \setminus \mathrm{im}\, g$. Recall that the degree of g relative to a (denoted $\deg_a g$) is defined as the degree of the map $y \mapsto \frac{g(y) - a}{|g(y) - a|}$ from N to S^{k-1}.

The symbol $\chi(N)$, as usual, will denote the Euler characteristic of N.

THEOREM 1. Let $f \in C_\infty^k(M)$ be a (vector-valued) Morse function. Then $\chi(M) = \deg \varphi_c$. If a vector $a \in R^k$ is not a critical value of f, then

$$\chi(f^{-1}(x)) = (-1)^k \deg_a f_c.$$

Proof. We may assume without loss of generality that $a = 0$.

LEMMA 4. For a typical Morse function $f \in C_\infty^k(M)$, the scalar function $|f_c|^2$ on C_f is a Morse function, and all its critical points are injectivity points of f_c.

Proof. A point $(\psi, x) \in C_f$ is critical for $|f_c|^2$ if and only if $(\ker f')_{\mathrm{ges}_x \psi f}^{\perp} \subset \ker f^\perp f'$. If (ψ, x) is an injectivity point of f_c (see Lemma 2), this condition is equivalent to $f^\perp f' = 0$. The Hessian of the function $(1/2) \cdot |f_c|^2$ reduces to the quadratic form $u \mapsto (f'u)^T f'u + f^T f''(x) \times (u, u)$, $\forall u \in (\ker f')_{\mathrm{ges}_x \psi f}^{\perp}$. Thus, the conditions stating that a critical point of $|f_c|^2$ is degenerate or "noninjective" can be written as equations on a 2-jet of the function f. Thus the proof of Lemma 4 reduces to calculating the parameters and applying Thom's Transversality Theorem.

Without loss of generality, we may assume that f satisfies the assumptions of Lemma 4. In particular, $|f_c|^2$ and $|f|^2$ have the same critical points.

LEMMA 5. Assume that an injectivity point (ψ, x) of the map f_c is a critical point of the function $|f|^2$, $|f(x)| \psi^\perp = f(x)$. Then

$$\mathrm{ind\,ges}_x |f|^2 = \mathrm{ind}(f^T \mathrm{ges}_x f) + \mathrm{ind\,ges}_{(\psi,x)} |f_c|^2.$$

Proof. If $f(x) = 0$ the right- and left-hand sides of the equality to be proved both vanish. Suppose, therefore, that $f(x) \neq 0$. Then $(1/2) \cdot \mathrm{ges}_x |f|^2 = f'(x)^\perp f'(x) + \mathrm{ges}_x \psi f$. Consequently,

$$\frac{1}{2} \mathrm{ges}_x |f|^2 |_{\ker f'(x)} = \mathrm{ges}_x \psi f |_{\ker f'(x)} = \psi\, \mathrm{ges}_x f.$$

The injectivity condition implies that the bilinear form $f^T \mathrm{ges}_x f$ is nonsingular (see Lemma 2). Thus, $\ker f' \times \cap (\ker f' \times)_{\mathrm{ges}_x |f|^2}^{\perp} = 0$ and, therefore,

$$\mathrm{ind}\,\mathrm{ges}_x |f|^2 = \mathrm{ind}\, f^T \mathrm{ges}_x f + \mathrm{ind}\left(\mathrm{ges}_x f|\,|^2 | (\ker f'(x))^{\perp}_{\mathrm{ges}_x |f|^2}\right).$$

It remains to be observed that $(\ker f'(x)_{\mathrm{ges}_x |f|^2}^{\perp} = (\ker f'(x))_{\mathrm{ges}_x \psi f}^{\perp}$ and $\mathrm{ges}_x(\psi, x)|f_c|^2 = \mathrm{ges}_x |f|^2 | (\ker f'(x))_{\mathrm{ges}_x \psi f}^{\perp}$ (cf. proof of Lemma 4).

Under the assumptions of Lemma 4, all critical points of $|f|^2$ lying in $M \setminus f^{-1}(0)$ are nondegenerate. Let $x_1, \ldots, x_m$ be all of them. Since $|f|^2 \geq 0$, Morse theory applied to the function $|f|^2$ yields

$$\chi(M) = \sum_{\alpha=1}^{m} (-1)^{\mathrm{ind}\,\mathrm{ges}_{x_\alpha} |f|^2} + \chi(f^{-1}(0)). \tag{1}$$

Let N be some smooth oriented d-dimensional manifold and $g_i : N \to S^d$ be smooth maps, $i = 1, 2$. Let $\Gamma(g_i) = \{y, g_i(y)) \in N \times S^d | y \in N\}$ denote the graphs of g_i, $i = 1, 2$, and $\Gamma(g_1) \cdot \Gamma(g_2)$ the intersection number of the m-dimensional oriented submanifolds $\Gamma(g_1)$ and $\Gamma(g_2)$ in $N \times S^m$. We have

$$\Gamma(g_1) \cdot \Gamma(g_2) = \deg g_2 + (-1)^d \deg g_1.$$

In particular,

$$\Gamma\left(\frac{f_c}{|f_c|}\right) \cdot \Gamma(\varphi_c) = \deg \varphi_c + (-1)^{k-1} \deg f_c. \tag{2}$$

At the same time, $\Gamma\left(\frac{f_c}{|f_c|}\right)$ and $\Gamma(\varphi_c)$ intersect at the points $\left(\left(\frac{f^T(x_\alpha)}{|f(x_\alpha)|}, x_\alpha\right), \frac{f(x_\alpha)}{|f(x_\alpha)|}\right)$, $\alpha = 1, \ldots, m$, and at no others. Define $\psi_\alpha = \frac{f^T(x_\alpha)}{|f(x_\alpha)|}$; then the intersection number at the point $((\psi_\alpha, x_\alpha), \psi_\alpha^T)$ is +1 if the linear map $D_{x_\alpha}\varphi_c - D_{x_\alpha}\frac{f_c}{|f_c|} = D_{x_\alpha}\varphi_c - \frac{1}{|f_c|} D_{x_\alpha} f_c$ takes the canonical orientation of the space $T_{(\psi_\alpha, x_\alpha)} C_f$ into the positive orientation of the space $T_{\psi_\alpha^T} \times S^{k-1}$, and -1 otherwise.

Let $(\xi_1, u_1), \ldots, (\xi_{k-1}, u_{k-1})$ be a canonical oriented basis in $T_{(\psi_\alpha, x_\alpha)} C_f$, where $u_1, \ldots, u_{k-1}$ are linearly independent. The required intersection number is precisely

$$\mathrm{sgn}\det\left(\psi_\alpha^T, \xi_1^T - \frac{1}{|f(x_\alpha)|} f'(x_\alpha) u_1, \ldots, \xi_{k-1}^T - \frac{1}{|f(x_\alpha)|} f'(x_\alpha) u_{k-1}\right) =$$
$$= \mathrm{sgn}\det(\psi_\alpha^T, |f(x_\alpha)|\xi_1^T - f'(x_\alpha) u_1, \ldots, |f(x_\alpha)|\xi_{k-1}^T - f'(x_\alpha) u_{k-1}) = \mathrm{sgn}\det(A + B).$$

It follows from Lemma 3 that $\mathrm{sgn}\det A = (-1)^{\mathrm{ind}\psi_\alpha f''(x_\alpha)}$. Consequently,

$$A = (\psi_\alpha^T, -f'(x_\alpha) u_1, \ldots -f'(x_\alpha) u_{k-1}),$$
$$B = (0, |f(x_\alpha)|\xi_1^T, \ldots, |f(x_\alpha)|\xi_{k-1}^T),$$

[we have taken into account that $\psi_\alpha \xi_i^T = \psi f'(x_\alpha) u_i = 0$, $i = 1, \ldots, k-1$.]

Differentiating the identity $\psi f'(x) = 0 \ \forall (\psi, x) \in C_f$ at the point (ψ_α, x_α), we get

$$\xi_j f'(x_\alpha) u_i + \psi_\alpha f''(x_\alpha)(u_i, u_j) = 0.$$

Thus,

$$\mathrm{sgn}\det(A + B) =$$
$$= (-1)^{\mathrm{ind}\psi_\alpha f''(x_\alpha)} \mathrm{sgn}\det(\{(f'(x_\alpha)u_i)^T f'(x_\alpha) u_j + f''(x_\alpha)(u_i, u_j)\}_{i,j=1}^{k-1}).$$

Since

$$(f'(x_\alpha) u_i)^T f'(x_\alpha) u_j + f(x_\alpha)^T f''(x_\alpha)(u_i, u_j) = \frac{1}{2} \mathrm{ges}_{(\psi_\alpha, x_\alpha)} |f_c|^2,$$

it follows in view of Lemma 5 that

$$\mathrm{sgn}\det(A + B) = (-1)^{\mathrm{ind}\,\mathrm{ges}_{x_\alpha} |f_c|^2 + \mathrm{ind}\psi_\alpha f''(x_\alpha)} = (-1)^{\mathrm{ind}\,\mathrm{ges}_{x_\alpha} |f|^2}.$$

Finally,

$$\Gamma\left(\frac{f_c}{|f_c|}\right)\cdot\Gamma(\varphi_c)=\sum_{\alpha=1}^{m}(-1)^{\mathrm{Indges}_{x_\alpha}|f|^*}.$$

Equalities (1), (2) imply

$$\chi(M)-\deg\varphi_c=\chi(f^{-1}(0))-\deg f_c(-1)^k.$$

Let $a\in\mathbf{R}^k$, where a is not a critical value of f. Replacing f by f − a, we obtain

$$\chi(M)-\deg\varphi_c=\chi(f^{-1}(a))-\deg_a f_c(-1)^k. \tag{3}$$

If $|a|$ is sufficiently large, then $f^{-1}(a)=\emptyset$. But the left-hand side of (3) is independent of a and, so, both sides must vanish. This completes the proof of the theorem.

COROLLARY 1. Let $f\in C_\infty^k(M)$ be a (vector-valued) Morse function, such that $0\in\mathbf{R}^k$ is not a critical value of f. Let $a_0\in\mathbf{R}^k\setminus 0$ be such that the map f_c is transverse to the ray $\{\tau a_0|\tau>0\}$ $f_c^{-1}(\{\tau a_0\ |\tau>0\})=\{(\pm\psi_1, x_1), \ldots, (\pm\psi_m, x_m)\}\subset C_f$, where $\psi_i f(x_i)>0$, $i=1, \ldots, m$. Then

$$\chi(f^{-1}(0))=((-1)^{n-k+1}-1)\sum_{i=1}^{m}(-1)^{\mathrm{ind}\psi_i f''(x_i)}.$$

Proof. The transversality condition guarantees regularity of the map $(f_c/|f_c|)$ at the points $(\pm\psi_i, x_i)$, $i=1, \ldots, m$. By Theorem 1, $\chi(f^{-1}(0))=\sum_{i=1}^{m}(-1)^{k+\ell_i}+(-1)^{k+\lambda_i}$, where the integers $\ell_i(\lambda_i)$ are even or odd according to the map

$$D_{(\psi_i,x_i)}\frac{f_c}{|f_c|}:T_{(\psi_i,x_i)}C_f\to T_{\frac{f(x_i)}{|f(x_i)|}}S^{k-1}\ \left(\text{the map } D_{(-\psi_i,x_i)}\frac{f_c}{|f_c|}\right)$$

takes the canonical orientation of the space $T_{(\psi_i,x_i)}C_f$ into the positive or negative orientation of $T_{\frac{f(x_i)}{|f(x_i)|}}S^{k-1}$. The signs of these orientations are easily determined by means of Lemma 3.

COROLLARY 2. Again let $f=(f_1, \ldots, f_k)^T\in C_\infty^k(M)$ be a (vector-valued) Morse function such that $0\in\mathbf{R}^k$ is not a critical value. Then

$$\chi(f^{-1}(0))=\int_{C_f}\frac{1}{k\sigma_k|f|^k}\sum_{i=1}^{k}(-1)^{k-i+1}f_i df_1\wedge\ldots d\widehat{f}_i\ldots\wedge df_k$$

$$\chi(M)=\int_{C_f}\frac{1}{k\sigma_k}\sum_{i=1}^{k}(-1)^{i-1}d\psi_1\wedge\ldots d\widehat{\psi}_i\ldots\wedge d\psi_k,$$

where σ_k is the volume of the unit k-sphere and the symbol $\wedge$ means that the relevant factor is omitted.

The map $(\psi, x)\mapsto x$ of the oriented manifold C_f into M defines a singular integral cycle in M. We will denote this cycle by $\overline{C}_f$ and call it the critical cycle of f. On the other hand, the choice of an orientation on M automatically determines an orientation of the submanifold $f^{-1}(0)\subset M$ and converts it into an integral cycle in M. At the same time,

$$\dim\overline{C}_f=k-1,\ \dim f^{-1}(0)=n-k,$$
$$\dim M=n=\dim\overline{C}_f+\dim f^{-1}(0)+1.$$

Let γ_1, γ_2 be two cycles in M with disjoint supports, $\dim\gamma_1+\dim\gamma_2+1=\dim M$. If γ_1 and γ_2 are homologous to zero, the linking coefficient $\ell(\gamma_1, \gamma_2)$ is well defined: if $\gamma_1=\partial\Gamma_1$, then $\ell(\gamma_1, \gamma_2)=\Gamma_1\cdot\gamma_2$ is the intersection number of the chains Γ_1 and γ_2 (that γ_2 is homologous to zero implies that the index is independent of the choice of Γ_1). It turns out that all this "works" in our situation.

Proposition 3. The critical cycle $\bar{C}_f$ of a (vector-valued) Morse function $f \in C_\infty^k(M)$ and the cycle $f^{-1}(0)$ are homologous to zero in M; the linking coefficient of these cycles is given by the formula

$$l(f^{-1}(0), \bar{C}_f) = -\deg f_c = (-1)^{k-1}\chi(f^{-1}(0)). \tag{4}$$

Proof. Choose $f_0 \in C_\infty(M)$ so that the vector-valued function $g = \binom{f_0}{f} \in C_\infty^{k+1}(M)$ is a Morse function. The critical manifold is

$$C_g = \{(\psi_0, \psi, x) \mid \psi_0 f_0(x) + \psi f(x) = 0,\ \psi_0^2 + |\psi|^2 = 1,\ x \in M\} \subset S^k \times M.$$

In C_g, consider the submanifold with boundary $C_g^- = \{(\psi_0, \psi, x) \in C_g \mid \psi_0 \leq 0\}$. Clearly, the boundary is $\partial C_g^- = \{(\psi_0, \psi, x) \in C_g \mid \psi_0 = 0\} = C_f$. The smooth map $(\psi_0, \psi, x) \mapsto x$ produces the cycle $\bar{C}_f$ and, so, $\bar{C}_f$ is homologous to zero in M.

Now assume that f is transverse to a ray $\{\tau a_0 \mid \tau > 0\}$, where $a_0 \in \mathbf{R}^k \setminus 0$ (by Sard's Theorem, such a ray always exists). Then $f^{-1}(\{\tau a_0 \mid \tau \geq 0\})$ is a smooth submanifold with boundary in M, with $\partial f^{-1}(\{\tau a_0 \mid \tau \geq 0\}) = f^{-1}(0)$. Consequently, the cycle $f^{-1}(0)$ is homologous to zero in M. Identity (4) follows at once from Corollary 1 to Theorem 2 and the definition of the linking coefficient, with due attention to orientations.

2. Let $x \in M$ and let J_x^2M be the space of 2-jets at x of smooth scalar functions on M. For arbitrary integers $r \geq 0$, define

$$\Sigma_r(x) = \{J_x^2 a \mid a \in C_\infty(M),\ d_x a = 0,\ \dim \ker \operatorname{ges}_x a = r\} \subset J_x^2 M.$$
$$\Sigma_r^0(x) = \{J_x^2 a \mid a \in C_\infty(M),\ J_x^2 a \in \Sigma_r(x),\ a(x) = 0\}.$$

Let $J^2M = \bigcup_{x \in M} J_x^2 M$ be the total space of the 2-jet bundle over M; define $\Sigma_r(M) = \bigcup_{x \in M} \Sigma_r^0(M) = \bigcup_{x \in M} \Sigma_r^0(x)$. It is easy to see that $\Sigma_r(M)$ and $\Sigma_r^0(M)$ are smooth submanifolds of J^2M for $r = 1, 2, \ldots$.

Definition. A vector-valued function $f \in C_\infty^k(M)$ is said to be strongly Morse if the map $(\psi, x) \mapsto J_x^2(\psi f)$ from $S^{k-1} \times M$ to J^2M is transverse to the submanifolds $\Sigma_r(M)$ and $\Sigma_r^0 \times (M)$, $\forall r \geq 0$.

It is clear that a strongly Morse vector-valued function is Morse and, moreover, it follows from Thom's Transversality Theorem that the strongly Morse functions constitute an open dense subset of $C_\infty^k(M)$.

Define

$$C_f^- = \{(\psi, x) \in C_f \mid \psi f(x) \leqslant 0\},$$
$$C_f^0 = \{(\psi, x) \in C_f \mid \psi f(x) = 0\}.$$

If f is strongly Morse then, as is readily seen, C_f^- is a smooth manifold with boundary C_f^0.

Let V be a smooth vector bundle over M such that $TM \oplus V$ is the trivial bundle, $TM \oplus V = M \times \mathbf{R}^N$, and let μ be a Euclidean metric on V, i.e., a smooth map $x \mapsto \mu(x) \in \mathcal{P}(V_x)$ such that $\mu(x) > 0\ \forall x \in M$. Define maps

$$Q_f : C_f^- \to \mathcal{P}(\mathbf{R}^N) \text{ and } Q_f^0 : C_f^0 \to \mathcal{P}(\mathbf{R}^N),$$

by putting

$$Q_f(\psi, x) = \operatorname{ges}_x(\psi f) \oplus \mu(x),\quad \forall (\psi, x) \in C_f^-,$$
$$Q_f^0 = Q_f \mid C_f^0. \tag{5}$$

LEMMA 6. If $f \in C_\infty^k(M)$ is a strongly Morse function, then $Q_f : C_f^- \to \mathcal{P}(\mathbf{R}^N)$ and $Q_f^0 : C_f^0 \to \mathcal{P}(\mathbf{R}^N)$ are transverse maps, such that the stable transversal homotopy class of Q_f and Q_f^0 is independent of the choice of the additional bundle V and metric μ (for the definition of stable transversal homotopy see end of Sec. 3.2).

The transversality of Q_f and $Q_f{}^0$ follows directly from the definitions; the rest of the lemma follows from the elementary theory of vector bundles. ▸

Let $x \in M$; define linear maps $\pi_1(x): J_x{}^2M \to J_2{}^1M$ and $\pi_0(x): J_x{}^2M \to J_x{}^0M = R$ by the formulas $\pi_1(x)(J_x^2a) = J_x^1a$, $\pi_0(x)(J_x^2a) = a(x)$. Since the jets of the constant functions induce the standard embedding of $J_m{}^0M$ into $J_m{}^kM$ for any k, we may assume that $J_x{}^0M \subset J_x{}^1M$. We have the obvious identifications $\ker \pi_1(x) = \mathcal{P}(T_xM)$,

$$\Sigma_r^0(x) = \Pi_r(T_xM) \setminus \Pi_{r+1}(T_xM),$$
$$\Sigma_r(x) = \Pi_r(T_xM) \setminus \Pi_{r+1}(T_xM) \oplus R \subset \ker(\pi_1(x) - \pi_0(x)).$$

For any $\alpha \in \ker(\pi_1(x) - \pi_0(x))$, we define a set $\Upsilon_\alpha \subset T^*J_x{}^2M \subset T_\alpha{}^*J^2M$

a) if $\alpha \notin \ker \pi_0(x) \cup \overline{\Sigma}_1(x)$, then $\Upsilon_\alpha = (\ker(\pi_1(x) - \pi_0(x)))^\perp$;

b) if $\alpha \in \ker \pi_0(x) \setminus \overline{\Sigma}_1{}^0(x)$, then $\Upsilon_\alpha = (\ker \pi_0(x))^\perp$;

c) if $\alpha \in \overline{\Sigma}_1{}^0(x) = \Pi_1(T_xM)$, then $\Upsilon_\alpha = (\ker \pi_0(x))^\perp \oplus \mathcal{N}_\alpha^+(T_xM)$;

d) if $\alpha \in \overline{\Sigma_1(x) \setminus \Sigma_1^0(x)}$, then $\Upsilon_\alpha = (\ker(\pi_1(x) - \pi_0(x)))^\perp \oplus \mathcal{N}_\alpha^+(T_xM)$.

We are now in a position to formulate a substantial generalization of the definition of strongly Morse functions. Let us say that a convex closed cone K in R^{k*} is piecewise smooth if $K \cap S^{k-1}$ is a submanifold with angles in S^{k-1}.

Definition. Let K be a piecewise smooth convex closed cone in R^{k*}, $f \in C_\infty{}^k(M)$. Let $F: (K \cap S^{k-1}) \times M \to J^2M$ be the map defined by the formula $F(\psi, x) = J_x{}^2(\psi f)$ and $F_{(\psi,x)}{}': T_\psi \times (K \cap S^{k-1}) \oplus T_xM \to T_{F(\psi,x)}J^2M$ the differential of F at the point (ψ, x). A vector-valued function f is said to be strongly Morse with respect to K if, for any $\psi \in K \cap S^{k-1}$, $x \in M$, whenever $d_x(\psi f)$, then

$$(\operatorname{im} F'_{(\psi,x)})^\circ \cap \Upsilon_{F(\psi,x)} = 0.$$

If $K = R^{k*}$, the definition of "strongly Morse with respect to K" is equivalent to our previous definition of "strongly Morse."

Using Thom's Transversality Theorem, we readily show that the vector-valued functions which are strongly Morse with respect to a fixed cone K constitute an open dense subset of $C_\infty{}^k(M)$. Moreover, if f is strongly Morse with respect to K, then the sets $C_f^-(K) = [(\psi, x) \in C_f^- | \psi \in K \cap S^{k-1}\}$ and $C_f{}^0(K) = \{(\psi, x) \in C_f{}^0 | \psi \in K \cap S^{k-1}\}$ are submanifolds with angles in C_f. As before (in the case $K = R^{k*}$) equality (5) defines a map

$$Q_f(K): C_f^-(K) \to \mathcal{P}(R^N), \quad Q_f^0(K) = Q_f | C_f^0(K).$$

LEMMA 7. Let $f \in C_\infty{}^k(M)$ be a strongly Morse function with respect to a piecewise smooth convex closed cone $K \subset R^{k*}$. Then (i) the map $f: M \to R^k$ is transverse to the cone K°, $K^\circ \subset R^k$; ii) $Q_f(K): C_f^-(K) \to \mathcal{P}(R^N)$ and $Q_f{}^0(K): C_f{}^0(K) \to \mathcal{P}(R^N)$ are transverse maps and, moreover, the stable transversal homotopy class of Q_f and Q_f^- is independent of the choice of the additional bundle V and metric μ.

The transversality relations are proved by direct calculation based on the definitions. As an example, we will verify part (i).

Suppose that f is not transverse to K° at a point $x \in M$. Then for some $\psi_0 \in K \cap S^{k-1}$ we have

$$d_x(\psi_0 f) = 0, \quad \psi_0 f(x) = 0, \quad \psi f(x) \leqslant 0, \quad \forall \psi \in K.$$

Consequently, $J_x{}^1(\psi_0 f) = 0$ and the differential of the function $J_y{}^2 a \mapsto a(y)$ on J^2M at the "point" $J_x{}^2(\psi_0 f) = F(\psi_0, x)$ is in the cone dual to $\operatorname{im} F'(\psi_0, x)$.

Proposition 4. Let $f \in C_\infty{}^k(M)$ be a strongly Morse function with respect to a piecewise smooth convex closed cone $K \subset R^{k*}$. If for any $(\psi, x) \in C_f^-(K)$ the quadratic form $\operatorname{hes}_x(\psi f)$ is nonsingular, then

$$\chi(f^{-1}(K^\circ)) = \chi(C_f^-(K)) - \chi(C_f^0(K)) - 2(\chi(Q_f(K)) - \chi(Q_f^0(K))) + \varepsilon\chi(M), \tag{6}$$

where

$$\varepsilon=\begin{cases}(-1)^{\dim K}, & K=-K\\ 0, & K\neq -K.\end{cases}$$

The proof follows the same lines as the proof of Theorem 4.1.

1) Suppose that K is a half-line, $K = \{\alpha\psi_0, \alpha \geq 0\}$. Then $C_f^-(K)$ is a finite set, $C_f^0 \times (K) = 0$, and Morse theory yields

$$\chi((\psi_0 f)^{-1}(-\infty, 0]) = \sum_{(\psi_0, x)\in C_f^-(K)} (-1)^{\operatorname{indges}_x \psi_0 f} = \#(C_f^-(K)) - 2\chi(Q_f(K)).$$

2) Let K be an arbitrary piecewise smooth cone and $\psi_0 \in K \cap S^{k-1}$. Equality (6) will be valid with K replaced by any sufficiently small conical neighborhood U_{ψ_0} of ψ_0 in K. Indeed, it follows from the assumptions of this proposition and from Lemma 2 that the map $\varphi_c\,|C_f^-(K): C_f^-(K) \to S^{k-1}$ defined by $\varphi_c(\psi, x) = \psi$ is a cover. Hence, again using the fact that $\operatorname{ges}_x(\psi f)$ is nonsingular, we infer that

$$\chi(C_f^-(U_{\psi_0})) - \chi(G_f^0(U_{\psi_0})) = \#\{(\psi_0, x) \mid \psi_0 f'(x) = 0,\ \psi_0 f(x) < 0\},$$
$$\chi(C_f^-(U_{\psi_0})) - \chi(C_f^0(U_{\psi_0})) - 2(\chi(Q_f(U_{\psi_0})) - \chi(Q_f^0(U_{\psi_0}))) =$$
$$= \sum_{\{x \mid \psi_0 f'(x)=0,\ \psi_0 f(x)<0\}} (-1)^{\operatorname{indges}_x \psi_0 f} = \chi((\psi_0 f)^{-1}(-\infty, 0)).$$

To prove the truth of our assertion it remains to be shown that the set $f^{-1}(U_{\psi_0}{}^\circ)$ is homotopically equivalent to the set $(\psi_0 f)^{-1}(-\infty, 0)$ if U_{ψ_0} is sufficiently small. Since 0 is a regular value of f, it follows, of course, that $f^{-1}(U_{\psi_2}{}^\circ)$ is homotopically equivalent to $f^{-1}(U_{\psi_0}{}^\circ \backslash 0)$. We claim that $f^{-1}(U_{\psi_1}{}^\circ \backslash 0)$ is a homotopic retract of the set $(\psi_0 f)^{-1}(-\infty, 0)$.

There is a subtle point here: zero may be a critical value of the function $\psi_0 f$. Let $x_1, \ldots, x_m$ be all the critical points of this function corresponding to the critical value zero and O_{x_i} be a neighborhood of x_i in which $\psi_0 f$ can be made into a quadratic form by a smooth change of variables, $i = 1, \ldots, m$. For some $\varepsilon > 0$ we have $|\operatorname{grad}_x \psi_0 f| \geq 2\varepsilon$ if $x \notin \bigcup_{i=1}^m O_{x_i}$.

Consequently, for any ψ in a sufficiently small neighborhood U_{ψ_0},

$$|\operatorname{grad}_x \psi_0 f| \geqslant \varepsilon, \quad x \notin \bigcup_{i=1}^m O_{x_i}.$$

Motion along trajectories of a vector field positively proportional to $(-\operatorname{grad}_x(\psi_0 f))$ (as in the usual Morse theory) corresponds to retraction of the set $(\psi_0 f)^{-1}(-\infty, 0)$ into the set $(\psi_0 f)^{-1}(-\infty, 0) \backslash \left(\bigcup_{i=1}^m O_{x_i}\right)$ and, thereafter, into the set $f^{-1}(U_{\psi_0}{}^\circ/0)$.

3) Let K_1, K_2 be piecewise smooth convex cones such that $K_1 \cup K_2$ is convex. If (6) is true for the cones K_1, K_2, $K_1 \cap K_2$, it is also true for $K_1 \cup K_2$. The proof is exactly the same as for the analogous assertion of Theorem 4.1 – use the additivity of the right- and left-hand sides of (6).

4) To end the proof, one uses induction on the dimension of the cone K. If the dimension is 1, then K is either a half-line (see part 1 of the proof) or a straight line. A straight line can be expressed as the union of two half-lines, and one then uses part 3. The inductive step is also carried out as in the proof of Theorem 4.1: cutting the cone K by several hyperplanes transverse to it and using part 3 of the proof, one reduces everything to the situation considered in the second part.

Remark. The condition that the quadratic forms $\operatorname{ges}_x \psi f$ are nonsingular for $\psi \in K\backslash 0$ is fairly stringent; we do not known whether (6) (or some modification thereof) is valid under weaker assumptions.

One often has to consider the case in which $M = S^n$, $f = F|S^n$, where $F \in C_\infty{}^k(\mathbf{R}^{n+1} \backslash 0)$ is a positively homogeneous function. We will therefore describe the singular set, Hessian, etc. of the vector-valued function f in terms of F in the case that F is positively homogeneous of degree $\nu \neq 0, 1$, i.e.,

$$F(tx)=t^{\nu}f(x),\ \forall t>0,\ x\in\mathbf{R}^{n+1}\setminus 0.$$

The basic identities are

$$(D_xF)x=\nu F(x),$$
$$D_x^2F(x, y)=(\nu-1)D_xFy,\ \forall x,\ y\in\mathbf{R}^{n+1}\setminus 0.$$

Let $\psi \in \mathbf{R}^{k*}$. The symmetric $(n + 1) \times (n + 1)$ matrix corresponding to the bilinear form $D_x{}^2\psi f$ will be denoted by $\psi F''(x)$. The critical set has the form $C_f = \{(\psi, x) \in S^{k-1} \times S^n | x$ is an eigenvector of the matrix $\psi F''(x)\}$.

Let $(\psi, x) \in C_f$, $\psi F''(x)x = \lambda x$. Then

$$\mathrm{ges}_x\psi f=\left(D_x^2\psi F-\frac{\lambda}{\nu-1}I\right)\Big|_{x^{\perp}}; \tag{7}$$

in particular, the map $\varphi_c:(\psi, x) \mapsto \psi$ is regular at (ψ, x) if and only if $[\lambda/(\nu - 1)]$ is not an eigenvalue of $\psi F''(x)|_{x^\perp}$. In addition, $d_x\psi F = [\lambda/(\nu - 1)]\cdot x$, $\psi F(x) = [\lambda/(\nu(\nu - 1))]$ and, so, λ is a smooth function on C_f and

$$C_f^-=\{(\psi,\ x)\in S^{k-1}\times S^n\,|\,\psi F''(x)x=\lambda x,\ \lambda\nu(\nu-1)\leqslant 0\}.$$

We illustrate the results of this section for the special case of quadratic maps.

<u>LEMMA 8.</u> Let $P \in \mathscr{P}(n + 1, k)$; then the following conditions are equivalent:

i) the map $p|S^n:S^n \to \mathbf{R}^k$ is not a Morse function;

ii) the zero in $\mathbf{R}^k$ is a critical value of the map $(x, y) \mapsto p(x, y)$, considered on the manifold $\{(x, y) \in S^n \times S^n | x \perp y\}$;

iii) for some $\psi \in S^{k-1}$, $x, y \in S^n$, $\lambda \in \mathbf{R}$ we have $\psi Px = \lambda x$, $\psi Py = \lambda y$, $x \perp y$, $p(x, y) = 0$.

The proof is by a direct check.

<u>LEMMA 9.</u> For a typical $p \in \mathscr{P}(n + 1, k)$, the map $p|S^n$ is Morse.

<u>Proof.</u> By the Transversality Theorem and Lemma 8, it will suffice to be proved that the zero of $\mathbf{R}^k$ is a regular value of the map

$$(p, x, y)\mapsto p(x, y),\ \text{where}\ p\in\mathscr{P}(n+1, k),$$
$$(x, y)\in\{(x, y)\in S^k\times S^k\,|\,x\perp y\}.$$

However, it is readily seen that this map need not have critical points.

Let $p \in \mathscr{P}(n + 1, k)$ be such that $p|S^*$ is Morse. Let C_p denote the critical set of the vector-valued function $p|S^n$ (this will not cause confusion) and $\pi_c:C_p \to S^{k-1}$ the map $(\psi, x) \mapsto \psi$. Then

$$C_p=\{(\psi,\ x)\in S^{k-1}\times S^n\,|\,(\psi P)x=\lambda x\ \text{for some}\ \lambda\in\mathbf{R}\}$$
$$C_p^-=\{(\psi,\ x)\in S^{k-1}\times S^n\,|\,(\psi P)x=\lambda x,\ \lambda\leqslant 0\}.$$

Let $(\psi, x) \in C_p$, $\psi Px = \lambda x$. A map π_c is a local diffeomorphism in a neighborhood of (ψ, x) if and only if $\mathrm{ges}_x(\psi p|S^n)$ is a nonsingular form (Lemma 2), i.e., λ is a simple eigenvalue of the matrix ψP [see (7)].

The symmetric matrix ψP has n + 1 real eigenvalues (counting multiplicities); arrange them in increasing order, $\lambda_1(\psi) \leq \ldots \leq \lambda_{n+1}(\psi)$.

Let $1 \leq i \leq n + 1$ and let $\lambda_i(\psi)$ be a simple eigenvalue, $S^n \ni x_i$, a corresponding eigenvector. Then the linear map $(-1)^{i-1}D_{(\psi,x_i)}\pi_i:T_{(\psi,x_i)}C_p \to T_\psi S^{k-1}$ carries the canonical orientation of the space $T_{(\psi,x_i)}C_p$ into the positive orientation of $T_\psi S^{k-1}$ (see remark after Lemma 3).

<u>LEMMA 10.</u> For a typical $p \in \mathscr{P}(n + 1, k)$, the set of critical values of the map $\pi_c: C_p \to S^{k-1}$ is an algebraic set of dimension at most $k - 3$.

Proof. Identifying the space of symmetric $(n+1)\times(n+1)$ matrices with $\mathcal{P}(\mathbf{R}^{n+1})$, we see that the set of all symmetric matrices with multiple eigenvalues is precisely $\Pi_1\times(\mathbf{R}^{n+1}) + \mathrm{span}\{I\}$ and it is an algebraic subset of codimension 2 in $\mathcal{P}(\mathbf{R}^{n+1})$. If the map $p^*:\psi\mapsto\psi p$, $\psi\in S^{k-1}$ is transverse to the submanifold $\Pi_m(\mathbf{R}^{n+1})\setminus\Pi_{m+1}(\mathbf{R}^{n+1}) + \mathrm{span}\{I\}$, $m = 1, \ldots, n+1$ in $\mathcal{P}(\mathbf{R}^{n+1})$ (which is not the case for typical p), the set of all $\psi\in S^{k-1}$ such that the matrix ψP has multiple eigenvalues is an algebraic subset of codimension at least two in S^{k-1}.

Any quadratic map $p\in\mathcal{P}(n+1, k)$ is even: $p(-x) = p(x)$. Identification of the points x and $-x$, $x\in S^n$, converts the sphere S^n into the projective space $\mathbf{RP}^n$, and, so, p uniquely defines a map $\tilde{p}:\mathbf{RP}^n\to\mathbf{R}^k$ by the rule $\tilde{p}(\{x, -x\}) = p(x)$. This map and its critical set $C_{\tilde{p}}$ are conveniently used along with p and C_p. Clearly, C_p is a double cover of $C_{\tilde{p}}$.

It follows from Lemma 10 that for a typical $p\in\mathcal{P}(n+1, 2)$ the matrix ψP has no multiple eigenvalues for any $\psi\in S^1$ and the manifold $C_{\tilde{p}}$ is the union of $n+1$ pairwise disjoint circles. When $k\ge 3$ the multiple eigenvalues are no longer "removable." When $k = 3$, for example, the most that can be achieved by a transformation to general position is a matrix ψP with one double eigenvalue for each of a finite set of points $\psi\in S^2$ and only simple eigenvalues for all other points ψ. Corresponding to the double eigenvalue we have a whole circle of eigenvectors of unit length.

Thus, in order to obtain the manifold $C_{\tilde{p}}$ for typical $p\in\mathcal{P}(n+1, 3)$, one must consider a set of $n+1$ two-dimensional spheres (corresponding to the differently numbered eigenvalues) and, then, "cut out" from this set several pairs of points, in such a way that points in one pair lie in consecutively numbered spheres. One then "glues" circles in place of the removed points and "glues together" pairwise corresponding spheres along these circles.

We see that if attention is confined to quadratic vector-valued functions $p|_{S^n}$, $p\in\mathcal{P}(n+1, k)$, then when $k = 2$ the assumptions of Proposition 4 are fulfilled in the typical situation; when $k\ge 3$ this is no longer the case: the cone K must be chosen in such a way that the matrix ψP has no multiple nonpositive eigenvalues for $\psi\in K\setminus 0$.

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