FLEXIBILITY OF LAPLACE–BELTRAMI EIGENFUNCTIONS UNDER METRIC VARIATION

AGRACHEV A. ANDREI, DA PONTE NICOLA, LERARIO ANTONIO, TESTA MATTEO

Abstract. We study the surjectivity of the differential of the Laplace–Beltrami operator with respect to the metric parameter. We use the obtained results to prove that every small deformation of a nonconstant eigenfunction can be realized with a small perturbation of the metric. As a consequence, we prove that every generic property of smooth functions which is invariant under multiplication by a nonzero scalar is generic for eigenfunctions of the Laplace–Beltrami operator, extending a result by Uhlenbeck.

The Laplace–Beltrami operator is a classical differential operator associated with a Riemannian manifold. Its spectral theory connects analysis, geometry and topology, and a natural question is to understand which properties of its eigenfunctions are stable, or generic, when the metric is allowed to vary. In [16], Uhlenbeck proved that on a smooth compact connected manifold of dimension at least two, the generic[1] C^k metric has simple eigenvalues and eigenfunctions which are Morse and transverse to zero. The aim of this paper is to prove a general form of this statement in the smooth category: informally, we will show that every generic property of smooth functions, in the sense of the Fréchet topology, is also generic for Laplace–Beltrami eigenfunctions.

We explain now the main ideas and statements in more details. Let M be a compact, connected, smooth manifold without boundary and of dimension at least two. We denote by $\mathcal{G}_+$ the space of smooth Riemannian metrics on M. The main analytic input is a submersivity result for the Laplace–Beltrami operator with respect to the metric variable, that we explain first. Consider the smooth map

$$\Delta : \mathcal{G}_+ \times C^\infty(M) \longrightarrow C^\infty(M), \qquad (g, u) \longmapsto \Delta_g u,$$

and denote its differential in the metric direction by

$$\dot{\Delta}_g[h](u) := \left.\frac{d}{dt}\right|_{t=0} \Delta_{g+th} u.$$

We introduce the open dense set $\mathcal{N} \subset \mathcal{G}_+ \times C^\infty(M)$ defined as

$$\mathcal{N} = \begin{cases} \{(g,u) : u \text{ is Morse and } \operatorname{crit}(u) \cap (\Delta_g u)^{-1}(0) = \varnothing\}, & \dim M = 2, \\ \{(g,u) : u \text{ is Morse}\}, & \dim M > 2. \end{cases}$$

In Section 2 we will prove that for every $(g_0, u_0) \in \mathcal{N}$ there is a neighborhood V of (g_0, u_0) such that the family

$$\dot{\Delta} : V \times \mathcal{G} \longrightarrow C^\infty(M), \qquad ((g,u), h) \longmapsto \dot{\Delta}_g[h](u),$$

is surjective and admits a smooth tame family of right inverses. See Theorem 2.2. Here we use the word "tame" in the sense of [10]. Notice that this statement does not require u to be an eigenfunction. In dimension at least three it therefore says, in particular, that for every Morse function u the map $g \mapsto \Delta_g u$ is locally open, see Corollary 2.3. The one-dimensional

[1]Here and throughout the paper, generic means belonging to a residual set, that is, to a set containing a countable intersection of open dense sets.

case is genuinely different: there the metric differential of the Laplace–Beltrami operator is never surjective for a nonconstant function. See Theorem 2.6.

We then apply Theorem 2.2 to the *eigenpair space*. Consider the space of simple eigenpairs

$$Q' = \{(g, \lambda, u) \in \mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M) : (\Delta_g + \lambda)u = 0,\ m_\lambda(g) = 1,\ u \neq 0\},$$

where $m_\lambda = \dim(\ker(\Delta_g + \lambda))$. We further define

$$Q = \{(g, \lambda, u) \in Q' : (g, u) \in \mathcal{N}\}.$$

(There is a technical reason to work with this set, that will become more transparent below.) Using Uhlenbeck's genericity theorem at every sufficiently high regularity, we show that Q is open and dense in Q'. The submersivity result above then gives the main "flexibility" statement.

Theorem A. Let M be a compact connected smooth manifold of dimension at least two. The projection

$$\pi : Q \longrightarrow C^\infty(M), \qquad (g, \lambda, u) \longmapsto u,$$

is an open map.

Equivalently, if $(g_0, \lambda_0, u_0) \in Q$, then every sufficiently small smooth deformation of u_0 is again an eigenfunction of a nearby metric, for a nearby eigenvalue. This is the sense in which eigenfunctions are flexible under metric variation, explaining the title of the paper. The statement is local on the eigenpair space, but the subset on which it holds is open and dense.

We explain now the genericity statement generalizing Uhlenbeck's result in the above sense. To proceed, since eigenfunctions are naturally defined up to multiplication by a nonzero scalar, it will be convenient to formulate the result using the projectivized space $\mathbb{P}C^\infty(M)$. Let $[u_k(g)]$ denote the k-th eigenspace of Δ_g, viewed as a point of $\mathbb{P}C^\infty(M)$ whenever it is one-dimensional. For $D \subset \mathbb{P}C^\infty(M)$, define

$$\mathcal{G}_+(k, D) := \{g \in \mathcal{G}_+ : \lambda_k(g) \text{ is simple and } [u_k(g)] \in D\}.$$

Theorem B. Let $k \geq 1$ be a positive integer and let $D \subset \mathbb{P}C^\infty(M)$. If D is open, dense, or residual, then $\mathcal{G}_+(k, D)$ is respectively open, dense, or residual in $\mathcal{G}_+$. In particular, if D is residual, then the set of metrics g such that every nonconstant eigenspace of Δ_g is simple and belongs to D is residual in $\mathcal{G}_+$.

Note that, this extends Uhlenbeck's theorem from the specific properties of being Morse and transverse to zero to *arbitrary* generic properties of *smooth* functions. Once Theorem A is known, the passage from generic properties of functions to generic properties of eigenfunctions is essentially topological: the preimage of a dense set under an open map is dense, and residual sets are handled by taking countable intersections.

Although our result is close in spirit to Uhlenbeck's, the proof uses a different mechanism. The argument in [16] is carried out in Banach spaces and relies on Sard–Smale transversality. In practice, this is particularly well suited to properties described by finite-order conditions. Our approach instead uses the Nash–Moser inverse function theorem in the category of tame Fréchet manifolds, following Hamilton[10]. The analytic work is concentrated in proving a single openness theorem for the eigenfunction projection; once this is done, arbitrary residual subsets of $\mathbb{P}C^\infty(M)$ are treated simultaneously, including properties which are not naturally encoded by finitely many jets. The proof of the analytic theorem has two local parts. Near a nondegenerate critical point, a finite-dimensional polynomial calculation solves the prescribed finite jet of the equation. In dimensions at least three, the remaining datum is solved by a characteristic integral along an attracting vector field. Finite-order vanishing and Lyapunov

estimates give a fixed loss of derivatives and smooth tame dependence. In dimension two, the additional condition in $\mathcal{N}$ instead gives an explicit conformal solution.

Away from the critical points, a conformal variation corrects the mean of the datum. A traceless variation then reduces the remaining equation to a divergence equation, solved by a fixed support-preserving Bogovskii operator. Patching the two constructions gives the required smooth tame right inverse.

Uhlenbeck's theorem provides an open dense locus of simple eigenpairs where this analytic construction applies. The Nash–Moser theorem then proves openness of the eigenfunction projection. All subsequent genericity statements follow from this openness and elementary category arguments.

We conclude by describing the organization of the paper. In Section 1 we recall the analytic and geometric background used throughout: the Fréchet topologies on spaces of smooth functions and metrics, the basic spectral theory of the Laplace–Beltrami operator, and the Nash–Moser inverse function theorem in Hamilton's tame category.

Section 2 is devoted to the submersivity of the Laplace–Beltrami operator with respect to the metric. We first solve the linearized equation on jets at the critical points and prove the surjectivity of the homogeneous polynomial model. We then solve the residual equation near critical points by the characteristic method and Lyapunov estimates, and solve the equation away from critical points using the Bogovskii operator. Finally, we prove that the set $\mathcal{N}$ is open and dense. This is done by perturbing the second jets at the finitely many Morse critical points. We also show directly that the analogous surjectivity statement fails in dimension one.

In Section 3 we turn to eigenfunctions. We recall the part of Uhlenbeck's work needed in the sequel and prove that the simple eigenpair space is a tame Fréchet manifold. The submersivity theorem from Section 2 then gives Theorem A. We use it to prove Theorem B and its simultaneous version for all eigenvalues. We also discuss two extensions. First, restricting to metrics induced by embeddings into Euclidean space and using Nash's theory of free embeddings [13], we obtain the analogous genericity statements for induced metrics. Second, we treat Schrödinger operators of the form

$$\Delta_g + V, \qquad (g, V) \in \mathcal{G}_+ \times C^\infty(M),$$

proving an analogue of the flexibility theorem on an open dense subset of the corresponding eigenpair space and deriving the same genericity consequences. The appendix contains the coordinate computation of the metric differential and the Lyapunov estimates used in the characteristic construction.

We expect that the methods developed here can also be adapted to manifolds with boundary, after imposing suitable boundary conditions. We do not pursue this direction in the present paper.

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1. Preliminaries

1.1. **Spaces of smooth functions and metrics.** Throughout the paper, we fix a compact connected smooth manifold M without boundary of dimension $n \geq 2$. Connectedness is retained throughout: the openness and Morse-density statements below do not extend verbatim to disconnected manifolds. The space of smooth real–valued functions on M is denoted by $C^\infty(M)$. Fix a finite atlas $\{(U_i, \varphi_i)\}_{i=1}^N$ of M and let $V_i \subset U_i$ be an open set with compact closure contained in U_i and such that $\{V_i\}_{i=1}^N$ covers M. For each $k \geq 0$ we define the C^k norm on M as

$$\|u\|_{C^k(M)} \coloneqq \max_i \left\|u \circ \varphi_i^{-1}\right\|_{C^k(\varphi_i(V_i))}, \tag{1.1}$$

where on each $\varphi_i(V_i) \subset \mathbb{R}^n$ we use the standard C^k norm. To simplify notation, we will often simply write $\|u\|_k = \|u\|_{C^k(M)}$ when there is no confusion. Observe that different choices of atlas give rise to equivalent norms. This is because the derivatives of the transition functions are bounded on a compact set. We endow $C^\infty(M)$ with the topology generated by the countable family of norms $(\|\cdot\|_{C^k})_{k\in\mathbb{N}}$. With this topology, $C^\infty(M)$ is a Fréchet space; see [10] for further details. It will also be useful to consider Sobolev norms $\|\cdot\|_{H_k(M)}$ on $C^\infty(M)$, since these are well suited to regularity estimates; see Theorem 1.1. Locally, for $u\colon U \subset \mathbb{R}^m \to \mathbb{R}$, the Sobolev norm is defined as

$$\|u\|_{H_k(U)} \coloneqq \left(\sum_{|\beta|\leq k} \int_U \partial^\beta u(x)^2 dx\right)^{\frac{1}{2}}.$$

Using the same finite atlas, one defines the corresponding Sobolev norms on M. Despite not being equivalent, the families of norms $\{\|\cdot\|_{C^k(M)}\}_{k\geq 0}$ and $\{\|\cdot\|_{H_k(M)}\}_{k\geq 0}$ induce the same topology on $C^\infty(M)$. In particular, the Sobolev norms also define a Fréchet topology on $C^\infty(M)$. We will also denote by $H_k(M)$ the space of square–integrable functions with distributional derivatives up to order k. Observe that on a compact manifold being square integrable does not depend on the choice of the underlying metric.

We denote by T^*M the cotangent bundle of M and by S^2T^*M the vector bundle of symmetric $(0,2)$–tensors on M. Among them, we denote by $S^2_+T^*M$ the subspace of the positive

definite ones. We will be especially interested in sections of these space. Thus, we denote by $\mathcal{G}$ the space of smooth sections of S^2T^*M and by $\mathcal{G}_+$ the space of smooth sections of $S^2_+T^*M$, which coincides with the space of smooth Riemannian metrics. Observe that $\mathcal{G}_+$ is an open cone in $\mathcal{G}$. As we did for smooth functions, we can define a Fréchet space structure on $\mathcal{G}$ whose topology is generated by the C^k norms. For each $g \in \mathcal{G}_+$ we denote by $\mathrm{d}\,\mathrm{vol}_g$ the Riemannian volume measure associated with g and by

$$\langle u_1, u_2\rangle_{L^2_g} \coloneqq \int_M u_1 u_2 \,\mathrm{d}\,\mathrm{vol}_g, \qquad \|u\|_{L^2_g} \coloneqq \left(\langle u, u\rangle_{L^2_g}\right)^{1/2},$$

the corresponding L^2–inner product and norm on $C^\infty(M)$. Observe that we could also define the norms $\|\cdot\|_{C^k_g(M)}$ using the metric g. Instead, we prefer to work with the (equivalent) fixed norm (1.1), which is independent of g and thus allows us to vary the metric while keeping the underlying Fréchet spaces and norms fixed. Moreover, we'll denote by $\mathcal{G}^k_+$ the space of C^k Riemannian metrics on M.

Analogously, we define a Fréchet space structure on the set of smooth vector fields on M whose topology is generated by the C^k norms.

1.2. **The Laplace–Beltrami operator.** Let g be a smooth Riemannian metric on M. We use the nonpositive sign convention for the Laplace–Beltrami operator:

$$\Delta_g \coloneqq \mathrm{div}_g \nabla_g,$$

so that Δ_g is nonpositive and self–adjoint on $H_2(M)$ with respect to the L^2_g inner product on $L^2_g(M)$. Since M is compact the spectrum of Δ_g consists of a discrete sequence,

$$0 = -\lambda_0(g) > -\lambda_1(g) \geq -\lambda_2(g) \geq \cdots \to -\infty,$$

where each eigenvalue is repeated according to its multiplicity. Thus, if $-\lambda$ is an eigenvalue of Δ_g, the corresponding eigenfunctions are precisely the nonzero solutions of

$$(\Delta_g + \lambda)u = 0.$$

The eigenspace corresponding to 0 consists of the constant functions. We shall mainly be interested in the negative eigenvalues. The multiplicity of an eigenvalue $-\lambda$ is denoted by $m_\lambda(g) \coloneqq \dim\ker(\Delta_g + \lambda)$ with the convention that $m_\lambda(g) = 0$ if $-\lambda$ is not an eigenvalue of Δ_g.

The range of the operator $\Delta_g + \lambda$ is described by the Fredholm alternative for selfadjoint operator. In particular if $g \in \mathcal{G}_+$, then for every $\lambda \in \mathbb{R}$, the kernel $\ker(\Delta_g + \lambda)$ is finite-dimensional and

$$(\Delta_g + \lambda)(C^\infty(M)) = \ker(\Delta_g + \lambda)^{\perp_{L^2_g}} \cap C^\infty(M).$$

We shall use the following elliptic estimates.

Theorem 1.1 ([7, Theorem 2, Chapter 6.3])**.** *Let M be a compact manifold and L an elliptic operator with C^{k+1} coefficients. Fix a scalar λ, a positive integer $k \geq 0$. There exists a positive constant $C = C(L)$ depending continuously on L such that*

$$\|u\|_{H_{k+2}(M)} \leq C\left(\|Lu\|_{H_k(M)} + \|u\|_{L^2(M)}\right),$$

for every $u \in H_{k+2}(M)$. Moreover if g is smooth, $u \in H_1(M)$ and $Lu \in C^\infty(M)$, then have $u \in C^\infty(M)$.

Remark 1.2. Theorem 1.1 holds for operators of the Laplace–Beltrami type. However, not that if g is a C^k metric, then Δ_g has C^{k-1} coefficients. This is an immediate consequence of tis definition.

Throughout this paper, we will always assume that k is large enough that $H_k(M)$ embedds into $C^2(M)$. Such an embedding is given by [7, Theorem 6, Chapter 5.6.3] for k greater than a fixed constant depending only on the dimension of M.

1.3. **Tame Fréchet geometry and the Nash–Moser theorem.** In this subsection we recall the notions of tame Fréchet geometry needed in the sequel. The purpose is twofold: first, to fix the notions of tame maps and tame submanifolds used below; second, to state the Nash–Moser inverse function theorem in the form in which it will be applied.

We begin with the underlying Fréchet-manifold structure, see [10, Definition 4.1.1 Sec 1].

Definition 1.3 (Fréchet manifold)**.** Let E be a Fréchet space. A *Fréchet manifold* modeled on E is a Hausdorff topological space endowed with an atlas of charts taking values in open subsets of E, whose transition maps are smooth in the Fréchet sense.

For the Nash–Moser inverse function theorem, however, this ordinary Fréchet structure is not sufficient. One must work with Fréchet spaces and maps satisfying an additional tameness assumptions. We will use the framework of tame Fréchet spaces in the sense of Hamilton, see [10], recalling only the parts of the theory that will be used explicitly. In particular, all Fréchet spaces considered below are assumed to be graded: their topology is induced by a family of increasing seminorms $\|\cdot\|_k$, satisfying

$$\|f\|_k \leq \|f\|_{k+1}$$

for every f and every k. The additional condition that a graded Fréchet space be tame is a compatibility condition with a model space of rapidly decreasing sequences. Refer [10, Definition 1.3.2 Sec 2] for the precise definition. The standard examples relevant here, such as spaces of smooth sections over compact manifolds endowed with their usual C^k-norms, are tame Fréchet spaces.

The notion of tameness for maps is expressed by estimates with a controlled loss of derivatives. Since we will have to verify such estimates later, we now recall the relevant definitions from [10].

Definition 1.4 (Tame maps)**.** Let F and G be Fréchet spaces, and let $L\colon F \to G$ be a linear map. We say that L is a *tame linear map* if there exist $r, b \in \mathbb{N}$ such that

$$\|Lf\|_k \leq C_k \|f\|_{k+r}$$

for every $k \geq b$, where the positive constant C_k depends only on k.
Let $U \subseteq F$ be open, and let $P\colon U \to G$ be a nonlinear continuous map. We say that P satisfies a *tame estimate* if there exist $r, b \in \mathbb{N}$ such that

$$\|P(f)\|_k \leq C_k\big(1 + \|f\|_{k+r}\big)$$

for every $k \geq b$, where the constant C_k depends only on k. The map P is said to be *tame* if it satisfies a tame estimate in a neighborhood of each point. A smooth tame map is a smooth map such that all its derivatives are tame. We say that the image of a smooth tame embedding is a smooth tame submanifold.

Remark 1.5. In practice, we will use the following simple criterion. If X is a smooth tame submanifold of a tame Fréchet space E, then to prove that a map $P\colon X \to F$ is tame it is enough to prove, locally on X, estimates of the form

$$\|P(f)\|_k \leq C_k(1 + \|f\|_{k+r}),$$

where the seminorms on X are induced from the ambient space E.

All partial differential operator are smooth and tame, in particular the Laplace–Beltrami operator

$$\begin{aligned}\Delta\colon \mathcal{G}_+ \times \mathbb{R} \times C^\infty(M) &\to C^\infty(M)\\ g, \lambda, u &\mapsto (\Delta_g + \lambda)u\end{aligned}$$

is smooth and tame. This is an immediate consequence of [10, Corollary 2.2.7 Sec 2].

The verification of tame estimates will often rely on standard interpolation inequalities. These inequalities allow us to estimate intermediate norms in terms of lower and higher norms, and to control products without losing more derivatives than necessary. We record the precise form that will be used below.

Proposition 1.6 ([10, Theorem 2.2.1, Corollary 2.2.3 Sec 2])**.** *Let M be a compact smooth manifold and $f, g \in C^\infty(M)$. If $k \le \ell \le m$, then*

$$\|f\|_\ell^{m-k} \le C_{k,\ell,m} \|f\|_k^{m-\ell} \|f\|_m^{\ell-k},$$

where $C_{k,\ell,m}$ is a positive constant depending on M, k, ℓ and m. This estimate also holds for Sobolev norms, possibly with different constants, and for smooth vector fields. Moreover, we have

$$\|fg\|_m \le C_{0,m}(\|f\|_0 \|g\|_m + \|f\|_m \|g\|_0),$$

where $C_{0,m}$ is a positive constant depending on M and m.

We will need to show that families of inverse maps are smooth and tame. The following criterion reduces this to proving continuity and tameness for the inverse family.

Proposition 1.7 ([10, Theorem 3.1.1 Sec 3])**.** *Let $L : (U \subseteq E) \times F \to G$ be a smooth tame family of linear maps. Suppose that the equation $L(e)f = g$ has a unique solution f for all e and g. Assume also that the family of inverses $V(e)g = f$ is continuous and tame as a map $V : U \times G \to F$. Then V is a smooth tame map.*

We are now ready to state the Nash–Moser inverse function theorem. The main difference from the classical Banach implicit function theorem, besides the additional tameness requirement, is that one must assume the differential to be invertible on a whole neighborhood of the point, rather than only at the point itself.

Theorem 1.8 ([10, Theorem 1.1.1, 1.1.2, 1.1.3 Sec 3])**.** *Let F, G be tame Fréchet spaces. Let U be an open subset of F and $P : U \subseteq F \to G$ be a smooth tame map. Then*

1. *Assume that $DP\colon U \times F \to G$ has a smooth tame right inverse, i.e. $RP\colon U \times G \to F$ such that for each $p \in U$, $f \in F$ we have $D_pP(R_pP(f)) = f$. Then P is locally surjective, i.e. in a neighborhood of any point it has a smooth tame right inverse.*
2. *Assume that $DP\colon U \times F \to G$ has a smooth tame left inverse. Then P is locally injective with a smooth tame left inverse.*
3. *Assume that $DP\colon U \times F \to G$ has a smooth tame inverse. Then P is locally invertible and the local inverse is a smooth tame map.*

2. Submersivity of the Laplacian

We can interpret the Laplacian operator as a smooth map

$$\Delta : \mathcal{G}_+ \times C^\infty(M) \to C^\infty(M),$$

where $\Delta(g, u) = \Delta_g(u)$. In this section, we'll study the differential of the Laplacian with respect to the metric component. To simplify notation, we'll denote such differential as $\dot{\Delta}_g[\cdot](u)$ at a point (g, u). We first introduce the space we'll be working with.

Definition 2.1 (Submersivity set)**.** Let M be a compact smooth manifold of $\dim(M) \geq 2$ and let k be a sufficiently large positive integer. We define the following subset of $\mathcal{G}^k_+ \times H_k(M)$ as

$$\mathcal{N}_{C^k} = \begin{cases} \{(g,u) : u \text{ Morse, } \operatorname{crit}(u) \cap (\Delta_g u)^{-1}(0) = \emptyset\} & \text{if } \dim(M) = 2 \\ \{(g,u) : u \text{ Morse}\} & \text{if } \dim(M) > 2. \end{cases}$$

We will denote its smooth part by

$$\mathcal{N} = \mathcal{N}_{C^k} \cap (\mathcal{G}_+ \times C^\infty(M)) \subseteq \mathcal{G}_+ \times C^\infty(M).$$

The aim of this section is to prove the following theorem, which will serve as the main tool prove the results in the next section.

Theorem 2.2. *Let M be a smooth compact connected manifold with $\dim(M) \geq 2$ and let $(g_0, u_0) \in \mathcal{N}$. Then there exists a neighborhood $\mathcal{V}$ of (g_0, u_0) in $\mathcal{G}_+ \times C^\infty(M)$ such that the map*

$$\dot{\Delta} \colon \mathcal{V} \times \mathcal{G} \to C^\infty(M)$$

is surjective with a smooth tame right inverse. Moreover, the set $\mathcal{N}$ is open dense.

As an immediate consequence, we obtain the following corollary.

Corollary 2.3. *Let M be smooth compact connected manifold with $\dim(M) \geq 3$. Fix a Morse function $u \in C^\infty(M)$. Then the map*

$$\Delta(u) : \mathcal{G}_+ \to C^\infty(M)$$

defined as $\Delta(u)(g) = \Delta_g(u)$ is open.

Proof. This result is an immediate consequence of Theorem 1.8 and Theorem 2.2. □

Proving Theorem 2.2 amounts to study the equation

$$\dot{\Delta}_g[\cdot](u) = v \tag{2.1}$$

when u is Morse and v is any smooth function. To do so, it will be convenient to give an explicit expression for the differential of the Laplacian. We will often omit the Riemannian metrics when denoting gradients if there is no ambiguity.

Recall that a smooth metric g on M induces a fiber metric for all tensor bundle, see [12, Lemma 3.1]. In particular for two smooth $(0,2)$–tensors h_1, h_2 we have

$$\langle h_1, h_2 \rangle_g = g^{ik} g^{j\ell} (h_1)_{ij} (h_2)_{k\ell}.$$

Also recall that one can define the divergence of a $(0,2)$–tensor h using a metric g. It is given by the $(0,1)$–tensor $\operatorname{div}_g(h) = \operatorname{tr}_g(\nabla h)$. Moreover, if α is a $(0,1)$–tensor and g is a Riemannian metric, we denote by $\alpha^\sharp$ the unique vector field such that $\langle \alpha^\sharp, \cdot \rangle_g = \alpha$.

Lemma 2.4 (Expression for the differential)**.** *Let (M, g) be a Riemannian manifold. Then*

$$\dot{\Delta}_g[h](u) = -\langle h, \nabla^2 u \rangle_g + \langle \frac{1}{2} \nabla(\operatorname{tr_g(h)}) - \operatorname{div}_g(h)^\sharp, \nabla u \rangle_g. \tag{2.2}$$

In particular, $\dot{\Delta}$ is smooth tame in q and h. For a conformal variation $h_{ij} = \psi g_{ij}$, then

$$\dot{\Delta}_g[\psi g](u) = -\psi \Delta_g u + \frac{n-2}{2} \langle \nabla \psi, \nabla u \rangle_g. \tag{2.3}$$

Moreover, for a traceless variation $g^{ij} h_{ij} = 0$ we have

$$\dot{\Delta}_g[h](u) = -\operatorname{div}_g(h(\nabla u, \cdot)) \tag{2.4}$$

Finally, let W be a coordinate chart and $\{dx^i\}_{i=1}^n$ be the corresponding coordinate 1–forms. Let $F = F_{ij}dx^i \otimes dx^j$ be a symmetric $(0,2)$–tensor whose entries are constant in coordinates and w be a function compactly supported in W. Then

$$\dot{\Delta}_g[wF](u) = a_q w + b_q(w), \tag{2.5}$$

where

$$a_q = -\langle F, \nabla^2 u\rangle_g + \langle \frac{1}{2}\nabla \operatorname{tr}_g F - \operatorname{div}_g F^\sharp, \nabla u\rangle_g, \quad b_q = \frac{1}{2}\operatorname{tr}_g F \nabla u - F(\nabla u, \cdot)^\sharp.$$

Moreover, a_q and b_q are smooth tame in q.

Proof. See the appendix. □

Remark 2.5 (Conformal equation in dimension 2)**.** Observe that (2.3) becomes significantly simpler when $n = 2$. This will be used in the proof of Proposition 2.8 in dimension 2.

Observe that Theorem 2.2 does not hold in dimension 1. Indeed, the differential of the Laplace–Beltrami operator is never surjective in this case.

Theorem 2.6. *Let M be a smooth compact connected manifold with $\dim(M) = 1$. Given a nonconstant function $u \in C^\infty(M)$, the metric differential $\Delta_g[\cdot](u)$ is not surjective in (g, u) for every $g \in \mathcal{G}_+$.*

Proof. Since every connected smooth compact manifold is diffeomorphic to the circle, we can assume $M = S^1$. Fix $(g, u) \in \mathcal{G}_+ \times C^\infty(S^1)$. Without loss of generality, we can assume $\operatorname{vol}_g(S^1) = 1$. Moreover, by fixing arclength coordinates for g, we can assume that g is the standard Euclidean metric. Also assume that we have fixed an orientation of S^1, so that it makes sense to treat g as a function on S^1. Since there are no nonzero traceless metric variations in dimension 1, the expression for $\Delta_g[\cdot](u)$ reduces to the conformal one

$$\Delta_g[\psi g](u) = -\psi \Delta_g u - \frac{1}{2}\langle \nabla\psi, \nabla u\rangle_g.$$

Under our assumptions, it further simplifies to

$$\Delta_g[\psi g](u) = -\psi u'' - \frac{1}{2}\psi' u' = -\frac{((u')^2\psi)'}{2u'}.$$

We want to show that there exists $f \in C^\infty(S^1)$ such that

$$((u')^2\psi)' = 2fu'$$

has no solution ψ. Let $\Omega \subseteq S^1$ be the set of points where u' is nonzero. Since u is nonconstant, this is a nonempty open set. Fix a connected component $\tilde{\Omega}$ of Ω. Since u is not constant, it has at least 2 critical points. In particular, the boundary of $\tilde{\Omega}$ is given by two distinct points x_1 and x_2. For every $\psi \in C^\infty(S^1)$ we have

$$\int_{\tilde{\Omega}} 2u'\Delta_g[\psi g](u) = -\int_{\tilde{\Omega}} ((u')^2\psi)' = \pm(u')^2(x_2)\psi(x_2) \mp (u')^2(x_1)\psi(x_1) = 0.$$

Thus, if $\Delta_g[\cdot](u) = f$ has a solution, then f must verify the equation

$$\int_{\tilde{\Omega}} u' f = 0.$$

This is clearly a nontrivial condition, thus $\Delta_g[\cdot](u)$ is not surjective. □

Remark 2.7. A finite regularity version of Theorem 2.6 can be proven in the same way.

2.1. **Solution on jets of the linearized Laplace–Beltrami operator at critical points.** The behavior of (2.1) is very different around critical points of u. Thus, we split the proof of Theorem 2.2 in 2 parts. First, we will build a smooth tame right inverse in a neighborhood of the critical points of u_0. In particular, we prove the following.

Proposition 2.8 (Solution around critical points)**.** *Let $q_0 = (g_0, u_0) \in \mathcal{N}$. Then there exists an open set $W_{\mathrm{crit}} \subseteq M$ with smooth boundary, a neighborhood $\mathcal{V}$ of q_0 in $\mathcal{G}_+ \times C^\infty(M)$, and a smooth tame map*

$$R_{\mathrm{crit}} \colon \mathcal{V} \times C^\infty(M) \to \mathcal{G}$$

as follows. For every $(g, u) \in \mathcal{V}$, the set $\mathrm{crit}(u)$ is contained in a fixed compact subset of W_{crit} with smooth boundary. Moreover,

$$\dot{\Delta} \circ R_{\mathrm{crit}} v(x) = v(x) \quad \text{for every } x \in W_{\mathrm{crit}}, v \in C^\infty(M). \tag{2.6}$$

We will prove Proposition 2.8 by constructing an appropriate solution of (2.1) using the method of characteristics. However, such a solution only converges when the d–th jet of the datum v in the critical points of u is 0 for a fixed d sufficiently large. In this subsection, we reduce the problem to this case by solving (2.1) in its jet. We first record the smooth and tame continuity of the critical points of u.

Lemma 2.9 (Persistence of Morse critical points)**.** *Let $u_0 \in C^\infty(M)$ be Morse, with critical points*

$$\mathrm{crit}(u_0) = \{x_1, \dots, x_m\}.$$

Then there exist pairwise disjoint open neighborhoods $W_i \ni x_i$, a neighborhood $\mathcal{V}$ of u_0, and smooth tame maps

$$x_i \colon \mathcal{V} \longrightarrow W_i, \qquad x_i(u_0) = x_i,$$

such that, for every $u \in \mathcal{V}$,

$$\mathrm{crit}(u) = \{x_1(u), \dots, x_m(u)\},$$

and each $x_i(q)$ is a nondegenerate critical point of u. In particular, $x_i(q)$ is the unique critical point of u in W_i.

Proof. Choose the neighborhoods W_i pairwise disjoint and so small that x_i is the only critical point of u_0 in $\overline{W_i}$. Working in a chart and writing the equation $Du(x) = 0$ in coordinates, the Morse condition says that the differential of the left-hand side with respect to x, namely $\nabla^2 u_0(x_i)$, is invertible. The implicit function theorem with parameters therefore produces, for q in a neighborhood of q_0, a unique solution $x_i(q) \in W_i$, depending smoothly on q. Since $x_i(q)$ is obtained by solving a finite-dimensional equation whose coefficients are finitely many derivatives of u, the map $q \mapsto x_i(q)$ is smooth tame. Shrinking $\mathcal{V}$ further, $\nabla^2 u(x_i(q))$ stays invertible by continuity.

Finally, Du_0 does not vanish on the compact set $M \setminus \bigcup_i W_i$; hence Du does not vanish there either, provided u is close enough to u_0 in $C^1(M)$. □

Let u be any Morse function and x_u be a nondegenerate critical point of u. Choose geodesic normal coordinates centered at p. For simplicity, identify p with the origin of $\mathbb{R}^n$. Set

$$H \coloneqq \nabla^2 u(0),$$

which at a critical point coincides with the Euclidean Hessian of the coordinate representation of u. Since u is Morse, it is a nonsingular symmetric matrix. In these coordinates,

$$g_{ij}(x) = \delta_{ij} + O(|x|), \qquad Du(x) = Hx + O(|x|^2), \qquad \nabla^2 u(x) = H + O(|x|).$$

For $d \geq 0$, let $\mathcal{P}_d$ denote the space of homogeneous polynomials of degree d on $\mathbb{R}^n$. Let $\mathcal{S}_d$ denote the space of symmetric-matrix-valued homogeneous polynomials of degree d, that is, of tensors $h = (h_{ij})$ with $h_{ij} = h_{ji} \in \mathcal{P}_d$. If $h \in \mathcal{S}_d$, then

$$\operatorname{tr}_g h = \operatorname{tr} h + O(|x|^{d+1}), \qquad \operatorname{div}_g h = \operatorname{div} h + O(|x|^d), \tag{2.7}$$

where the operations on the right-hand side are the Euclidean ones. Substituting these expansions into (2.2) gives

$$\dot{\Delta}_g[h](u) = A_{H,d}h + O(|x|^{d+1}), \qquad h \in \mathcal{S}_d, \tag{2.8}$$

where

$$A_{H,d}h := -\langle H, h\rangle + \tfrac{1}{2}\big\langle Hx, D(\operatorname{tr} h)\big\rangle - \big\langle \operatorname{div} h, Hx\big\rangle. \tag{2.9}$$

This is because $D(\operatorname{tr} h)$ and $\operatorname{div} h$ are homogeneous of degree $d-1$, while Hx is homogeneous of degree 1. Thus, $A_{H,d}h$ is exactly the homogeneous component of degree d of $\dot{\Delta}_g[h](u)$, and (2.9) defines a linear map

$$A_{H,d}\colon \mathcal{S}_d \longrightarrow \mathcal{P}_d.$$

Observe that $A_{H,d}$ depends only on H: all the information carried by g and by the higher-order Taylor coefficients of u has been discarded. This is the model to which the equation reduces at the critical point, and the whole subsection rests on the following algebraic fact.

Proposition 2.10 (The homogeneous polynomial model)**.** *Let $H \in \operatorname{Sym}(n)$ be nonsingular, $n \geq 2$. Then*

$$A_{H,d}\colon \mathcal{S}_d \longrightarrow \mathcal{P}_d$$

is surjective for every $d \geq 0$. Moreover, if H_0 is fixed, right inverses $R_{H,d}$ of $A_{H,d}$ can be chosen so as to depend smoothly on H in a neighborhood of H_0, with locally uniformly bounded operator norm.

Proof. For $Q \in O(n)$, define $(Q^*h)(x) = Q^T h(Qx)Q$. A direct change of coordinates gives

$$A_{Q^T HQ,d}(Q^*h) = (A_{H,d}h) \circ Q.$$

We may therefore assume that

$$H = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \qquad \lambda_i \neq 0 \qquad \text{for every } i.$$

Since the monomials x^α with $|\alpha| = d$ span $\mathcal{P}_d$, it suffices to prove that each of them lies in the image of $A_{H,d}$. Throughout the proof we set

$$S_\alpha := \sum_{k=1}^{n} \lambda_k \alpha_k.$$

We first consider diagonal tensors. Fix i and let $h \in \mathcal{S}_d$ be the tensor whose only nonvanishing component is $h_{ii} = x^\alpha$. Then

$$\operatorname{tr} h = x^\alpha, \qquad \langle H, h\rangle = \lambda_i x^\alpha, \qquad (\operatorname{div} h)_j = \delta_{ij}\, \partial_i x^\alpha,$$

hence

$$\tfrac{1}{2}\langle Hx, Dx^\alpha\rangle = \tfrac{1}{2} S_\alpha\, x^\alpha, \qquad \langle \operatorname{div} h, Hx\rangle = \lambda_i \alpha_i\, x^\alpha.$$

Therefore

$$A_{H,d}h = c_i(\alpha)\, x^\alpha, \tag{2.10}$$

where $c_i(\alpha) := \frac{1}{2}S_\alpha - \lambda_i(\alpha_i + 1)$. If $c_i(\alpha) \neq 0$ for some i, then x^α belongs to the image of $A_{H,d}$. Call α *exceptional* for H when this fails, that is when

$$\lambda_i(\alpha_i + 1) = \tfrac{1}{2}S_\alpha \qquad \text{for every } i = 1, \dots, n. \tag{2.11}$$

Assume now that α is exceptional. We want to show that x^α belongs to the image of $A_{H,d}$. Since the λ_i are nonzero, (2.11) implies $S_\alpha \neq 0$. Multiplying the i-th identity by $\frac{\alpha_i}{\alpha_i+1}$ gives

$$\lambda_i \alpha_i = \frac{S_\alpha}{2}\,\frac{\alpha_i}{\alpha_i+1}.$$

Summing over i and dividing by $S_\alpha \neq 0$ gives

$$\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_i+1} = 2. \tag{2.12}$$

Each summand lies in $[\frac{1}{2}, 1)$ when $\alpha_i \geq 1$ and vanishes otherwise, so (2.12) has at least three and at most four nonzero entries. In particular $n \geq 3$, $|\alpha| \geq 4$, and there is an index j with $\alpha_j \geq 1$.

Fix a j as above, and pick any index $i \neq j$. Define

$$\gamma := \alpha + e_i - e_j \in \mathbb{N}^n,$$

and let $h \in \mathcal{S}_d$ be the tensor whose only nonvanishing components are $h_{ij} = h_{ji} = x^\gamma$. Since H is diagonal and h is off-diagonal,

$$\operatorname{tr} h = 0, \qquad \langle H, h \rangle = 0,$$

while

$$(\operatorname{div} h)_i = \partial_j x^\gamma, \qquad (\operatorname{div} h)_j = \partial_i x^\gamma, \qquad (\operatorname{div} h)_k = 0 \quad (k \neq i, j).$$

If $\alpha_j = 1$, direct computation gives $A_{H,d}h = -\lambda_j(\alpha_i+1)x^\alpha$, which proves the claim. Thus, suppose $\alpha_j \geq 2$ and define $\beta = \alpha + 2e_i - 2e_j \in \mathbb{N}^n$. Using $x_i\,\partial_j x^\gamma = (\alpha_j - 1)x^\beta$ and $x_j\,\partial_i x^\gamma = (\alpha_i+1)x^\alpha$, we get

$$A_{H,d}h = -\lambda_j(\alpha_i+1)\,x^\alpha - \lambda_i(\alpha_j-1)\,x^\beta. \tag{2.13}$$

The coefficient $a = \lambda_j(\alpha_i+1)$ is nonzero, and $|\beta| = d$.

We claim that β is not exceptional. Suppose it were. Since $n \geq 3$ we may choose $k \notin \{i, j\}$, which gives $\beta_k = \alpha_k$. Then (2.11) applied to β and to α gives

$$\tfrac{1}{2}S_\beta = \lambda_k(\beta_k+1) = \lambda_k(\alpha_k+1) = \tfrac{1}{2}S_\alpha.$$

On the other hand, the identities for the index i give

$$\tfrac{1}{2}S_\beta = \lambda_i(\beta_i+1) = \lambda_i(\alpha_i+3), \qquad \tfrac{1}{2}S_\alpha = \lambda_i(\alpha_i+1).$$

This implies $2\lambda_i = 0$, a contradiction since H_0 is nonsingular. Therefore $c_\ell(\beta) \neq 0$ for some ℓ, and Step 1 provides $\widetilde{h} \in \mathcal{S}_d$ with $A_{H,d}\widetilde{h} = x^\beta$. Setting $b := \lambda_i(\alpha_j - 1)$, (2.13) yields

$$A_{H,d}\Big(-\tfrac{1}{a}\big(h + b\,\widetilde{h}\big)\Big) = x^\alpha.$$

This proves that every monomial of degree d is in the image, and hence that $A_{H,d}$ is surjective.

We now prove the last part of the statement. Fix a right inverse $R_{H_0,d}$ of $A_{H_0,d}$. The map $H \mapsto A_{H,d}$ is linear, hence continuous, so for H close to H_0 the endomorphism $A_{H,d}R_{H_0,d}$ of $\mathcal{P}_d$ is close to the identity and therefore invertible. Consequently

$$R_{H,d} := R_{H_0,d}\big(A_{H,d}R_{H_0,d}\big)^{-1}$$

is a right inverse of $A_{H,d}$; it depends smoothly on H and its operator norm is bounded on a neighborhood of H_0. □

We now pass from homogeneous polynomials to jets. For a smooth function or tensor field F defined near the origin, we denote by $[F]_d$ the homogeneous component of degree d of its Taylor expansion at the origin, and by $j_x^N F$ the N-jet of F at x. Let $\mathcal{P}_{\le N}$ and $\mathcal{S}_{\le N}$ denote respectively scalar and symmetric-tensor-valued polynomials of degree at most N. From now till the end of this subsection, we'll implicitly always used the coordinates above.

Lemma 2.11 (Triangularity of the linearization)**.** *Let $u \in C^\infty(M)$ and x_u be a nondegenerate critical point of u. If $h = h_0 + \cdots + h_N$ with $h_d \in \mathcal{S}_d$, then for every $0 \le d \le N$*

$$\big[\dot\Delta_g[h](u)\big]_d = A_{H,d} h_d + \Phi_d(h_0, \dots, h_{d-1}),$$

where Φ_d is linear in its arguments and its coefficients are polynomial expressions in the Taylor coefficients of g, g^{-1} and u at p, of order at most $d+2$.

Proof. Since $Du(p) = 0$, the operator $h \mapsto \dot\Delta_g[h](u)$ does not decrease the order of vanishing at p: the zero-order term $-\langle h, \nabla^2 u\rangle_g$ clearly does not, and in the first-order term the derivatives of h are multiplied by $Du = O(|x|)$. Hence each h_e contributes only to the homogeneous components of degree $\ge e$. By (2.8) its contribution in degree exactly e is $A_{H,e} h_e$. Collecting the terms of degree d gives the stated formula, with Φ_d accounting for the contributions of $h_0, \dots, h_{d-1}$. The last assertion follows by inspection of (2.2), since $\dot\Delta_g[h](u)$ involves g, g^{-1}, Dg, Du and $\nabla^2 u$. □

Proposition 2.12 (Solution on jets)**.** *Let $q_0 = (g_0, u_0) \in \mathcal{N}$, and let $N \ge 0$. Then there exist a neighborhood $\mathcal{V}$ of q_0 in $\mathcal{N}$ and a map*

$$R_{\mathrm{jet},N} \colon \mathcal{V} \times C^\infty(M) \longrightarrow \mathcal{G},$$

linear in the second variable and smooth tame, such that

$$j_x^N\Big(\dot\Delta_g\big[R_{\mathrm{jet},N}(q,v)\big](u)\Big) = j_x^N v \qquad \text{for every } x \in \mathrm{crit}(u), \tag{2.14}$$

for every $q = (g,u) \in \mathcal{V}$ and every $v \in C^\infty(M)$. Moreover, $R_{\mathrm{jet},N}(q,v)$ is supported in a fixed compact neighborhood of $\mathrm{crit}(u_0)$, and for every $m \ge 0$ there is a constant $C_{m,N} > 0$, independent of $q \in \mathcal{V}$, such that

$$\big\|R_{\mathrm{jet},N}(q,v)\big\|_{C^m(M)} \le C_{m,N}\, \|v\|_{C^N(M)}. \tag{2.15}$$

Proof. We first work near a single critical point. Let W_i, $\mathcal{V}$ and $x_i \colon \mathcal{V} \to W_i$ be as in Lemma 2.9, and fix i. Let x be a fixed system of g_0-normal coordinates centered at $x_i(q_0)$ and defined on W_i, and for $q \in \mathcal{V}$ use the translated coordinates

$$y \coloneqq x - x\big(x_i(q)\big),$$

which are centered at the critical point $x_i(q)$ of u. Define

$$L_N(q) \colon \mathcal{S}_{\le N} \longrightarrow \mathcal{P}_{\le N},$$

where $L_N(q)h$ is the Taylor polynomial of degree N of $\dot\Delta_g[h](u)$ at $y = 0$.

Write $h = h_0 + \cdots + h_N$ and $P = P_0 + \cdots + P_N \in \mathcal{P}_{\le N}$. By Lemma 2.11, the equation $L_N(q_0)h = P$ is equivalent to the triangular system

$$A_{H,d} h_d = P_d - \Phi_d(h_0, \dots, h_{d-1}), \qquad d = 0, \dots, N,$$

where $H = \nabla^2 u_0(x_i(q_0))$. Since u_0 is Morse, H is nonsingular, and Proposition 2.10 solves each equation of the system. Solving them recursively, starting from $d = 0$, produces h with $L_N(q_0)h = P$. Hence $L_N(q_0)$ is surjective; let R_N^0 be a right inverse.

All the spaces involved are finite-dimensional, and by Lemma 2.11 the operator $L_N(q)$ depends only on the jets $j^{N+1}_{x_i(q)} g$ and $j^{N+2}_{x_i(q)} u$, evaluated at the point $x_i(q)$, which varies smoothly

and tamely with q by Lemma 2.9. Consequently $q \mapsto L_N(q)$ is smooth, and continuous for the C^{N+2} topology. Shrinking $\mathcal{V}$ to a C^{N+2}-ball around q_0, we may assume that $L_N(q)R_N^0$ is invertible on $\mathcal{P}_{\leq N}$ for every $q \in \mathcal{V}$, with inverse bounded uniformly in q. Then

$$R_N(q) \coloneqq R_N^0\big(L_N(q)R_N^0\big)^{-1}$$

is a right inverse of $L_N(q)$, depending smoothly on q, with operator norm bounded uniformly on $\mathcal{V}$.

Let ρ_i be a cutoff function supported in W_i and identically 1 on a neighborhood K_i of $x_i(\mathcal{V})$, fixed once and for all, and set

$$R_{\mathrm{jet},N}(q,f) \coloneqq \sum_{i=1}^{m} \rho_i \cdot R_N^{(i)}(q)\Big(j^N_{x_i(q)}f\Big),$$

where $R_N^{(i)}(q)$ is the operator constructed above at the i-th critical point, and where the polynomial tensor $R_N^{(i)}(q)(j^N_{x_i(q)}f)$, written in the coordinates y, is regarded as a smooth symmetric tensor on W_i and extended by zero. The map is linear in f by construction, and (2.14) holds because ρ_i is identically 1 near $x_i(q)$ and the neighborhoods W_i are pairwise disjoint.

The tensor $R_{\mathrm{jet},N}(q,f)$ is a polynomial of degree at most N multiplied by a fixed cutoff, and its coefficients depend linearly on $j^N_{x_i(q)}f$, with a matrix bounded uniformly on $\mathcal{V}$. Since all the C^m norms on the finite-dimensional space $\mathcal{S}_{\leq N}$ restricted to the fixed compact sets $\operatorname{supp}\rho_i$ are equivalent, and since $|j^N_x f| \leq \|f\|_{C^N(M)}$, we obtain (2.15). In particular, $R_{\mathrm{jet},N}$ is a tame map, with loss of N derivatives. Its derivatives with respect to q are again of the same form, namely finite-dimensional expressions in the jets of g and u at $x_i(q)$ composed with the jet evaluation map; hence they satisfy analogous estimates, and $R_{\mathrm{jet},N}$ is smooth tame. □

2.2. **Right inverse of the linearized Laplace–Beltrami operator at critical points.** We can now give a complete solution of (2.1) in a neighborhood of critical point of (g_0, u_0). We will assume $\dim(M) > 2$ for now. The 2–dimensional case is significantly different and will be treated separately.

A key step in proving smoothness of the solution will be estimating appropriately the flow of a vector field. These estimates are given by Lyapunov theory, and are recorded in the following lemma.

Lemma 2.13 (Lyapunov estimates)**.** *Let X_0 and X_1 be smooth vector fields on $\mathbb{R}^n$. Let x_0 be an isolated critical point of X_0 and $\mu > 0$. Assume all eigenvalues of $JX_0(x_0)$ have real parts strictly smaller than $-\mu$. Then there exists $\delta > 0$ depending only on X_0 and a bounded neighborhood U of x_0 as follows. Denoting by Φ_t^1 the flow of X_1, we have that, if $\|X_0 - X_1\|_{C^1} < \delta$, the the following hold:*

1. *(Positive invariance) For every $t \geq 0$, we have $\Phi_t^1(x) \in U$. In particular, $\Phi_t^1(x)$ is defined for every positive t;*
2. *(Uniqueness of critical point) X_1 has a unique critical point x_1 in U;*
3. *(Exponential estimates) For every $m \geq 1$, there exists $c_m = c(m, X_0, \delta) > 0$ such that, if $x \in U$ and $\|X_0 - X_1\|_{C^1} < \delta$, then*

$$\|\Phi_t^1(x) - x_1\| \leq c_0 e^{-\mu t} \quad \text{and} \quad \|D^m\Phi_t^1(x)\| \leq c_m e^{-\mu t}(1 + \|X_1\|_m), \tag{2.16}$$

where $\|\cdot\|_m$ is the C^m norm on $\overline{U}$.

A neighborhood U with these properties is said to be Φ_t^1-positively invariant.

Proof. See the appendix. □

We will also need the following technical lemma on matrices. Denote by Sym_n^* the set of invertible symmetric matrices of size n.

Lemma 2.14. *Let $n \neq 2$ and H be an invertible symmetric matrix of size n. Then there exists a matrix F such that the eigenvalues of matrix*

$$B(H,F) = \frac{1}{2}\operatorname{tr}(F)H - FH,$$

are all distinct and strictly negative. Moreover, there exists an open neighborhood $\mathcal{H}$ of H in Sym_n and a positive constant $\mu > 0$ such that all the eigenvalues of $B(H',F)$ are strictly smaller than $-\mu$ for every $H' \in \mathcal{H}$.

Proof. Let H be a nonsingular symmetric matrix. We look for F as in the statement that diagonalizes with H. In particular, we can assume that both F and H are diagonal. Let $F = \operatorname{diag}(f_1,\dots,f_n)$ and $H = \operatorname{diag}(h_1,\dots,h_n)$. In particular, the eigenvalues of $B(H,F)$ are given by

$$\sigma_j = \frac{1}{2}h_j \sum_{i=1}^{n} f_i - h_j f_j. \tag{2.17}$$

The vector σ is obtained from f by an invertible linear map. Indeed denote by J be the $n \times n$ matrix where every entry is 1. The matrix $\frac{1}{2}J - \mathrm{Id}$ is invertible if and only if n is different from 2. Observing that σ is given by

$$\sigma = H\left(\frac{1}{2}J - \mathrm{Id}\right) f$$

proves the first part of the statement. The second part of the statement is given by the continuity of $B(\cdot,F)$ in the first entry. □

We now ready for the proof of Proposition 2.8. We need to construct a solution of

$$\dot{\Delta}_g[h](u) = v$$

in a neighborhood of the critical points of u. The strategy is the following. First, with Proposition 2.12, we reduce to the case where $j_x^N v = 0$ at each critical point x of u. Then we reduce to a scalar equation using the ansatz $h = wF$, where w is a smooth compactly supported function and F is a constant symmetric matrix. The function w will be found using the method of characteristics. Since the characteristics degenerate in a sink, we will need to prove that the constructed function is smooth. Tame estimates will follow naturally.

proof of Proposition 2.8 for $\dim(M) > 2$. As for the proof of Proposition 2.12, we reduce the problem to a neighborhood of a single critical point x_{u_0} of u_0. Let (W,φ) be a geodesic normal chart with smooth boundary centered at x_{u_0}. For now, to simplify notation, we will treat W as a bounded open subset of $\mathbb{R}^n$. Let $\mathcal{V}$ be a neighborhood of q_0. We can assume that W contains a single critical point x_u of u for every $q = (g,u)$ in $\mathcal{V}$. We can further assume that every x_u is contained in a fixed compact set $K \subseteq W$ with smooth boundary and nonempty interior. Let F be a constant symmetric matrix. We look for a solution of (2.1) in the form wF, where w is a compactly supported smooth function. By (2.5), we can express (2.1) as

$$a_q w + b_q(w) = v, \tag{2.18}$$

where a_q is a function, b_q is a vector field and $v \in C^\infty(W)$. By Lemma 2.14, we can pick F such that b_{q_0} is nondegenerate in x_u and its Jacobian is negative definite. Up to shrinking $\mathcal{V}$, we can take a uniform constant $\mu > 0$ such that the eigenvalues of $B_q = Db_q(x_u)$ are distinct and smaller than $-\mu$ for every $q \in \mathcal{V}$. Since we will mainly be interested in solving (2.18) in K, we can assume that $v \in C^\infty(\overline{W})$ up to shrinking W. For now, we will simply treat

these functions as functions on a subset of $\mathbb{R}^n$. Moreover, to relax notation, we will omit the dependence on F unless relevant. Denote by $\Phi_{q,t}$ the flow of b_q. Up to a shrinking of $\mathcal{V}$, Lemma 2.13 gives a bounded neighborhood U of 0 which is $\Phi_{q,t}$–positively invariant for every $q \in \mathcal{V}$. We can assume $U = W$. Thus, for each integer $m \geq 1$, there exists a positive constant c_m such that

$$\|\Phi_{q,t}(x) - x_u\| \leq c_0 e^{-\mu t} \quad \text{and} \quad \|D^m \Phi_{q,t}(x)\| \leq c_m e^{-\mu t}(1 + \|b_q\|_m). \tag{2.19}$$

Moreover, x_u is the only critical point of b_q for every $q = (g, u) \in \mathcal{V}$. We can also assume that the C^1–norms of a_q and b_q are uniformly bounded by a positive constant C.
We now solve equation (2.18) on W. First, assume that v verifies $j^d_{x_u}(v) = 0$ for a fixed integer $d > \frac{C}{\mu}$. Define

$$I_q(t, x) \coloneqq \exp\left(\int_0^t a_q(\Phi_{q,\tau}(x))d\tau\right).$$

We show that the following function is a well defined smooth solution of (2.18) in W

$$w_q(v)(x) = -\int_0^\infty v(\Phi_{q,t}(x))I_q(t, x)dt. \tag{2.20}$$

Smoothness in x: Let m be a non-negative integer. Let $x \in W$. We want to show that

$$\int_0^\infty \|D^m(v(\Phi_{q,t}(x))I_q(t, x))\|dt \tag{2.21}$$

converges in W. This will prove that w_q is a smooth function in x. From now on, we will denote by C_m every constant depending only on m, μ, n, C and W. We will also denote the C^j norms on $C^\infty(\overline{W})$ by simply $\|\cdot\|_j$. We have

$$\|D^m(I_q(t, x)v(\Phi_{q,t}(x)))\| \leq C_m \sum_{m_1+m_2=m} \|D^{m_1} I_q(t, x)\| \cdot \|D^{m_2}(v(\Phi_{q,t}(x))\|. \tag{2.22}$$

We start by bounding the derivatives of I_q. First, observe

$$I_q(t, x) \leq e^{Ct}.$$

We now assume $m_1 \geq 1$. We have

$$\|D^{m_1} I_q(t, x)\| \leq C_m \sum_{k=1}^{m_1} I_q(t, x) \sum_{j_1+\ldots+j_k=m_1} \int_0^t \|D^{j_1} G_{q,\tau}(x)\|d\tau \cdots \int_0^t \|D^{j_k} G_{q,\tau}(x)\|d\tau, \tag{2.23}$$

where $G_{q,\tau}(x) = a_q(\Phi_{q,\tau}(x))$ and the j-indices are positive integers. Consider now the derivatives of G_q. Since composition of maps is a degree 1 tame operation [10, Lemma 2.3.4], we get

$$\|D^j G_q(x, \tau)\| \leq C_m(1 + \|\Phi_{q,\tau}\|_j + \|a_q\|_j) \leq C_m(1 + e^{-\mu\tau}\|b_q\|_j + \|a_q\|_j).$$

Integrating, we obtain

$$\int_0^t \|D^j G_q(x, \tau)\|d\tau \leq C_m \int_0^t (1 + e^{-\mu\tau}\|b_q\|_j + \|a_q\|_j)d\tau \leq C_m(t + \|b_q\|_j + t\|a_q\|_j).$$

Thus,

$$\int_0^t \|D^{j_1} G_{q,\tau}(x)\|d\tau \cdots \int_0^t \|D^{j_k} G_{q,\tau}(x)\|d\tau \leq C_m \prod_{i=1}^k (t + \|b_q\|_{j_i} + t\|a_q\|_{j_i}) \tag{2.24}$$

We want to interpolate between the C^j norms of a_q. We distinguish 2 cases. First, assume $m_1 = 1$. Then $k = m_1 = 1$. Thus, there is a single j to consider. Then (2.24) becomes

$$\int_0^t \|D^1 G_{q,\tau}(x)\| d\tau \leq C_m(t + \|b_q\|_1 + t\|a_q\|_1) \leq C_m P_1(t)(1 + \|b_q\|_1 + \|a_q\|_1),$$

where $P_1(t) = 1 + t$.

Assume now $m_1 > 1$. Given a positive integer i and a finite set P, we denote by $\binom{P}{k}$ the collection of subsets of i of cardinality i. If $P = \{1, \dots, k\}$ is the collection of the first k natural numbers, we'll write $\binom{P}{k}$ as $\binom{k}{i}$. We can write (2.24) as

$$\begin{aligned} \prod_{i=1}^k \int_0^t \|D^{j_i} G_{q,\tau}(x)\| d\tau &\leq C_m \prod_{i=1}^k (t + \|b_q\|_{j_i} + t\|a_q\|_{j_i}) \\ &= C_m \left(t^k + \sum_{i=1}^k t^{k-i} \sum_{P \in \binom{k}{i}} \prod_{f \in P} (\|b_q\|_{j_f} + t\|a_q\|_{j_f}) \right). \end{aligned} \tag{2.25}$$

Fix $P \in \binom{k}{i}$. We have

$$\prod_{f \in P} (\|b_q\|_{j_f} + t\|a_q\|_{j_f}) = \prod_{f \in P} \|b_q\|_{j_f} + \sum_{\ell=1}^i t^\ell \sum_{P' \in \binom{P}{\ell}} \left(\prod_{\ell' \in P'} \|a_q\|_{j_{\ell'}} \cdot \prod_{\ell' \in (P')^c} \|b_q\|_{j'_\ell} \right) \tag{2.26}$$

By Proposition 1.6, we obtain

$$\|a_q\|_j \leq C_m \|a_q\|_1^{\frac{m_1 - j}{m_1 - 1}} \|a_q\|_{m_1}^{\frac{j-1}{m_1-1}} \leq C_m \|a_q\|_{m_1}^{\frac{j-1}{m_1-1}},$$

where we have incorporated $\|a_q\|_1 \leq C$ in the constant in the last step. The same inequality holds for the j–norms of b_q. This means that for $P' \in \binom{P}{\ell}$ we have

$$\prod_{\ell' \in P'} \|a_q\|_{j_{\ell'}} \prod_{\ell' \in (P')^c} \|b_q\|_{j'_\ell} \leq C_m \|a_q\|_{m_1}^{\frac{-\ell + \sum_{\ell' \in P'} j_{\ell'}}{m_1 - 1}} \|b_q\|_{m_1}^{\frac{-i + \ell + \sum_{\ell' \in (P')^c} j_{\ell'}}{m_1 - 1}} = C_m \|a_q\|_{m_1}^{S(P')} \|b_q\|_{m_1}^{S((P')^c)}, \tag{2.27}$$

where $S(P')$ and $S((P')^c)$ are the corresponding exponents. We have the inequality

$$S(P') + S((P')^c) \leq 1.$$

Indeed, only set of indices j with total sum smaller than m_1 appear in the summands for $S(P')$ and $S((P')^c)$. This means that

$$(m_1 - 1)(S(P') + S((P')^c)) = -\ell + \sum_{\ell' \in P'} j_{\ell'} - i + \ell + \sum_{\ell' \in (P')^c} j_{\ell'} = -i + \sum_{f \in P} j_f \leq m_1 - 1.$$

Then

$$1 - S(P') \geq S(P') + S((P')^c) - S(P') = S((P')^c).$$

This gives

$$\begin{aligned} \|a_q\|_{m_1}^{S(P')} \|b_q\|_{m_1}^{S((P')^c)} &\leq \|a_q\|_{m_1}^{S(P')} \left(1 + \|b_q\|_{m_1}^{1 - S(P'))}\right) \\ &\leq \|a_q\|_{m_1}^{S(P')} + C_m(\|a_q\|_{m_1} + \|b_q\|_{m_1}) \\ &\leq 1 + \|a_q\|_{m_1} + C_m(\|a_q\|_{m_1} + \|b_q\|_{m_1}) \\ &\leq C_m(1 + \|a_q\|_{m_1} + \|b_q\|_{m_1}), \end{aligned} \tag{2.28}$$

where in the second inequality we have used

$$\|a_q\|_{m_1}^{S(P')}\|b_q\|_{m_1}^{1-S(P')} \leq C_m(\|a_q\|_{m_1} + \|b_q\|_{m_1}).$$

This is a consequence of the convexity of $-\log$. Plugging everything into equation (2.23), we obtain

$$\|D^{m_1} I_q(t,x)\| \leq C_m(1 + \|a_q\|_{m_1} + \|b_q\|_{m_1})P_{m_1}(t)e^{tC}, \tag{2.29}$$

where $P_{m_1}(t)$ is a polynomial. The same inequality also holds for $m_1 = 0$.

We now study the terms of type $\|D^{m_2}(v(\Phi_{q,t}(x))\|$. Assume $m_2 \geq 1$. The case $m_2 = 0$ will be a direct consequence of the following computations. We have

$$\|D^{m_2}(v(\Phi_{q,t}(x))\| \leq C_m \sum_{k=1}^{m_2} \sum_{j_1+\ldots+j_k=m_2} \|D^k v(\Phi_{q,t}(x))\| \cdot \|D^{j_1}\Phi_{q,t}(x)\| \cdots \|D^{j_k}\Phi_{q,t}(x)\|,$$

where the j are positive integers. Applying Lemma 2.13 gives

$$\|D^{m_2}(v \circ \Phi_{q,t})(x)\| \leq C_m \sum_{k=1}^{m_2} \sum_{j_1+\ldots+j_k=m_2} \|D^k v(\Phi_{q,t}(x))\| e^{-k\mu t} \prod_{i=1}^{k}(1 + \|b_q\|_{j_i}). \tag{2.30}$$

Consider the products in the right hand side (2.30). It is a sum of terms in the form $\|b_q\|_{j_{i_1}} \cdots \|b_q\|_{j_{i_\ell}}$, where the j are positive integers with $j_{i_1} + \cdots + j_{i_\ell} \leq m_2 - (k - \ell)$. The $k - \ell$ term is given because all the $k - \ell$ indices that don't appear in the summand are positive. Assume that m_2 is different from 1. The case $m_2 = 1$ give the same result with easier computations. As in (2.27), we have

$$\|b_q\|_{j_{i_1}} \cdots \|b_q\|_{j_{i_\ell}} \leq C_m \|b_q\|_{m_2}^{\frac{-\ell+\sum j_f}{m_2-1}} \leq C_m \left(1 + \|b_q\|_{m_2}^{\frac{m_2-k}{m_2-1}}\right), \tag{2.31}$$

where the last inequality follows by $j_{i_1} + \cdots + j_{i_\ell} \leq m_2 - (k - \ell)$. We now give estimates to the k-th derivative of v. By Proposition 1.6, we obtain

$$\|D^k v(\Phi_{q,t}(x))\| \leq \|v\|_k \leq C_m \|v\|_1^{\frac{m_2-k}{m_2-1}} \|v\|_{m_2}^{\frac{k-1}{m_2-1}}.$$

We are interested only in tameness and convergence of (2.21). Thus, we can assume $\|v\|_1$ to be bounded by a constant. This gives

$$\|D^k v(\Phi_{q,t}(x))\| \leq C_m \|v\|_{m_2}^{\frac{k-1}{m_2-1}}. \tag{2.32}$$

In particular,

$$\begin{aligned} \|v\|_{m_2}^{\frac{k-1}{m_2-1}} \left(1 + \|b_q\|_{m_2}^{\frac{m_2-k}{m_2-1}}\right) \leq & \|v\|_{m_2}^{\frac{k-1}{m_2-1}} + \|v\|_{m_2}^{\frac{k-1}{m_2-1}} \|b_q\|_{m_2}^{\frac{m_2-k}{m_2-1}} \\ \leq & 1 + \|v\|_{m_2} + \|v\|_{m_2}^{\frac{k-1}{m_2-1}} \|b_q\|_{m_2}^{\frac{m_2-k}{m_2-1}} \\ \leq & C_m(1 + \|v\|_{m_2} + \|b_q\|_{m_2}), \end{aligned} \tag{2.33}$$

where in the last equality we used

$$\|v\|_{m_2}^{\frac{k-1}{m_2-1}} \|b_q\|_{m_2}^{\frac{m_2-k}{m_2-1}} \leq C_m(\|v\|_{m_2} + \|b_q\|_{m_2}).$$

The estimate (2.33) gives a bound for the summands in (2.30). However, we need a different estimate for $k \leq d$. The condition $j_{x_u}^d(v) = 0$ gives

$$\|D^k v(x)\| \leq C_m \|x - x_u\|^{d-k} \|v\|_d.$$

We can assume the d-norm of v to be bounded by a constant C_m. This is because we are only interested in tameness and convergence of (2.21). Therefore,

$$\begin{aligned}\|D^k v(\Phi_{q,t}(x))\|e^{-k\mu t} \leq & C_m\|v\|_d\|\Phi_{q,t}(x)-x_u\|^{d-k}e^{-k\mu t}\\ \leq & C_m e^{-d\mu t},\end{aligned} \tag{2.34}$$

where the last inequality follows from (2.16). Observe that the right hand side does not depend on k. Then, applying (2.33) and (2.34) to (2.30) gives

$$\begin{aligned}\|D^{m_2}(v\circ\Phi_{q,t})(x)\| &\leq C_m\left(de^{-d\mu t}(1+\|b_q\|_{m_2})+\sum_{k=d+1}^{m_2}e^{-k\mu t}(1+\|b_q\|_{m_2}+\|v\|_{m_2})\right)\\ &\leq C_m(1+\|b_q\|_{m_2}+\|v\|_{m_2})\left(de^{-d\mu t}+\sum_{k=d+1}^{m_2}e^{-k\mu t}\right).\end{aligned} \tag{2.35}$$

The bound in (2.35) also holds for $m_2=0$ by (2.34). Combining (2.29) and (2.35) gives the following bound for $\|D^{m_1}I_q(t,x)D^{m_2}(v\circ\Phi_{q,t})(x)\|$

$$C_m(1+\|a_q\|_{m_1}+\|b_q\|_{m_1})(1+\|b_q\|_{m_2}+\|v\|_{m_2})P_{m_1}(t)\left(de^{(C-d\mu)t}+\sum_{k=d+1}^{m_2}e^{(C-k\mu)t}\right). \tag{2.36}$$

Since $P_{m_1}(t)$ is a polynomial and all the exponents appearing in (2.36) are strictly negative, (2.21) converges. The right hand side of (2.36) is integrable on $[0,+\infty)$. Therefore, by dominated convergence $w_q(v)$ is a smooth function on W. Moreover,

$$\|w_q(v)\|_m \leq C_m(1+\|a_q\|_{m_1}+\|b_q\|_{m_1})\,(1+\|b_q\|_{m_2}+\|v\|_{m_2})\,. \tag{2.37}$$

This is obtained by simply integrating (2.36) in t. We estimate $(1+\|a_q\|_{m_1})(1+\|b_q\|_{m_2})$ further. By Lemma 2.4, a_q and b_q are tame in g. Since $m_1+m_2=m$, applying Proposition 1.6 gives

$$\|a_q\|_{m_1}\|b_q\|_{m_2}\leq C_m(\|a_q\|_m+\|b_q\|_m)$$

and analogue estimates for the other product of norms in (2.37). Thus, we have tame estimates

$$\|w_q(v)\|_m \leq C_m(1+\|a_q\|_m+\|b_q\|_m+\|v\|_m)\leq C_m(1+\|q\|_{m+r}+\|v\|_m), \tag{2.38}$$

where r is a nonnegative integer. In the second inequality we used that a_q and b_q are tame in q by Lemma 2.4. *$w_q(v)$ is a solution:* We now show that $w_q(v)$ verifies (2.18). By the method of characteristics, a smooth function w is solution of equation (2.18) if and only if

$$w(x)=I_q(s,x)w(\Phi_{q,s}(x))-\int_0^s I(t,x)v(\Phi_{q,t}(x))dt,\quad \text{for each } s\geq 0.$$

Note that this formula is well defined, since we are working in a Φ_q–positively invariant set. Let $x\in W$ and $s\geq 0$. We have

$$\begin{aligned}-w_q(v)(\Phi_{q,s}(x)) &= \int_0^\infty \exp\left(\int_0^t a_q(\Phi_{q,\tau}(\Phi_{q,s}(x)))d\tau\right)v(\Phi_{q,t}(\Phi_{q,s}(x)))dt\\ &= \int_0^\infty \exp\left(\int_0^t a_q(\Phi_{q,\tau+s}(x))d\tau\right)v(\Phi_{q,t+s}(x))dt\\ &= \int_0^\infty \exp\left(\int_s^{t+s} a_q(\Phi_{q,\tau}(x))d\tau\right)v(\Phi_{q,t+s}(x))dt\\ &= \int_s^\infty \exp\left(\int_s^t a_q(\Phi_{q,\tau}(x))d\tau\right)v(\Phi_{q,t}(x))dt\\ &= \int_s^\infty \frac{I_q(t,x)}{I_q(s,x)}v(\Phi_{q,t}(x))dt.\end{aligned}$$

Thus, $w_q(v)(\Phi_s(x))I_q(s,x) = -\int_s^\infty I_q(t,x)v(\Phi_{q,t}(x))dt$. Then $w_q(v)$ solves (2.18).

Smoothness in the parameter: It only remains to prove that $w_q(v)$ is smooth in q. We obtain this as an application of Proposition 1.7. Define $E_d \subseteq C^\infty(\overline{W})$ as the set of smooth functions with d–th jet in the origin equal to 0. Since E_d is a closed vector subspace of $C^\infty(\overline{W})$ of finite codimension, it is a tame Fréchet space. For every $q = (g,u) \in \mathcal{V}$ and smooth function f, we denote T_{x_u} as $T_{x_u}f(x) = f(x + x_u)$. By Lemma 2.9, this is a smooth tame operator. We denote by $\tilde{a}_q(x) = T_{x_u}a_q$ and by $\tilde{b}_q = T_{x_u}b_q$. Up to shrinking the neighborhoods we are working with, we can assume that they are well defined and all the estimates we have proven so far also hold the translated functions we'll use. For example, we could shrink $\mathcal{V}$ and take $W_1 \subset W_2 \subset W$ s.t. $W_1 + x_u \subset W_2$, where W_2 is a $\phi_{q,t}$-positive invariant neighborhood of W_1 such that $W_2 - x_u \subseteq W$ for every $(g,u) \in \mathcal{V}$. Consider the linear map

$$L : \mathcal{V} \times E_d \to E_d$$

defined as $L_q(v) = \tilde{a}_q v + \tilde{b}_q(v)$. Since $\tilde{b}_q(0) = b_q(x_u) = 0$, this map is well defined i.e. $j_0^d L_q(v) = 0$ for every (q,v) in the domain of L. Recall from Lemma 2.9 and Lemma 2.4 that x_u, a_q and b_q are smooth tame in q. Thus, the map L is also smooth and tame. Let now $q \in \mathcal{V}$. Consider the corresponding fiber

$$L_q : E_d \to E_d.$$

We define

$$\tilde{w}_q(v)(x) = -\int_0^\infty v(\Phi_{q,t}(x + x_u) - x_u)I_q(t, x + x_u)dt \tag{2.39}$$

As we argued in the first part of the proof, we have

$$D^m\tilde{w}_q(v)(x) = -\int_0^\infty D^m\left(v(\Phi_{q,t}(x + x_u) - x_u)I_q(t, x + x_u)\right)dt$$

for every integer m. In particular, for $x = 0$ the integrand is sum of terms containing a derivative of v in 0 of order less or equal than m. These derivatives are all 0 for $m \leq d$, because v belongs to E_d. Thus, $\tilde{w}_q(v) \in E_d$ for every $v \in E_d$. Moreover, $w_q(v)$ solves (2.18), which implies that $\tilde{w}_q$ is a linear right inverse of L_q. Thus, we have shown that L_q is surjective. We now show that it is injective. Since it is linear, it is enough to prove that $L_q(v) = 0$ implies $v = 0$. Since the d–th jet of v in 0 is null, there exists a positive constant C_d such that

$$\|v(x)\| \leq C_d\|x\|^d$$

for every $x \in \overline{W}$. Moreover, by the method of the characteristics, $L_q(v) = 0$ implies that

$$v(x) = v(\Phi_{q,s}(x + x_u) - x_u)I_q(s, x + x_u) \quad \text{for each } s \geq 0, x \in \overline{W}.$$

$$\begin{aligned}|v(x)| &\leq |I_q(s, x + x_u)| \cdot |v(\Phi_{q,s}(x + x_u) - x_u)| \\ &\leq e^{Cs}C_d\|\Phi_{q,s}(x + x_u) - x_u\|^d \\ &\leq e^{Cs}C_d e^{-d\mu s} = C_d e^{(C-d\mu)s},\end{aligned}$$

which goes to 0 as s goes to infinity for $d > \frac{C}{\mu}$. Thus, v is constantly 0, implying injectivity of L_q. Moreover, the family $\tilde{w}_q$ of inverses tame by (2.38). It only remains to prove that it is also continuous. Let $(q_n, v_n) \in \mathcal{V} \times E_d$ be a sequence converging in $(q,v) \in \mathcal{V} \times E_d$. By the continuous dependence of the flow in the vector field, $\Phi_{q_n,t}(\cdot + x_{u_n})$ converges to $\Phi_{q,t}(\cdot + x_u)$ in the smooth topology. Then the corresponding integrand in (2.39) converges in the smooth topology uniformly on every bounded interval. Moreover, (2.36) applied to the translated functions gives an uniform integrable bound for the integrands of (2.39) corresponding to the sequence (q_n, v_n). Then, by dominated convergence we have $\tilde{w}_{q_n}(v_n) \to \tilde{w}_q(v)$.

We can then apply Proposition 1.7. This gives that $\tilde{w} : \mathcal{V} \times E_d \to E_d$ is a smooth map. Observe now that $w_q(v) = T_{-x_u}\tilde{w}_q(T_{x_u}(v))$ for every (q, v) such that $j^d_{x_u} v = 0$. Thus, w_q is smooth in (q, v).

Definition of R_{crit}: We can finally defined R_{crit}. Let $\{x_1, \dots, x_m\}$ be the critical points of u_0. Let d be sufficiently large. Consider sufficiently small g_0 geodesic charts $\varphi_i : W_i \to U_i \subset \mathbb{R}^n$ around them and for each one the corresponding compact set K_i such that K_i contains exactly one critical point of u for every $(g, u) \in \mathcal{V}$. Let $w_{i,q}(v) : U_i \to \mathbb{R}$ be the corresponding solutions of (2.18) obtained as above, each corresponding to a matrix F_i. Define the map

$$R_{\text{crit}} : \mathcal{V} \times C^\infty(M) \to \mathcal{G}$$

as

$$R_{\text{crit}}(q, v) = R_{\text{jet},d}(q, v) + \sum_i \rho_i \varphi_{i,*} \left(w_{i,q} \left(\varphi_{i,*}^{-1} \left(v - \dot{\Delta} R_{\text{jet},d}(q, v) \right) \right) F \right),$$

where $R_{\text{jet},d}$ is defined as in Proposition 2.12. Since pullbacks are smooth tame operations by [10, Lemma 2.3.4], this is a smooth tame map. The map R_{crit} is linear in v. Moreover, it verifies (2.6) with $W_{\text{crit}} = \cup_i \text{int}(K_i)$ by construction. Indeed, for $x \in K_i$ we have

$$\begin{aligned} \dot{\Delta} R_{\text{crit}}(q, v)(x) &= \dot{\Delta} R_{\text{jet},d}(q, v)(x) + \dot{\Delta}\varphi_{i,*} \left(w_{i,q} \left(\varphi_{i,*}^{-1} \left(v - \dot{\Delta} R_{\text{jet},d}(q, v) \right) \right) F_i \right) \\ &= \dot{\Delta} R_{\text{jet},d}(q, v)(x) + v - \dot{\Delta} R_{\text{jet},d}(q, v) = v, \end{aligned}$$

where in the second equality we have used that w_q solves (2.18) induced by the matrix F_i. □

A key step in the proof of Proposition 2.8 was the fixing of a suitable matrix F with Lemma 2.14. This is not always possible in dimension 2. However, the proof of Proposition 2.8 is significantly simpler in dimension this case. Indeed, we can find an explicit solution using Lemma 2.4.

proof of Proposition 2.8 in $\dim(M) = 2$. Let $\mathcal{V}$ be a neighborhood of q_0 and W_{crit} open in M as follows. For every $(g, u) \in \mathcal{V}$, the set $\text{crit}(u)$ is contained in W_{crit}. Moreover, $\Delta_g u$ is never 0 in W, where W is a fixed open neighborhood of W_{crit}. Let ρ be a smooth function compactly supported in W which is 1 on W_{crit}. Let v be a smooth function on M. We look for a conformal solution of (2.1) for ρv. By Lemma 2.4, this amounts to solving

$$-\psi \Delta_g u = \rho v.$$

We simply define $R_{\text{crit}}(q, v) = -\frac{\rho v}{\Delta_g u} g$. This is a smooth tame map verifying the required property. □

2.3. **Right inverse of the linearized Laplace–Beltrami operator at regular points.** Proposition 2.8 gives a smooth tame solution of (2.1) in a fixed open neighborhood of critical points of u. We want to prove an analogue of Proposition 2.8 outside such a neighborhood. Let $W \subseteq M$ be open connected with smooth boundary. We denote by $C^\infty_0(W)$ the space of smooth function vanishing at ∂W with all derivatives. Similarly, we denote by $\mathfrak{X}_0(W)$ and by $\Gamma^\infty_0(S^2T^*W)$ the set of smooth vector fields and smooth $(0, 2)$-tensors vanishing at ∂W with all derivatives. On the set $\mathfrak{X}_0(W)$ we define C^k norms as we did for smooth metrics in the introduction. This gives both vector spaces a structure of tame Fréchet spaces.

Proposition 2.15 (Solution away from critical points)**.** *Let $q_0 = (g_0, u_0) \in \mathcal{N}$. Then there exists a neighborhood $\mathcal{V}$ of q_0 in $\mathcal{G}_+ \times C^\infty(M)$ as follows. Let $W_{\text{reg}} \subseteq M$ be an open connected set with smooth boundary, $\text{crit}(u) \cap \overline{W}_{\text{reg}} = \emptyset$ for every $(g, u) \in \mathcal{V}$, and $\Delta_{g_0} u_0$ is not identically 0 in $\overline{W}_{\text{reg}}$. Then there exists a smooth tame map*

$$R_{\text{reg}} \colon \mathcal{V} \times C^\infty_0(W_{\text{reg}}) \to \Gamma^\infty_0(S^2T^*M)$$

such that $\dot{\Delta} \circ R_{\mathrm{reg}} = \mathrm{Id}$.

It is very important to work with objects vanishing at the boundary. This is because $C_0^\infty(W_{\mathrm{reg}})$ can be embedded in $C^\infty(M)$, while $C^\infty(W_{\mathrm{reg}})$ cannot. The same holds for smooth metrics. This allows us to combine the solutions obtained from Proposition 2.8 and Proposition 2.15 appropriately. By using Lemma 2.4, we will reduce proving Proposition 2.15 to solving an equation of the type

$$\mathrm{div}_g X = \psi \tag{2.40}$$

where ψ is a function supported on W_{reg} and X is an unknown vector field. A solution of (2.40) supported again on W_{reg}, is given by the Bogovskii operator. The following is a simplified version of [8, Theorem III.3.3].

Theorem 2.16 (Bogovskii operator)**.** *Let $U \subseteq \mathbb{R}^n$ be open, connected and bounded with Lipschitz boundary. Assume that $n \geq 2$. Then for every $f \in C_0^\infty(U)$ satisfying $\int_U f = 0$ there exists $\mathcal{B}(f) \in \mathfrak{X}_0(U)$ such that $\mathrm{div}\,\mathcal{B}(f) = f$. Moreover, $\mathcal{B}(f)$ can be chosen to be linear and such that*

$$\|\mathcal{B}(f)\|_{H_{l+1}(U)} \leq C(\Omega, \ell)\|f\|_{H_l(U)}. \tag{2.41}$$

for every positive integer ℓ, where $C(\Omega, \ell)$ is a positive constants depending only on Ω and ℓ.

We need a generalization of this result for manifold with boundaries. The boundary assumption cannot be dropped in our method. This is because we aim to apply this result to $M \setminus W_{\mathrm{crit}}$, with W_{crit} coming from Proposition 2.8. This manifold always has a nonempty boundary. After making this generalization, we will discuss the tameness of the constructed Bogovskii operator. First, we define the appropriate spaces where this operator should be defined. Let W be a compact connected smooth manifold with boundary. We denote the set of couples of metric and function with 0 mean by

$$Z = \left\{ (g, f) \in \mathcal{G}_+ \times C_0^\infty(W) : \int_W f \,\mathrm{d}\,\mathrm{vol}_g = 0 \right\}.$$

We will denote its fibers by

$$Z(g) = \{ f \in C_0^\infty(W) \mid (g, f) \in Z \}.$$

We briefly argue for the tameness of these spaces.

Lemma 2.17 (Tameness of 0 mean)**.** *The sets Z and $Z(g)$ are smooth tame Fréchet submanifolds.*

Proof. The set $Z(g)$ is a closed finite codimension vector subspace of $C_0^\infty(W)$. Thus, it is tame Fréchet. Fix a smooth metric g_0 on W. We denote by ρ_g the unique positive smooth function such that $\mathrm{d}\,\mathrm{vol}_{g_0} = \rho_g \mathrm{d}\,\mathrm{vol}_g$. We first claim that the map

$$\rho : \Gamma^\infty(S_+^2 T^* W) \to C^\infty(W)$$

defined as $\rho(g) = \rho_g$ is smooth tame. Indeed, by definition we have $\rho_g = \frac{\sqrt{\det(g_0)}}{\sqrt{\det(g)}}$. Thus, it is enough to prove that the map assigning a metric g to the function $\det(g)$ is smooth tame. This immediately follows from the definition of C^k norms. Consider now the map

$$\chi : \mathcal{G}_+ \times C_0^\infty(W) \to \mathcal{G}_+ \times C_0^\infty(W)$$

defined as $\chi(g, f) = (g, \rho_g f)$. This is a smooth tame embedding. By the definition of ρ_g, we have $Z = \chi(\mathcal{G}_+ \times Z(g_0))$. □

We have proven that the domains of the Bogovskii operators we will construct are tame. We will now define such operators. First consider the case where the metric is fixed.

Theorem 2.18 (Global Bogovskii operator)**.** *Let (W, g) be a compact connected Riemannian manifold with boundary. Assume* $\dim W \geq 2$*. Then, there exists a smooth tame linear map*

$$\mathcal{B}_g : Z(g) \to \mathfrak{X}_0(W)$$

such that $\operatorname{div}_g \mathcal{B}_g(f) = f$*. In particular, for every positive integer ℓ there exists a positive constant $C(W, g, \ell)$ depending only on W, g and ℓ such that*

$$\|\mathcal{B}_g(f)\|_{H_{\ell+1}(W)} \leq C\|f\|_{H_\ell(W)}.$$

Proof. Choose a finite coordinate cover $\{(U_i, \varphi_i)\}_{i=1}^N$ such that each $V_i = \varphi_i(U_i)$ is either a unit ball or a half unit ball. As in the proof of [8, Lemma III.3.4], we can find functions f_i satisfying

- $f_i \in C_0^\infty(U_i)$,
- $\int_{U_i} f_i d\operatorname{vol}_g = 0$,
- $\sum_i f_i = f$.
- $\|f_i\|_{H_l(W)} \leq c_1\|f\|_{H_l(W)}$ for some positive constant $c_1 = c_1(W, \ell)$.

Moreover, we can choose the f_i to be linear in f. We solve the divergence equation in local coordinates. It becomes:

$$\frac{1}{\sqrt{|g|}} \operatorname{div}\left(\sqrt{|g|} v_i\right) = f_i, \tag{2.42}$$

where div is the Euclidean divergence. Multiplying both sides by $\sqrt{|g|}$ and then setting $w_i = \sqrt{|g|}v_i$ gives $\operatorname{div} w_i = \sqrt{|g|}f_i$. The condition $\int_{U_i} f_i d\operatorname{vol}_g = 0$ in local coordinates becomes $\int_{V_i} f_i\sqrt{|g|} = 0$. By Theorem 2.16, there exists $X_i(f_i) \in \mathfrak{X}_0(W)$ satisfying (2.42). Since multiplication by $\sqrt{|g|}$ and its inverse are bounded on Sobolev spaces, we also have

$$\|X_i(f_i)\|_{H_{\ell+1}(W)} \leq C_i\|f_i\|_{H_\ell(W)}.$$

Consider $\mathcal{B}_g(f) = \sum_i X_i(f_i)$. We have $\operatorname{div}_g \mathcal{B}_g(f) = f$ by construction. Moreover,

$$\|\mathcal{B}_g(f)\|_{H_{\ell+1}(W)} \leq \sum_i \|X_i(f_i)\|_{H_{\ell+1}(W)} \leq C(W, g, \ell)\|f\|_{H_\ell(W)},$$

where C is a positive constant depending only on ℓ, W and g. □

Remark 2.19 (Regularity improvement)**.** Observe that the Bogovskii operator is more than just tame. Indeed, it improves regularity. For our purposes, tameness will be enough.

Theorem 2.18 is not sufficient for us. This is because we need to build a local right inverse of $\dot{\Delta}_g$. Thus, we must take into account a variation of the underlying metric. This is obtained in the following theorem.

Theorem 2.20 (Augmented Bogovskii operator)**.** *Let (W, g_0) be a compact connected Riemannian manifold with boundary. Assume* $\dim W \geq 2$*. Then, the map*

$$\mathcal{B}(g_0) : Z \to \mathfrak{X}_0(W)$$

defined as $\mathcal{B}(g_0)(g, f) = \rho_g B_{g_0}(\rho_g^{-1} f)$ is smooth and tame, where ρ_g is the unique positive smooth function such that $\mathrm{d}\operatorname{vol}_{g_0} = \rho_g \mathrm{d}\operatorname{vol}_g$*. Moreover,*

$$\operatorname{div}_g \mathcal{B}(g_0)(g, f) = f. \tag{2.43}$$

Proof. Observe that this map is well defined. Indeed,

$$\int_W f \,\mathrm{d}\,\mathrm{vol}_g = \int_W f\rho_g^{-1}\mathrm{d}\,\mathrm{vol}_{g_0} = 0,$$

where in the last line we used that f belongs to Z. We will now prove tameness of $\mathcal{B}(g_0)$. Recall that the map $g \to \rho_g$ is smooth tame. This was proven in Lemma 2.17. Thus, we can restrict proving tameness of $\mathcal{B}(g_0)$ to an open neighborhood of (g_0, f_0) where the 1-norm of ρ_g and ρ_g^{-1} are bounded by a fixed constant. This is because tameness is a local property. In this neighborhood, multiplication by ρ_g and ρ_g^{-1} are smooth tame maps by [10, Lemma 2.3.4]. Observe that applying a fixed linear continuous tame map such as $\mathcal{B}_{g_0}$ preserves smoothness and tame estimates up to a fixed constant. Thus, $\mathcal{B}(g_0)$ is smooth tame. Finally, we prove (2.43). Observe that for every smooth vector field X on M and smooth metric g the following holds

$$\mathrm{div}_g(X) = \rho_g \,\mathrm{div}_{g_0}(\rho_g^{-1}X).$$

This is obtained directly from the definition of divergence. Thus, for every $(g, f) \in Z$ we have

$$\mathrm{div}_g(\mathcal{B}(g_0)(g, f)) = \rho_g \,\mathrm{div}_{g_0}(\rho_g^{-1}\rho_g\mathcal{B}_{g_0}(\rho_g^{-1}f)) = f.$$

□

We are finally ready to prove Proposition 2.15.

Proof of Proposition 2.15. We divide the map R_{reg} as a sum of a conformal variation and a traceless variation, i.e.:

$$R_{\mathrm{reg}}(q, v) = R_1(q, v) + R_2(q, v),$$

where R_1 is a conformal variation and R_2 is a traceless variation, both with respect to g.

Let $y_0 \in W_{\mathrm{reg}}$ be such $\Delta_{g_0}u_0(y_0) \neq 0$. Then there exists a neighborhood V of y in W_{reg} such that $\Delta_g u(y) \neq 0$ for every $(g, u) \in \mathcal{V}$ and $y \in V$. Let $\rho \geq 0$ be a nonzero bump function supported in V. Consider a conformal variation in the form $R_1(q, v) = c(v)\rho\Delta_g u \cdot g$, where $c = c(q, v) \in \mathbb{R}$. We look for c such that

$$\int_{W_{\mathrm{reg}}} \dot{\Delta}_g[R_1](u)\, d\,\mathrm{vol}_g = \int_{W_{\mathrm{reg}}} v\, d\,\mathrm{vol}_g\,. \tag{2.44}$$

By Lemma 2.4, we get $\dot{\Delta}_g[R_1](u) = -c\rho(\Delta_g u)^2 + \frac{n-2}{2}\langle\nabla(c\rho\Delta_g u), \nabla u\rangle_g$. Thus, integrating by parts gives

$$\begin{aligned}\int_{W_{\mathrm{reg}}} \dot{\Delta}_g[R_1](u)\, d\,\mathrm{vol}_g &= \int_{W_{\mathrm{reg}}} -c\rho(\Delta_g u)^2 + \frac{n-2}{2}\langle\nabla(c\rho\Delta_g u), \nabla u\rangle_g \, d\,\mathrm{vol}_g \\ &= \int_{W_{\mathrm{reg}}} -c\rho(\Delta_g u)^2 - \frac{n-2}{2}c\rho(\Delta_g u)^2 \, d\,\mathrm{vol}_g \\ &= -\frac{n}{2}c\int_{W_{\mathrm{reg}}} \rho(\Delta_g u)^2 \, d\,\mathrm{vol}_g\end{aligned}$$

In particular, $\rho(\Delta_g u)^2$ is a nonnegative function which is positive on an open subset of W_{reg}. Therefore, we can choose

$$c = -\frac{2\int_{W_{\mathrm{reg}}} v d\,\mathrm{vol}_g}{n\int_{W_{\mathrm{reg}}} \rho(\Delta_g u)^2 \, d\,\mathrm{vol}_g}$$

so that (2.44) holds. It follows also that the map R_1 defined in this way is smooth and tame, being composition of integrals and multiplication by smooth functions.

Let $\tilde{v} \coloneqq v - \dot{\Delta}_g[R_1((q, v)](u)$. We now search for a traceless variation $R_2(q, \tilde{v})$ such that

$$\dot{\Delta}_g[R_2(q, \tilde{v})](u) = \tilde{v}.$$

By Lemma 2.4, this amounts to solve

$$-\operatorname{div}_g(Du \lrcorner R_2^*) = \tilde{v}, \tag{2.45}$$

where R_2^* is the tensor R_2 but all the indexes are raised by g. By Theorem 2.20, we can write (2.45) as

$$Du \lrcorner R_2^* = \mathcal{B}(g_0)(g, -\tilde{v}). \tag{2.46}$$

Denote by $B := \mathcal{B}(g_0)(g, -\tilde{v}) \in \mathfrak{X}_0(W_{\mathrm{reg}})$ and by $B^\perp := B - \frac{\langle \nabla u, B\rangle_g}{\|\nabla u\|^2}\nabla u$ its projection on $(\nabla u)^\perp$. Notice that $B^\perp$ is a smooth vector field because ∇u never vanishes on W_{reg}. Define R_2^* as:

$$R_2^* := \frac{\nabla u \otimes B^\perp + B^\perp \otimes \nabla u}{\|\nabla u\|^2} + \frac{n}{n-1}\frac{\langle B, \nabla u\rangle_g}{\|\nabla u\|^2}\left(\frac{\nabla u \otimes \nabla u}{\|\nabla u\|^2} - \frac{1}{n}g^*\right) \tag{2.47}$$

Since $V \equiv 0$ in a neighborhood of the critical points of u then R_2^* is smooth. Notice that R_2^* is symmetric. A simple computation shows that it is traceless:

$$\operatorname{tr} R_2^* = n\frac{\langle B, \nabla u\rangle}{\|\nabla u\|^2}(1-1) = 0.$$

Also (2.46) is satisfied:

$$\begin{aligned} Du \lrcorner R_2^* &= B^\perp - \cancel{Du(B^\perp)\frac{\nabla u}{\|\nabla u\|^2}} - n\frac{\langle B, \nabla u\rangle_g}{\|\nabla u\|^2}\left(-\nabla u + \frac{1}{n}\nabla u\right) \\ &= B^\perp + \frac{\langle B, \nabla u\rangle_g}{\|\nabla u\|^2}\nabla u \\ &= B. \end{aligned}$$

Since the Bogovskii operator is smooth and tame by Theorem 2.20, it follows that

$$R_2 \colon \mathcal{V} \times C_0^\infty(W_{\mathrm{reg}}) \to \Gamma_0^\infty(S^2T^*W_{\mathrm{reg}}) \subseteq \Gamma^\infty(S^2T^*M)$$

is a smooth tame map since it is a composition of smooth tame maps. Finally we define $R_{\mathrm{reg}} = R_1 + R_2$, so that

$$\begin{aligned} \dot{\Delta}_g[R_{\mathrm{reg}}((g,u),v)](u) &= \dot{\Delta}_g[R_1((g,u),v)](u) + \dot{\Delta}_g[R_2(g,u),v)](u) \\ &= \dot{\Delta}_g[R_1((g,u),v)](u) + v - \dot{\Delta}_g[R_1((g,u),v)](u) \\ &= v. \end{aligned}$$

□

Remark 2.21. Contrary to the one of R_{crit}, the smoothness in q of R_{reg} can be easily derived from its explicit expression.

2.4. **Density of the submersivity set.** So far, we have essentially constructed the map R in the statement of Theorem 2.2. To prove the theorem, it is still necessary to show that $\mathcal{N}$ is open dense. We will need a technical transversality result to prove do so. The following can be found in [1, 17.2, 19.1, 19.2].

Theorem 2.22. *Let X and Y be C^1 Banach manifolds with a countable base. let $W \subset Y$ be C^1 Banach submanifold of finite codimension c, and let M be a smooth manifold of dimension n. Consider a C^1 map $F \colon X \times M \to Y$, and assume that $n < c < +\infty$. If $F \pitchfork W$, then denoting $F_x := F(x, \cdot) : M \to Y$, we have that the set*

$$\left\{ x \in X \;\middle|\; \operatorname{Im}(F_x) \cap W = \emptyset \right\}$$

is residual in X.

Proof. The proof is a consequence of standard transversality theory in Banach manifolds. Since this result is standard and is not used elsewhere in the present work, we do not recall the definitions of transversality and Fredholm maps for Banach manifolds. We only give a sketch of the proof, and we refer the interested reader to [1] for the precise theory and definitions.

Since $F \pitchfork W$, from [1, Corollary 17.2] we obtain that $F^{-1}(W)$ is a C^1 Banach submanifold of $X \times M$ of codimension c. Consider the restriction of the projection on the X component

$$\pi|_{F^{-1}(W)} \colon F^{-1}(W) \to X.$$

Then by [1, Lemma 19.2] $\pi|_{F^{-1}(W)}$ is a C^1 Fredholm map of index $n - c < 0$. By the Sard–Smale theorem [1, Theorem 16.2] the set of regular values of $\pi|_{F^{-1}(W)}$ is residual in X. We now observe that, since the Fredholm index is negative, a regular value of $\pi|_{F^{-1}(W)}$ cannot belong to its image. Thus the set of regular values is precisely

$$X \setminus \pi\big(F^{-1}(W)\big).$$

Finally, $x \notin \pi\big(F^{-1}(W)\big)$ is equivalent to $F_x(M) \cap W = \emptyset$. Hence

$$\left\{ x \in X \;\middle|\; \operatorname{Im}(F_x) \cap W = \emptyset \right\}$$

is residual in X. □

To apply Theorem 2.22, it will be convenient to work with C^k spaces. The following topological lemma guarantees that this is possible.

Lemma 2.23. *Let $\{X_k\}_{k \geq k_0}$ be a nested family of Banach manifolds with continuous dense inclusions*

$$X_\ell \hookrightarrow X_k, \qquad \ell \geq k,$$

and let

$$X_\infty := \bigcap_{k \geq k_0} X_k$$

be endowed with the induced Fréchet topology. Assume that the inclusion $X_\infty \hookrightarrow X_k$ is dense for every k. Let $D \subset X_k$ and assume that

$$D_\ell = D \cap X_\ell, \qquad \ell \geq k$$

is open dense in X_ℓ for every sufficiently large ℓ. Then

$$D_\infty := D \cap X_\infty$$

is open dense in X_∞.

Proof. Openness is clear, since X is endowed with the Freéchet topology induced by the limit of the X_ℓ. It remains to prove density. Let $U \subset X_\infty$ be a nonempty open set. By definition of the Fréchet topology on X_∞, there exists ℓ and a nonempty open set $V \subset X_\ell$ such that $V \cap X_\infty \subseteq U$. We can assume that ℓ is large enough that D_ℓ is dense in X_ℓ. Then, the set $V \cap D_\ell$ is a nonempty open subset of X_ℓ. Since X_∞ is dense in X_ℓ, the intersection $X_\infty \cap V \cap D_\ell = V \cap D_\infty$ is nonempty. Since $V \cap X_\infty \subseteq U$, this completes the proof. □

Remark 2.24 (Generic properties and regularity)**.** Lemma 2.23 allows to work in the Banach setting when studying some open dense properties of smooth objects. However, not all open dense or residual properties of smooth functions come from finite regularity ones. This justifies working in the Fréchet setting.

We can now prove that $\mathcal{N}$ is open dense.

Proposition 2.25. *The set $\mathcal{N}$ is open dense. In particular, it is a tame Fréchet variety.*

Proof. For $\dim(M) > 2$, this result is equivalent to Morse functions being open dense in $C^\infty(M)$. Then we can assume $\dim(M) = 2$. By Lemma 2.23, it is enough to prove that the set $\mathcal{N}_{C^k}$ is open dense for every k sufficiently large. Since the set of Morse functions is open dense, it is enough to study the second condition. We first prove density as an application of Theorem 2.22. Consider the map

$$F\colon \left(\mathcal{G}^k_+ \times H_k(M)\right) \times M \to \left(\mathcal{G}^k_+ \times H_k(M)\right) \times T^*M \oplus \mathbb{R}$$
$$(g,u), x \mapsto (g,u), D_x u, \Delta_g u.$$

Let Z be the 0 section of $\left(\mathcal{G}^k_+ \times H_k(M)\right) \times T^*M \oplus \mathbb{R}$ and define $W = \left(\mathcal{G}^k_+ \times H_k(M)\right) \times Z$. Then W is exactly the set we need to avoid generically. Since $\operatorname{codim}(W) = 3 > 2 = \dim(M)$, we need only to prove that $F \pitchfork W$ to apply Theorem 2.22. To show this, simply observe that we can always pick a smooth function on M with a prescribed jet at a fixed point. Finally, $\mathcal{N}$ is open because M is compact. Thus, the set of $q \in \mathcal{G}^k_+ \times H_k(M)$ such that $F(q,x)$ avoids Z is open. □

We can finally prove Theorem 2.2. It is simply a combination of Proposition 2.8 and Proposition 2.15.

Proof of Theorem 2.2. We apply Proposition 2.8 to q_0. This gives a neighborhood $\mathcal{V}_1$ of q_0, an open set $W_{\text{crit}} \subseteq M$ and a map R_{crit} as in Proposition 2.8. Let $K \subseteq W_{\text{crit}}$ be a compact set with smooth boundary containing $\operatorname{crit}(u_0)$ in its interior. Up to shrinking $\mathcal{V}_1$, we can assume that $\operatorname{crit}(u)$ is contained in K for every (g,u) in $\mathcal{V}_1$. Moreover, we can assume that $M \setminus K$ is connected since $n \geq 2$. This is because $\operatorname{crit}(u_0)$ is finite. Since u_0 is not constant, $\Delta_{g_0} u_0$ is nonzero in an open subset of M. Up to shrinking $\mathcal{V}_1$ and K, we can assume that it is not constantly zero in the complement of K.

We can now apply Proposition 2.15 with $W_{\text{reg}} = M \setminus K$. This gives a neighborhood $\mathcal{V}_2$ of q_0 and a map R_{reg}. We can assume $\mathcal{V} := \mathcal{V}_1 = \mathcal{V}_2$. Consider the open set $W_{\text{reg}} = M \setminus K$. We define

$$R\colon \mathcal{V} \times C^\infty(M) \to \mathcal{G}$$

as $R(q,v) = R_{\text{crit}}(q,v) + R_{\text{reg}}(q, v - \dot{\Delta} R_{\text{crit}}(v))$. This map is well defined. This is because $v - \dot{\Delta} R_{\text{crit}}(v) \in C^\infty_0(W_{\text{reg}})$ by Proposition 2.8. Moreover, it is a sum of smooth tame maps. Thus, it is smooth tame. Finally,

$$\begin{aligned}\dot{\Delta} R(q,v) &= \dot{\Delta} R_{\text{crit}}(q,v) + \dot{\Delta} R_{\text{reg}}(v - \dot{\Delta} R_{\text{crit}}(q,v)) \\ &= \dot{\Delta} R_{\text{crit}}(q,v) + v - \dot{\Delta} R_{\text{crit}}(q,v) = v.\end{aligned}$$

□

3. Generic properties of eigenfunctions of the Laplace–Beltrami operator

Theorem 2.2 can be interpreted as a sort of flexibility property for the Laplace–Beltrami operator. It states that every tangent direction can be realized by Δ only using a variation of the metric. In this section, we discuss the consequence of this result on the generic properties of the eigenfunctions of second order selfadjoint elliptic operators. We will call a set containing a countable intersection of open dense sets residual. We will say that a property is generic if it is verified by every element in a residual set. To even begin to discuss whether a property of selfadjoint elliptic operators is generic, it is necessary to equip them with a topology. In this case, the topology will be induced by a parametrization of the kind of elliptic operators we want to consider. First, we'll consider properties of operators of the Laplace–Beltrami type.

They will be parametrized naturally with the identification $g \to \Delta_g$. This case represents the most direct consequence of Theorem 2.2. Afterwards, we'll consider operators of the same type, but parametrize them with metrics induced by embeddings of M into the Euclidean space. Finally we'll consider consider the family of Schrödinger operators with independent metric and potential parameters.

3.1. **Genericity properties after Uhlenbeck.** Generic properties of eigenfunctions of the Laplace–Beltrami operator were previously studied by K. Uhlenbeck in [16] in the Banach setting. Her argument is carried out in a finite-regularity Banach setting and is based on the Sard–Smale theorem and parametric transversality for Fredholm maps between Banach manifolds. We briefly recall the part of her work that is relevant to our setting.

Denote by $\mathcal{G}_+^k$ the space of C^k sections on C^k symmetric $(0,2)$–tensors on M. Each element of $\mathcal{G}_+$ induces a Laplace–Beltrami operator $\Delta_g : H_k(M) \to H_{k-2}(M)$. Consider the following properties of element in $\mathcal{G}_+^k$

(1) Every eigenvalue of Δ_g is simple.
(2) Every eigenfunction of Δ_g has 0 as regular value.
(3) Every eigenfunction of Δ_g is Morse.

One can show the following.

Theorem 3.1 ([16, Theorem 8])**.** *Let M be a compact connected smooth manifold with $\dim(M) = n \geq 2$. Let $k > n + 3$ be a positive integer. Then, the set*

$$\{g \in \mathcal{G}_+^k : \Delta_g \text{ satisfies (1), (2), (3) on nonconstant eigenfunctions}\}$$

is residual in $\mathcal{G}_+^k$.

Theorem 3.1 was originally proven using transversality techniques on Banach manifolds, which are not available in this form in the Fréchet setting. This justifies the different approach of our work, which instead relies on Theorem 1.8. However, notice that Theorem 3.1 will play a key role in the proof of Proposition 3.10. This density input is essential for the main theorem; its fixed-metric potential version is also used for the Schrödinger extension. In this sense Uhlenbeck's study of the Banach setting provide the groundwork to treat the smooth one.

Remark 3.2. Observe that there is no analogue of Theorem 3.1 in dimension 1. Indeed, every 1-dimensional compact connected smooth Riemannian manifold is isometric to S^1 with a uniform metric and every nonzero eigenvalue of the corresponding Laplace–Beltrami operator has multiplicity 2.

Uhlenbeck's method also applies to other families of operator. We will use the following to study a special class of operators, called Schrödinger operators, in Section 3.5.

Theorem 3.3 ([16, Theorem 7])**.** *Let M be smooth compact connected manifold of any dimension and $k \geq 3$ be a positive integer. Denote by L a fixed second order self–adjoint elliptic operator on M with smooth coefficients. Then the set*

$$\{v \in C^k(M) : L + v \text{ satisfies (1), (2), (3)}\}$$

is residual in $C^k(M)$.

Remark 3.4 (Eigenpair space formulation)**.** Despite being different from our methods, the proofs of Theorem 3.1 and Theorem 3.3 still pass through the eigenpair space, although Sobolev spaces are used in the original work. In particular, Uhlenbeck's results could also be stated as density of properties (1) to (3) for triples in the eigenpair space. This formulation will be more convenient for us and will be implicitly used in many occasions.

To conclude this section, we explain why only operators of order 2 are considered in [16]. This is due to the the use the *weak unique continuation theorem*, which is a key tool in the proofs of Theorem 3.1.

Theorem 3.5 ([16, Weak Unique Continuation Theorem])**.** *Let L be a second order elliptic operator with C^3 coefficients. Let D be a nonempty open connected subset of M. If $Lv = 0$ in D and $v = 0$ on a nonempty open subset of D, then v is identically 0 in D.*

Notice how Theorem 3.5 requires the order of the operator to be 2. Indeed, ellipticity alone does not guarantee a unique continuation of solution and Theorem 3.5 it does not hold for higher order operators. Since the present paper requires Theorem 3.1, we also restrict ourselves to studying operators of second order.

3.2. **Geometry of the eigenpair space.** Consider the eigenpair set

$$\mathcal{Q}' = \left\{ (g, \lambda, u) \in \mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M) \;\middle|\; (\Delta_g + \lambda)u = 0,\ m_\lambda(g) = 1, u \neq 0 \right\}.$$

Near a simple eigenpair, the eigenvalue and the corresponding eigenfunction are determined by the metric up to multiplication by a nonzero scalar. Thus one expects $\mathcal{Q}'$ to be locally parametrized by the metric together with one real normalization parameter. Our goal is to make this precise by showing that the space $\mathcal{Q}'$ is a tame Fréchet submanifold of the space $\mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M)$, modeled on $\mathcal{G} \times \mathbb{R}$. The basic idea is to encode the eigenvalue equation and the normalization of the eigenfunction in a single map. Consider

$$\begin{aligned} \Psi \colon \mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M) &\longrightarrow \mathcal{G}_+ \times \mathbb{R} \times C^\infty(M), \\ (g, \lambda, u) &\longmapsto \left(g, \frac{1}{2}\|u\|^2_{L^2_g}, (\Delta_g + \lambda)u \right) \end{aligned}$$

Let (g, λ, u) be a point in $\mathcal{Q}'$ and assume that $-\lambda$ is the k–th eigenvalue of Δ_g. Locally, the space $\mathcal{Q}'$ can be described as the inverse image under Ψ of $U \times \mathbb{R} \times \{0\}$, where U is an open neighborhood of g in $\mathcal{G}_+$. Thus, if Ψ is locally invertible near a simple eigenpair, the desired manifold structure on $\mathcal{Q}'$ follows from the corresponding product structure in the target.

This reduces the problem to an application of the inverse function theorem. Since the spaces involved are Fréchet spaces, the ordinary Banach-space inverse function theorem does not apply. We will instead use the Nash–Moser inverse function theorem, i.e. Theorem 1.8.

To prove the required tame estimates for the inverse of $D\Psi$, it is convenient to use Sobolev norms. The analytic input is the following refinement of the elliptic estimates, with a bound also on the metric parameter.

Lemma 3.6 (Local elliptic estimates)**.** *Let $q_0 = (g_0, \lambda_0, u_0) \in \mathcal{Q}'$. Then there exists a neighborhood $\mathcal{V}$ of q_0 and positive constants $\{C_n\}_{n\in\mathbb{N}}$ such that for all $q = (g, \lambda, u) \in \mathcal{V}$, $v \in C^\infty(M)$ and $\eta \in \mathbb{R}$ we have*

$$\|v\|_{H_2} + |\eta| \leq C_0 \left(\|(\Delta_g + \lambda)v + \eta u_0\|_{H_0} + \left| \langle u_0, v \rangle_{L^2_{g_0}} \right| \right).$$

Moreover, $\|v\|_{H_{n+2}}$ is bounded by

$$C_n \left(\|(\Delta_g + \lambda)v + \eta u_0\|_{H_n} + \|(\Delta_g + \lambda)v + \eta u_0)\|_{H_0} \|q\|_{n+1} + (1 + \|q\|_{n+1}) \left| \langle u_0, v \rangle_{L^2_{g_0}} \right| \right)$$

for every $q = (g, \lambda, u) \in \mathcal{V}$ and $n \in \mathbb{N}$.

Proof. The following result is a particular case [10, Lemma 3.3.4 Sec 2]. With the notation of the original statement, it is obtained by considering the map $i(\dot\lambda) = \dot\lambda u_0$ and the map $j(v) = \langle u_0, v \rangle_{L^2_{g_0}}$. □

We are ready to show that $\mathcal{Q}'$ is a Fréchet submanifold.

Theorem 3.7 ($\mathcal{Q}'$ is tame Fréchet)**.** *Let $q_0 = (g_0, \lambda_0, u_0) \in \mathcal{Q}'$. Then, there exists a neighborhood $\mathcal{V}$ of q_0 such that $\mathcal{Q}' \cap \mathcal{V}$ is a smooth tame Fréchet submanifold of $\mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M)$.*

Proof. Consider the smooth tame map between Freéchet manifolds

$$\Psi \colon \mathcal{G}_+ \times \mathbb{R}_{>0} \times C^\infty(M) \to \mathcal{G}_+ \times \mathbb{R} \times C^\infty(M), \tag{3.1}$$

defined as $\Psi(g, \lambda, u) = \big(g,\ \frac{1}{2}\|u\|^2_{L^2_g}, (\Delta_g + \lambda)u\big)$. Its differential in a point $q = (g, \lambda, u)$ can be written as

$$D_q\Psi(h, \eta, v) = \Big(h,\ \frac{1}{4}\int_M u^2 \operatorname{tr}_g(h)\, d\mathrm{vol}_g + \langle u, v\rangle_{L^2_g}, \dot{\Delta}_g(u)[h] + (\Delta_g + \lambda)v + \eta u\Big),$$

where $\operatorname{tr}_g(h)$ is the trace with respect to g and we used $d(\mathrm{vol}_g)[h] = \frac{1}{2}\operatorname{tr}_g(h)\, d\mathrm{vol}_g$. To apply Theorem 1.8, we need to prove that the differential is locally invertible with a family of continuous inverses. This amounts to showing that the equation

$$D_q\Psi(h, \eta, v) = (h', \eta', v') \tag{3.2}$$

has a unique solution for every h', η', v'. Then, we need to prove that this family of solutions is smooth and tame in q. Let $\mathcal{V}$ be an open neighborhood of $q_0 = (g_0, \lambda_0, u_0)$. If λ_0 is a simple k-th eigenvalue we can assume that all the metrics that appear in $\mathcal{V}$ have simple k–th eigenvalue by [4, Theorem 2.2]. We now argue that it is enough to study only the second and third component of $D\Psi$. Indeed, (3.2) forces $h = h'$. Thus, the equation reduces to

$$D_q\Psi(0, \eta, v) = D_q\Psi(h, \eta, v) - D_q\Psi(h, 0, 0) = (0, \eta'', v''). \tag{3.3}$$

Observe that the first component of $D_q\Psi(h, \eta, v)$ depends only on h. In particular, requiring for it to be 0 imposes no condition on η and v. This implies also that the tameness of the solution map of (3.2) follows from the tameness of the solution map of (3.3). Therefore we can reduce the problem to studying

$$D\tilde{\Psi} \colon \mathcal{V} \times \mathbb{R} \times C^\infty(M) \to \mathbb{R} \times C^\infty(M), \quad D_q\tilde{\Psi}(\eta, v) = \Big(\langle u, v\rangle_{L^2_g}, (\Delta_g + \lambda)v + \eta u\Big).$$

A simple computation shows that $D_{q_0}\tilde{\Psi}$ is invertible. We first prove that we can take $\mathcal{V}$ such that $D\tilde{\Psi}$ is invertible with a family of continuous inverses. Fix a sufficiently large positive integer ℓ. Consider the extension

$$D\tilde{\Psi}_\ell : \mathcal{V} \times \mathbb{R} \times H_{\ell+2}(M) \to \mathbb{R} \times H_\ell(M),$$

defined as in the smooth case. The assignment q to $D_q\tilde{\Psi}_\ell$ is a continuous map from $\mathcal{V}$ to $\mathcal{L} = \mathcal{L}(\mathbb{R} \times H_{\ell+2}, \mathbb{R} \times H_\ell)$. Observe that $\mathcal{L}$ is the set of continuous maps between 2 Banach spaces. In particular, the subset of invertible operators is open. Thus, we can take $\mathcal{V}$ such that $D_q\tilde{\Psi}_\ell$ is invertible for every q in $\mathcal{V}$. Moreover, the total inverse

$$D\tilde{\Psi}_\ell^{-1} \colon \mathcal{V} \times \mathbb{R} \times H_\ell(M) \to \mathbb{R} \times H_{\ell+2}(M)$$

is continuous. This is because taking inverses is a continuous operation in Banach spaces. By Theorem 1.1, the restriction

$$D\tilde{\Psi}^{-1} := D\tilde{\Psi}_\ell^{-1}|_{C^\infty(M)} \colon \mathcal{V} \times \mathbb{R} \times C^\infty(M) \to \mathbb{R} \times C^\infty(M)$$

is s.t. $D_q\tilde{\Psi}^{-1}$ is invertible for every $q \in \mathcal{V}$. We want to show that the total map $D\tilde{\Psi}^{-1}$ is continuous. Let (q_n, c_n, f_n) be a sequence in $\mathcal{V} \times \mathbb{R} \times C^\infty(M)$ converging in the smooth topology to $(q, c, f) \in \mathcal{V} \times \mathbb{R} \times C^\infty(M)$. Denote by $D\tilde{\Psi}^{-1}_{q_n}(c_n, f_n) = (\eta_n, v_n)$. Continuity in the H_ℓ topology already gives η_n converges to η and v_n to v, the latter only in the H_ℓ topology. Consider the PDEs

$$(\Delta_{g_n} + \lambda_n)v_n + \eta_n u_n = f_n$$

and

$$(\Delta_g + \lambda)v + \eta u = f.$$

By subtracting them, we obtain

$$(\Delta_{g_n} + \lambda_n)(v_n - v) = (f_n - f) + (\eta_n u_n - \eta u) + ((\Delta_g + \lambda) - (\Delta_{g_n} + \lambda_n))\, v. \tag{3.4}$$

We have that η_n converges to η and (q_n, c_n, f_n) to (q, c, f) in the smooth topology. Then, the right hand side of (3.4) goes to 0 in the smooth topology. This means that so does the left hand side. Observe now that, since q_n converges to q, by Theorem 1.1 there exists a uniform positive constant C such that

$$\|v_n - v\|_{H_{k+2}(M)} \leq C \left(\|(\Delta_{g_n} + \lambda_n)(v_n - v)\|_{H_k(M)} + \|v_n - v\|_{L^2(M)} \right) \tag{3.5}$$

for every sufficiently large n and $k \geq \ell$. Since both terms on the right hand side of (3.5) go to 0 for every k, v_n converges to v in the smooth topology. This proves continuity of the total map $D_q\tilde{\Psi}^{-1}$.

By construction, $D_q\tilde{\Psi}^{-1}$ is a linear inverse of $D_q\tilde{\Psi}$ for every q. We now want prove that $D\tilde{\Psi}^{-1}$ is smooth and tame in a neighborhood of q_0 in $\mathcal{V}$. By Proposition 1.7, it suffices to prove that it is tame. Tameness is a local property. Thus, it is enough to restrict to an open set in the codomain where $\|q\|_0, |\langle u, v\rangle_{L^2_g}|$ and $\|(\Delta_g + \lambda)v + \eta u\|_{H_0}$ are bounded by a constant C. We first make estimates for $n = 0$. By Lemma 3.6, we have

$$\|v\|_{H_2} + |\eta| \leq C_0 \left(\|(\Delta_g + \lambda)v + \eta u_0\|_{H_0} + \left| \langle u_0, v\rangle_{L^2_{g_0}} \right| \right). \tag{3.6}$$

Fix a positive constant $\epsilon < \frac{1}{C_0}$. Then,

$$\begin{aligned} \|(\Delta_g + \lambda)v + \eta u_0\|_{H_0} \leq & \|(\Delta_g + \lambda)v + \eta u\|_{H_0} + |\eta| \cdot \|u - u_0\|_{H_0} \\ \leq & \|(\Delta_g + \lambda)v + \eta u\|_{H_0} + \epsilon|\eta| \\ \leq & C + \epsilon|\eta| \end{aligned} \tag{3.7}$$

for (g, λ, u) in a small neighborhood of q_0. Thus, (3.6) gives

$$\|v\|_{H_2} + |\eta| \leq \tilde{C}_0 \tag{3.8}$$

in the chosen open set. In particular, (3.7) gives a uniform bound on $\|(\Delta_g + \lambda)v + \dot{\lambda}u_0\|_{H_0}$. This gives estimates for $n = 0$. Let now n be strictly positive. By Lemma 3.6, we have

$$\|v\|_{H_{n+2}} \leq C_n \left(1 + \|(\Delta_g + \lambda)v + \dot{\lambda}u_0\|_{H_n} + \|(\Delta_g + \lambda)v + \eta u_0\|_{H_0} \|q\|_n + \|q\|_n \right), \tag{3.9}$$

where we have already applied $\left| \langle u_0, v\rangle_{L^2_{g_0}} \right| \leq C$. As in (3.7), we obtain

$$\begin{aligned} \|(\Delta_g + \lambda)v + \eta u_0\|_{H_n} \leq & \|(\Delta_g + \lambda)v + \eta u\|_{H_n} + |\eta| \cdot \|u - u_0\|_{H_n} \\ \leq & \|(\Delta_g + \lambda)v + \eta u\|_{H_n} + \tilde{C}\|q\|_n \end{aligned}$$

for a positive constant $\tilde{C}$. Thus, (3.9) gives

$$\|v\|_{H_{n+2}} \leq \tilde{C}_n \left(1 + \|(\Delta_g + \lambda)v + \eta u\|_{H_n} + \|q\|_n \right),$$

for a positive constant $\tilde{C}_n$ depending only on n. This proves tameness. By Theorem 1.8, there is a smooth tame right inverse of Ψ on $\mathcal{V}$. Therefore, $\mathcal{V} \cap \Psi^{-1}(U \times I \times \{0\}) = \mathcal{V} \cap \mathcal{Q}'$ is the image of a smooth tame embedding, where U is the projection of $\mathcal{V}$ on $\mathcal{G}_+$ and $I \subseteq \mathbb{R}$ is a sufficiently small open interval containing $\frac{1}{2}\|u_0\|^2_{L^2_{g_0}}$. Since this projection is an open map, U is open. Therefore, $\mathcal{Q}'$ is a smooth tame Frećhet submanifold. □

From Theorem 3.7 it follows the smooth dependence of the eigenfunction on the metric. In particular, the following is an immediate consequence of the previous proof.

Corollary 3.8 (Tame local dependence of simple eigenpairs)**.** *Let $q_0 = (g_0, \lambda_0, u_0) \in \mathcal{Q}'$, and assume that λ_0 is the k–th eigenvalue of g_0. Then there exist an open neighborhood $U \subset \mathcal{G}_+$ of g_0, an open interval $I \subseteq \mathbb{R}$, and an open neighborhood $\mathcal{V} \subset \mathcal{Q}'$ of q_0 such that*

$$\Psi|_{\mathcal{V}} : \mathcal{V} \longrightarrow U \times \mathbb{R} \times \{0\}$$

is a smooth tame diffeomorphism onto its image. In particular, for every $c \in I$, the map

$$\mathrm{eig}_{k,c} : U \longrightarrow \mathcal{Q}', \qquad \mathrm{eig}_{k,c}(g) := (\Psi|_{\mathcal{V}})^{-1}(g, c, 0),$$

is smooth and tame.

If $c = 1/2$, then $\mathrm{eig}_{k,1/2}(g)$ gives a local smooth tame choice of L^2_g-normalized eigenfunction. We now describe the geometry of the connected components of $\mathcal{Q}'$. In particular, we show that the eigenpair space is divided in a family of subspaces corresponding to the index of the eigenvalue. Let k be a positive integer. Denote by $\mathcal{Q}'_k$ the set of triples (g, λ, u) in $\mathcal{Q}'$ where λ is the k-th eigenvalue of g.

Proposition 3.9 (Connected components of $\mathcal{Q}'$)**.** *For every positive integer k, the subspace $\mathcal{Q}'_k$ is union of connected components of $\mathcal{Q}'$. In particular, it is a tame Fréchet submanifold.*

Proof. By continuity of the k–th eigenvalue with respect to the metric, $\mathcal{Q}'_k$ is open in $\mathcal{Q}'$. This follows from [4, Theorem 2.2]. Since $\mathcal{Q}'$ is the disjoint union of $\{\mathcal{Q}'_k\}_{k\in\mathbb{N}}$, then $\mathcal{Q}'_k$ is also closed in $\mathcal{Q}'$. Therefore these sets are union of connected components of $\mathcal{Q}'$. In particular, they are tame Fréchet submanifolds of codimension 0. □

The right inverse constructed in Section 2 is available on $\mathcal{N}$. We therefore restrict the eigenpair space to the following subset

$$\mathcal{Q} = \left\{ (g, \lambda, u) \in \mathcal{Q}' \,\middle|\, (g, u) \in \mathcal{N} \right\}.$$

The following is an immediate consequence of Theorem 3.1.

Proposition 3.10. *The set $\mathcal{Q}$ is open dense in $\mathcal{Q}'$. In particular, it is a smooth tame Fréchet submanifold of $\mathcal{G} \times \mathbb{R}_{>0} \times C^\infty(M)$.*

Proof. If $\dim(M) > 2$, then this statement only claims that the generic eigenfunction of the Laplace–Beltrami operator is Morse, which follows from Theorem 3.1. Assume now that $\dim(M) \geq 2$. Observe that if $(g, u) \in \mathcal{N}$ and u is a nonconstant eigenfunction of Δ_g, then the condition $\mathrm{crit}(u) \cap (\Delta_g)^{-1}(0) = \emptyset$ is equivalent to having u transverse to 0. Then the statement follows from Theorem 3.1 in the same way. □

We can now prove Theorem A. The rest of this work will be dedicated to discussing its topological consequences.

Theorem 3.11 (Flexibility of eigenfunctions)**.** *The projection map $\pi\colon \mathcal{Q} \to C^\infty(M)$ is an open map.*

Proof. By Theorem 3.7, $\mathcal{Q}$ is a tame Fréchet variety. Let q_0 in $\mathcal{Q}$ and $\mathcal{V}$ a neighborhood of q_0 as in Theorem 2.2. We want to show that π is a smooth tame submersion with a smooth tame right inverse of $D\pi\colon T\mathcal{V} \to C^\infty(M)$. Notice that a triple $(h, \eta, v) \in \mathcal{G} \times \mathbb{R} \times C^\infty(M)$ belongs to $T_q\mathcal{Q}$, for $q = (g, \lambda, u)$, if and only if

$$\dot{\Delta}_g[h](u) + \eta u + (\Delta_g + \lambda)v = 0. \tag{3.10}$$

Let R be a smooth tame right inverse of $\dot{\Delta}$ in $\mathcal{V}$ given by Theorem 2.2, i.e. $\dot{\Delta}_g[R(q,v)](u) = v$ for each $v \in C^\infty(M)$. Then setting

$$\nu(q,v) = (R(q, -(\Delta_g + \lambda)v), 0, v)$$

we have that $\nu(q,v)$ satisfies (3.10). Therefore

$$\nu \colon \mathcal{V} \times C^\infty(M) \to T\mathcal{V}$$

is a smooth tame right inverse of $D\pi$ and it is linear on the second component. We can conclude using Theorem 1.8. The same proof applies to the projection on the eigenvalue and on both the eigenvalue and the eigenfunction. □

Remark 3.12 (Banach setting)**.** Theorem 3.11 might not hold if we set up the problem with C^k metrics and C^k eigenfunction. This is because the function $w_q(v)$ defined in the proof of Proposition 2.8 might not be sufficiently regular. Indeed, the map a_q in (2.18) would only be C^{k-1}, which means the solution constructed in (2.20) might yield only a C^{k-1} metric. The obstruction here is given by a loss of regularity.

Remark 3.13 (Image of the projection)**.** As one would expect, the projection on $C^\infty(M)$ is not a surjective map. Its image is not even dense. Indeed, continuous maps with dense image send dense sets in dense sets. However, every eigenfunction of a Laplace–Beltrami operator has nonempty zero set. This is clearly not a dense property of smooth functions in general.

3.3. **Generic properties of eigenfunctions.** For a smooth metric g on M, denote by $[u_k(g)]$ its k-th eigenspace and by $\lambda_k(g)$ its k–th eigenvalue. We now aim to prove Theorem B. In other words, that every generic properties for smooth functions is also generic for eigenfunctions of the Laplace–Beltrami operator. To do so, it will be convenient to work with the projectivization of the eigenpair spaces.

Definition 3.14. We define the projectivization of the eigenpair space of the Laplace–Beltrami operator as

$$\tilde{\mathcal{Q}} = \{(g, \lambda, [u]) : (g, \lambda, u) \in \mathcal{Q}\} \subseteq \mathcal{G}_+ \times \mathbb{R}_{>0} \times \mathbb{P}C^\infty.$$

For $k \geq 1$, denote by $\tilde{\mathcal{Q}}_k$ the projectivization of $\mathcal{Q}_k$.

Remark 3.15 (Connected components of $\tilde{\mathcal{Q}}$)**.** By Proposition 3.9, $\tilde{\mathcal{Q}}_k$ is union of connected components of $\tilde{\mathcal{Q}}$.

We are now ready to prove Theorem B.

Theorem 3.16 (Generic properties of the k–th eigenfunction)**.** *Let M be a smooth compact connected manifold of dimension $\dim(M) \geq 2$ and k be a positive integer. Let $D \subset \mathbb{P}C^\infty(M)$ and define*

$$\mathcal{G}_+(k, D) = \{g \in \mathcal{G}_+ \colon \lambda_k(g) \text{ is simple}, [u_k(g)] \in D\}.$$

The following hold

(1) if D is open, $\mathcal{G}_+(k,D)$ is open.
(2) if D is dense, $\mathcal{G}_+(k,D)$ is dense.
(3) if D is residual, $\mathcal{G}_+(k,D)$ is residual.

Proof. The first part of the statement is an immediate consequence of the continuity of the k–th eigenspace and k–th eigenvalue on the metric. This is given by Corollary 3.8. Let now $\bar{g}$ be a fixed smooth metric on M. Define $\mathcal{Q}_{\bar{g},k} = \{(g, \lambda, u) \in \mathcal{Q}_k : \|u\|_{L^2_{\bar{g}}} = 1\}$. Consider the restriction and corestriction of the projection π

$$\pi_k : \mathcal{Q}_{\bar{g},k} \to S_{L^2_{\bar{g}}}C^\infty(M).$$

Since π is an open map by Theorem 3.11, π_k is open as well. Indeed, if $f : X \to Y$ is an open continuous map and $Z \subseteq Y$, then $f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z$ is also open continuous. Define the map

$$p_k \colon \mathcal{Q}_{\bar{g},k} \to \mathcal{G}_+ \times \mathbb{R}_{>0} \times \mathbb{P}C^\infty(M)$$

as $p_k(g, \lambda, u) = (g, \lambda, p(u))$, where $p(u)$ is the vector span of u. Observe that $\tilde{\mathcal{Q}}_k = p_k(\mathcal{Q}_{\bar{g},k})$. The map p_k is a local homeomorphism with its image. Denote by $\tilde{\pi}_k : \tilde{\mathcal{Q}}_k \to \mathbb{P}C^\infty(M)$ the projection on the last component. We have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{Q}_{\bar{g},k} & \xrightarrow{\pi_k} & S_{L^2_{\bar{g}}}C^\infty(M) \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p} \\ \tilde{\mathcal{Q}}_k & \xrightarrow[\tilde{\pi}_k]{} & \mathbb{P}C^\infty(M) \end{array}$$

The maps p_k and p are local homeomorphism. Then $\tilde{\pi}_k$ is open if and only if π_k is open. This is because being open is a local property. Thus, $\tilde{\pi}_k$ is open. Consider now the following diagram,

$$\begin{array}{ccccc} & & \tilde{\mathcal{Q}}_k & & \\ & \swarrow^{P_k} & & \searrow^{\tilde{\pi}_k} & \\ \mathcal{G}_+ & & & & \mathbb{P}C^\infty(M) \end{array}$$

where P_k is the projection on the first component. The map P_k is a homeomorphism with its image. Indeed, it is continuous and injective. To show that the inverse is continuous as well, we need to show that the k-th eigenvalue and eigenspace depend continuously on the metric. This follows from Corollary 3.8. Then

$$\tilde{\pi}_k \circ P_k^{-1} : P_k(\tilde{\mathcal{Q}}_k) \subseteq \mathcal{G}_+ \to \mathbb{P}C^\infty(M)$$

is an open continuous map. Observe that $P_k(\tilde{\pi}_k^{-1}(D)) \subseteq \mathcal{G}_+(k, D)$ by definition. Assume now that D is dense. Standard topology gives that the preimage of a dense set under an open map is dense. Thus, $P_k(\tilde{\pi}_k^{-1}(D))$ is dense in $P_k(\tilde{\mathcal{Q}}_k)$. Since $P_k(\tilde{\mathcal{Q}}_k)$ is dense in $\mathcal{G}_+$, the set $P_k(\tilde{\pi}_k^{-1}(D))$ is dense in $\mathcal{G}_+$. This proves the second part of the statement. For the third part, choose open dense sets O_j with $\bigcap_j O_j \subset D$. Each $\mathcal{G}_+(k, O_j)$ is open dense by the first two parts, and their intersection is contained in $\mathcal{G}_+(k, D)$. This proves residuality with the convention used here. □

We can use Theorem 3.16 to prove an similar result for metrics of finite regularity. Recall that we denote by $\mathcal{G}^\ell_+$ the set of C^ℓ Riemannian metrics on M. By [9, Theorem 6.17], eigenfunctions of Laplace–Beltrami operators associated to one such metric are in $C^\ell(M)$.

Theorem 3.17 (Generic properties of the k–th eigenfunction for C^ℓ– metrics)**.** *Let M be a smooth compact connected manifold of dimension $\dim(M) = n \geq 2$ and k be a positive integer and $\ell \geq 2$ be a nonnegative integer. Let $D \subset \mathbb{P}C^\ell(M)$ and define*

$$\mathcal{G}^\ell_+(k, D) = \{g \in \mathcal{G}^\ell_+ \colon \lambda_k(g) \text{ is simple}, [u_k(g)] \in D\}.$$

The following hold

(1) if D is open, $\mathcal{G}^\ell_+(k, D)$ is open.
(2) if D is open and $i_\ell^{-1}(D)$ is dense in $\mathbb{P}C^\infty(M)$, then $\mathcal{G}^\ell_+(k, D)$ is open dense.
(3) Suppose D contains $\bigcap_{j\geq 1} D_j$, where every D_j is open in $\mathbb{P}C^\ell(M)$ and every $i_\ell^{-1}(D_j)$ is dense in $\mathbb{P}C^\infty(M)$. Then $\mathcal{G}^\ell_+(k, D)$ is residual.

Proof. We first define the following subset of $\mathcal{G}^k_+ \times \mathbb{R}_{>0} \times \mathbb{P}C^\ell$

$$\hat{\mathcal{Q}}_{k,C^\ell} = \{(g, \lambda, [u]) : (\Delta_g + \lambda)u = 0,\ m(g, \lambda) = 1, (g, u) \text{ is nonresonant, } \lambda \text{ is the } k-th \text{ eigenvalue}\}.$$

Consider the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{Q}}_k & \xrightarrow{\tilde{\pi}} & \mathbb{P}C^\infty \\ \downarrow{\scriptstyle i_\ell} & & \downarrow{\scriptstyle i_{\mathbb{P}_\ell}} \\ \hat{\mathcal{Q}}_{k,C^\ell} & \xrightarrow[\tilde{\pi}_{C^\ell}]{} & \mathbb{P}C^\ell \end{array}$$

where the π arrows are projections onto the third component and the i arrows are inclusions. Observe that all these maps are continuous and the vertical ones are injections with dense image. Moreover, $\tilde{\pi}$ is open. This was shown in the proof of Theorem 3.16. Consider now a set $D \subseteq \mathbb{P}C^\ell$. By identifying $\tilde{\mathcal{Q}}_{k,C^\ell}$ the with the set of C^ℓ metrics, as we did in the proof of Theorem 3.16, we reduce to proving the statement for $\tilde{\pi}^{-1}_{C^\ell}(D)$. If D is open, then $\tilde{\pi}^{-1}_{C^\ell}(D)$ is open by continuity of the horizontal map. This proves the first part of the statement. Assume now that D is open dense. Since

$$\tilde{\pi}^{-1}_{C^\ell}(D) \supseteq i_\ell(i_\ell^{-1}(\tilde{\pi}^{-1}_{C^\ell}(D))) = i_\ell(\tilde{\pi}^{-1}(i^{-1}_{\mathbb{P}_\ell}(D))), \tag{3.11}$$

where in the second equality we used the commutativity of the diagram. By hypothesis, the preimage $i^{-1}_{\mathbb{P}_\ell}(D)$ is open dense. Then, since $\tilde{\pi}$ is open, $\tilde{\pi}^{-1}(i^{-1}_{\mathbb{P}_\ell}(D))$ is also open dense. This is because the preimage of a dense set over an open map is still dense. Finally, since i_ℓ is a continuous map with dense image, $i_\ell(\tilde{\pi}^{-1}(i^{-1}_{\mathbb{P}_\ell}(D)))$ is dense in $\tilde{\mathcal{Q}}_{k,C^\ell}$. Then, by (3.11), $\tilde{\pi}^{-1}_{C^\ell}(D)$ is dense. This proves the second part of the statement. The third part of the statement immediately follow from the second after taking countable intersections. □

Remark 3.18. The second hypothesis in (2) of Theorem 3.17 cannot be dropped. Indeed, fix $0 < \alpha < 1$, $\ell \geq 2$ and a smooth reference metric $\bar{g}$. Define the following subsets of $\mathbb{P}C^\ell(M)$

$$K_m = \left\{ [u] : \frac{\|u\|_{C^{\ell,\alpha}}}{\|u\|_{L^2_{\bar{g}}}} \leq m \right\}, \qquad m \geq 1.$$

It is easy to see that K_m is compact using Arzelà–Ascoli [3, Theorem 3.25]. Moreover, an infinite-dimensional Banach manifold has no compact subset with nonempty interior. Thus

$$D = \mathbb{P}C^\ell(M) \setminus \bigcup_{m\geq 1} K_m$$

is residual. Nevertheless, elliptic regularity [9, Theorem 6.17] implies that no eigenfunction of any C^ℓ metric belongs to D. Thus $\mathcal{G}^\ell_+(k, D) = \emptyset$. This is why the compatibility assumption in Theorem 3.17 is essential.

Taking a countable intersection gives the following corollary.

Theorem 3.19 (Generic properties of every eigenfunction)**.** *Let M be a smooth compact connected manifold of dimension $\dim(M) \geq 2$. Let $D \subset \mathbb{P}C^\infty(M)$ be residual and define*

$$\mathcal{G}_+(D) = \{g \in \mathcal{G}_+ \colon \lambda_k(g) \text{ is simple and } [u_k(g)] \in D \text{ for every } k \geq 1\}.$$

Then, $\mathcal{G}_+(D)$ is residual.

The remaining extensions are stated in the smooth category. Their genericity assertions use the smooth openness theorem.

3.4. **Generic properties of metrics induced by an embedding.** A natural way of equipping M with a Riemannian metric is to embed it into the Euclidean space and pull back the Euclidean metric. Let $\dim(M) = n$ and $N \geq 2n+1$ be a positive integer. Denote by $\mathrm{Emb}(M, \mathbb{R}^N)$ the space of smooth embeddings of M into $(\mathbb{R}^N, g_N)$, where g_N is the Euclidean metric on $\mathbb{R}^N$. This is an open dense subset of $C^\infty(M, \mathbb{R}^N)$ by the standard transversality theorem for embeddings. Given an embedding f_0 of M, we can consider the metric $f_0^* g_N$ on M given by the pullback of the Euclidean metric. This induces a map between Fréchet manifolds

$$\mathcal{P} : \mathrm{Emb}(M, \mathbb{R}^N) \to \mathcal{G}^+ \tag{3.12}$$

defined as $\mathcal{P}(f) = f^* g_N$. This map was extensively studied by J. Nash in [13] to prove his celebrated isometric embedding theorem.

Theorem 3.20 (Nash's isometric embedding theorem [6, Theorem 3.1.1])**.** *Let $3 \leq k \leq \infty$, $n \geq 1$ and $N = \frac{n(3n+11)}{2}$. If (M, g) is a compact C^k Riemannian manifold of dimension n with a C^k metric, then there exists a C^k isometric embedding $f : (M, g) \to (R^N, g_N)$.*

We refer to [2] and [6] for a survey of this result. A key ingredient in Nash's proof is the local openness of the metric-inducing map at free embeddings. We aim to use that approach to prove the following.

Theorem 3.21 (Generic properties of induced metrics)**.** *Let M be a smooth compact connected manifold of dimension $\dim(M) = n \geq 2$. Let N and k be positive integers and assume $N \geq \frac{n(n+5)}{2}$. Let $D \subset \mathbb{P}C^\infty(M)$ and define*

$$E(k, D) = \{f \in \mathrm{Emb}(M, \mathbb{R}^N) \colon [u_k(f^* g_N)] \text{ is simple}, [u_k(f^* g_N)] \in D\}.$$

The following hold

(1) if D is open, $E(k, D)$ is open.
(2) if D is dense, $E(k, D)$ is dense.
(3) if D is residual, $E(k, D)$ is residual.

Moreover, if D is residual, then the set

$$E(D) = \{f \in \mathrm{Emb}(M, \mathbb{R}^N) \colon \lambda_k(f^* g_N) \text{ is simple}, [u_k(f^* g_N)] \in D \text{ for every } k\}$$

is residual.

To prove Theorem 3.21, it is enough to show that the pullback map is open in an open dense subset of $\mathrm{Emb}(M, \mathbb{R}^N)$ under the hypothesis of the theorem. Since the preimage of an open dense set under an open continuous map is still open dense, the result then follows directly from Theorem 3.16 and Theorem 3.19. Following Nash's work, we define the following property.

Definition 3.22 (Free embedding)**.** Let f_0 be a smooth embedding of M in $\mathbb{R}^N$. We say that f_0 is free if, for every $x \in M$ and every coordinate system $\{x_1, \dots, x_n\}$ in a neighborhood of x, the following vectors are linearly independent

$$\frac{\partial f_0}{\partial x_i}(x), \frac{\partial^2 f_0}{\partial x_i x_j}(x) \quad 1 \leq i \leq j \leq n. \tag{3.13}$$

Remark 3.23. Observe that freedom is an intrinsic condition on the 2–jets of f_0. In particular, it does not depend on the choice of atlas on M. Since M is compact, the set of free embeddings is open in the C^2 topology, and thus in the smooth one as well.

No free embeddings can exist if $N < \frac{n(n+3)}{2}$, which is the number of vectors in (3.13). However, we need to ask for a stricter bound on the dimension of the target space. This stronger bound is needed because the proof of Theorem 3.21 requires free embeddings to be dense. We refer to [5] for the following theorem although it should be attributed to J. Nash and M. Gromov.

Proposition 3.24 ([5, Theorem N1, Theorem N2, Theorem IFT])**.** *Let M be a compact smooth manifold of dimension n and let $N \geq \frac{n(n+5)}{2}$ be a positive integer. Then the set of free embeddings of M in $\mathbb{R}^N$ is open dense in* $\mathrm{Emb}(M, \mathbb{R}^N)$*. Moreover, the pullback map $\mathcal{P}$ is open when restricted to the set of free embeddings.*

Proposition 3.24 only gives openness of the pullback over free embeddings. However, since this space is open dense, Theorem 3.21 immediately follows.

3.5. **Schrödinger operators.** So far we've restricted our study to the Laplace–Beltrami operators. In this section we consider a more general family of self–adjoint second order elliptic operators on M.

Definition 3.25 (Schrödinger operator)**.** Let M be a smooth manifold. We say that a partial differential operator on M is a Schrödinger operator if it is in the form

$$\Delta_g + v,$$

where g is a smooth metric and v is a smooth function on M

Schrödinger operators are clearly selfadjoint and elliptic. Moreover, their spectrum is a nonincreasing sequence

$$-\lambda_1(g, v) \geq -\lambda_2(g, v) \geq \cdots \to -\infty.$$

Thus, it makes sense to say which is the k–th eigenvalue of a Schrödinger operator. If λ is a scalar, we will denote by $m(g, v, \lambda) = \dim(\ker(\Delta_g + v + \lambda))$. We will parametrize them naturally as

$$(g, v) \to \Delta_g + v$$

and study the generic properties of this family. In particular, we want to prove an analogue of Theorem 3.11 for Schrödinger operators. We first define the corresponding eigenpair space

$$\mathcal{S}' = \{(g, v, \lambda, u) : (\Delta_g + v + \lambda)u = 0, m(g, v, \lambda) = 1, u \neq 0\} \subseteq \mathcal{G}_+ \times C^\infty(M) \times \mathbb{R} \times C^\infty(M).$$

The proof of Theorem 3.7 applies in the same way, giving that S' is a smooth tame submanifold of the ambient space. Moreover, we define

$$\mathcal{S} = \{(g, v, \lambda, u) \in \mathcal{S}' : (g, u) \in \mathcal{N}\}.$$

This is the subset of the Schrödinger eigenpair space where the differential of the Laplace–Beltrami operator is locally surjective with smooth tame right inverse. We want to show that $\mathcal{S}$ is open dense in $\mathcal{S}'$. In preparation for what follows, it will be convenient to define S'_ℓ to be

the set of (g, v, λ, u) in $\mathcal{S}'$ where λ is the ℓ–th eigenvalue of $\Delta_g + v$. Specifying the index of the eigenvalue this way allows us to identify the generic operator with its eigenspace. As we did in Proposition 3.9, one can show that $\mathcal{S}'_\ell$ is union of connected components of $\mathcal{S}'$. In particular, it is open. We will prove that $\mathcal{S}_\ell$ is open dense in $\mathcal{S}'_\ell$ for every ℓ. The following result will be useful in this regard. We only state a simplified version, which will be enough for our purposes. Recall that a Baire space is a topological space where every countable intersection of open dense sets is dense. Since every space we work with is Baire, we will not discuss this property further.

Theorem 3.26 (Kuratowski-Ulam, [15, Theorem 3.5.16])**.** *Let X and Y be second countable Baire spaces and $A \subseteq X \times Y$ be open. Then the following are equivalent:*

(1) A is dense.
(2) The set $\{x \in X : A_x$ is residual$\}$ is residual, where

$$A_x = \{y \in Y : (x, y) \in A\}.$$

Theorem 3.26 allows to reduce global density to density in the fibers. To prove the following result, we define the finite-regularity eigenpair space of the Schrödinger operators as follows.

Definition 3.27. Let $k \geq 2$ be a positive integer. We define the C^k eigenpair space of Schrödinger operators as

$$\mathcal{S}'_{C^k} = \{(g, v, \lambda, u) : (\Delta_g + v + \lambda)u = 0, m(g, v, \lambda) = 1, u \neq 0\} \subseteq \mathcal{G}^k_+ \times C^{k-1}(M) \times \mathbb{R} \times H_k(M).$$

Remark 3.28. Recall that the eigenfunctions of an elliptic operator with C^{k-1} coefficients are in H_k by Theorem 1.1. This justifies the spaces chosen in the definition above. Also observe that $\mathcal{S}'_{C^k}$ is a Banach manifold. This can be proven as in Theorem 3.7.

Proposition 3.29. *Let M be a smooth compact connected manifold with $\dim(M) \geq 2$. For the generic $q \in \mathcal{S}'$, u has nonsingular zero set, is Morse, and $(\Delta_g u)^{-1}(0) \cap \mathrm{crit}(u) = \emptyset$. Thus, the set $\mathcal{S}$ is open dense in $\mathcal{S}'$. In particular, it is a tame Fréchet submanifold.*

Proof. Let ℓ be a positive integer and $\dim(M) = n$. We will prove that the set of q as in the statement is open dense in $\mathcal{S}'_\ell$. Let $k \geq n + 3$ be a positive integer. Consider the space $\mathcal{G}_+ \times C^{k-1}(M)$ of Schrödinger operators given by a smooth metric and a $C^k(M)$ function. Denote by $A_{\ell,C^k} \subseteq \mathcal{G}_+ \times C^k(M)$ the set of Schrödinger operators with simple ℓ–th eigenvalue and the corresponding eigenfunction Morse and transverse to 0. This set is open. The fact that having simple ℓ-th eigenvalue is an open condition is proven exactly like it is for the Laplace-Beltrami operator. By Theorem 3.3, every fiber $A_{\ell,C^k,g} \subseteq C^k(M)$ is residual. Then, by Theorem 3.26, the set $A_{\ell,C^k} \subseteq \mathcal{G}_+ \times C^k(M)$ is also residual. It is also open by compactness of M. Since this holds for every k sufficiently large, using Lemma 2.23 we obtain that

$$A_\ell = A_{\ell,C^k} \cap (\mathcal{G}_+ \times C^\infty(M)) \subseteq \mathcal{G}_+ \times C^\infty(M)$$

is also open dense. Consider now the continuous map

$$P_\ell : \mathcal{S}'_\ell \to \mathcal{G}_+ \times C^\infty(M)$$

defined as $P_\ell(g, v, \lambda, u) = (g, v)$. Since the space of Schrödinger operators with simple ℓ–th eigenvalue is open and P_ℓ is a projection, this map is open. In particular, the preimage of an open dense set will still be open dense. Thus, $P_\ell^{-1}(A_\ell)$ is open dense.

It remains only to prove that the generic element in S' avoids the third condition of the statement. Consider the map

$$F : \mathcal{S}'_{C^k} \times M \to T^*M \oplus \mathbb{R}$$

defined as $F(q,x) = (x, d_x u, \Delta_g u(x))$. We want to show that F is transverse to the set

$$Z = \{(x,0,0) : x \in M\}$$

and then apply Theorem 2.22. Assume then that $(g,v,\lambda,u) \in \mathcal{S}'_{C^k}$ and $x \in M$ are such that $F((v,\lambda,u),x) \in Z$. By arguing as in the first part of the proof, we can show that for every $(\tilde{g},\tilde{v},\tilde{\lambda},\tilde{u})$ in an open dense subset of $\mathcal{S}'_{C^k}$, $\tilde{u}$ is transverse to 0. To complete the proof, it is enough to prove that the restriction of F to this set is transverse to Z. Thus, we can assume that there exists an open neighborhood V of x where u is never vanishing. Observe that the tangent to $\mathcal{S}'_{C^k}$ in (g,v,λ,u) is defined by the equation

$$\dot{\Delta}_g[\dot{g}](u) + (\dot{v} + \dot{\lambda})u + (\Delta_g + v + \lambda)\dot{u} = 0.$$

To prove submersivity of F, it is enough to consider variations with $\dot{g}|_V = 0$ and $\dot{\lambda} = 0$. Indeed, let w be any smooth function compactly supported in V. Consider the function

$$\dot{v}(w) = -\frac{(\Delta_g + v + \lambda)w}{u}.$$

By construction, the element $(0,\dot{v}(w),0,w)$ verifies the equation characterizing the tangent of $\mathcal{S}'_{C^k}$. To prove that it belongs to the tangent, we need only to prove that $\dot{v}(w) \in C^{k-1}(M)$. Recall that throughout this work we always assume k to be sufficiently large such that $u \in C^2(M)$. Then, since u is a C^2 eigenfunction of an elliptic operator with C^{k-1} coefficients, we have $u \in C^k$. This is a consequence of [9, Theorem 6.17]. Moreover, $(\Delta_g + v + \lambda)w$ is also a C^{k-1} function. This is because w is smooth and $\Delta_g + v + \lambda$ has C^{k-1} coefficients. This proves that $(0,\dot{v}(w),0,w) \in T_q\mathcal{S}'_{C^k}$. Then F is a submersion in (q,x), since we can always consider a smooth function w compactly supported in V with prescribed differential on $\Delta_g w$ in x. Since $\operatorname{codim}(Z) = n+1 > \dim(M)$, by Theorem 2.22 we have that $\mathcal{S}'_{C^k} \cap \{q : F(q,M) \cap Z = \emptyset\}$ residual in $\mathcal{S}'_{C^k}$. It is also open by compactness of M. Since this holds for every k sufficiently large, by Lemma 2.23 we obtain that $\left(\mathcal{S}'_{C^k} \cap \{q : F(q,M) \cap Z = \emptyset\}\right) \cap \mathcal{S}'$ is open dense in S'. It is exactly the set of elements in $\mathcal{S}'$ avoiding the third condition in the statement. This concludes the proof. □

Remark 3.30. Observe than the generic Schrödinger operator has simple eigenvalue by Theorem 3.3 even in dimension 1. This is very different from the behaviour of the Laplace–Beltrami operators, whose nonzero eigenvalues always have multiplicity 2.

We can finally state the main result of this section. The proof is an immediate consequence of Theorem 2.2 and Proposition 3.29. It essentially coincides with the one of Theorem 3.11.

Theorem 3.31 (Flexibility of eigenfunctions for Schrödinger operators)**.** *Let M be a smooth compact connected manifold* $\dim(M) \geq 2$*. Then, the projection on the eigenfunction*

$$\pi_{\mathcal{S}} : \mathcal{S} \to C^\infty(M)$$

is open. In particular, it is open in an open dense subset of $\mathcal{S}'$.

Analogues of Theorem 3.16 and Theorem 3.19 follow immediately with the same proofs. In the same way, one can also prove generic properties of Schrodinger operators induced by embeddings. We denote by $\lambda_k(f^*g, v)$ the k–th eigenvalue of the operator $\Delta_g + v$.

Theorem 3.32 (Generic properties of induced Schrödinger operators)**.** *Let M be a smooth compact connected manifold with* $\dim(M) = n$ *and assume $n \geq 2$. Let N and k be positive integers and assume $N \geq \frac{n(n+5)}{2}$. Let $D \subset \mathbb{P}C^\infty(M)$ and define*

$$\mathcal{S}(k,D) = \{(f,v) \in \operatorname{Emb}(M,\mathbb{R}^N) \times C^\infty(M) \colon \lambda_k(f^*g_N, v) \text{ is simple}, [u_k(f^*g_N, v)] \in D\}.$$

The following hold

(1) if D is open, $\mathcal{S}(k, D)$ is open.
(2) if D is dense, $\mathcal{S}(k, D)$ is dense.
(3) if D is residual, $\mathcal{S}(k, D)$ is residual.

Moreover, if D is residual, then the set

$$\mathcal{S}(D) = \{(f, v) \in \mathrm{Emb}(M, \mathbb{R}^N) \times C^\infty(M) \colon \lambda_k(f^* g_N, v) \text{ is simple}, [u_k(f^* g_N, v)] \in D \text{ for every } k\}$$

is residual.

Remark 3.33. One could also consider different families of second order self–adjoint elliptic operators and apply the methods used in this paper. An interesting one is given by

$$\{-4\Delta_g + \mathrm{Scal}_g : g \in \mathcal{G}_+\},$$

where Scal_g is the scalar curvature associated to the metric g. This gives an important class of Schrödinger operators, which are studied in [14]. However, since the function involved in the operator depends on the metric, one cannot expect to prove openness of the projection by simply applying Theorem 3.11.

4. Appendix

4.1. **Proof of Lemma 2.4.** Consider a curve of metrics $g_t = g + th$. We make computations in coordinates. Thus, we can write

$$\Delta_{g_t} u = g_t^{ij}(\partial_i \partial_j u - \Gamma_{ij}^k(t)\partial_k u), \tag{4.1}$$

where $\Gamma_{ij}^k(t)$ are the Christoffel symbols w.r.t. the metric g_t. This expression is obtained directly from the definition of Δ. We have

$$\frac{d}{dt} g_t^{ij}\Big|_{t=0} = -g^{ik} g^{j\ell} h_{k\ell} = -h^{ij}.$$

This gives the first summand of (2.2). We now differentiate the Christoffel symbols. Recall

$$\Gamma_{ij}^k(t) = \frac{1}{2} g_t^{k\ell}(\partial_i g_{t,j\ell} + \partial_j g_{t,i\ell} - \partial_\ell g_{t,ij}).$$

We will omit the time when $t = 0$ to simplify notation. Differentiating gives

$$\begin{aligned}\dot\Gamma_{ij}^k &= -\frac{1}{2} h^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) + \frac{1}{2} g^{k\ell}(\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij}) \\ &= -h^{k\ell} g_{a\ell} \Gamma_{ij}^a + \frac{1}{2} g^{k\ell}(\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij}).\end{aligned}$$

We want to simplify the second term of the summand. We have

$$\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij} = \nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij} + C_{i,j,\ell},$$

where $C_{i,j,\ell}$ is the correction given by the Christoffel symbols. In particular,

$$C_{i,j,\ell} = \Gamma_{ij}^a h_{a\ell} + \Gamma_{i\ell}^a h_{aj} + \Gamma_{j\ell}^a h_{ai} + \Gamma_{ij}^a h_{a\ell} - \Gamma_{i\ell}^a h_{aj} - \Gamma_{j\ell}^a h_{ai} = 2\Gamma_{ij}^a h_{a\ell}.$$

Moreover, observe that

$$h^{k\ell} g_{a\ell} = g^{ik} g^{j\ell} h_{ij} g_{a\ell} = g^{ik} h_{ia} = g^{k\ell} h_{a\ell}.$$

Thus, we have obtained

$$\begin{aligned}\dot\Gamma_{ij}^k &= \Gamma_{ij}^a(g^{k\ell} h_{a\ell} - h^{k\ell} g_{a\ell}) + \frac{1}{2} g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \\ &= \frac{1}{2} g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}).\end{aligned}$$

Now we want to compute $-g^{ij}\dot\Gamma^k_{ij}\partial_k u$. We have

$$\begin{aligned} g^{ij}\dot\Gamma^k_{ij}\partial_k u &= \frac{1}{2}g^{ij}g^{k\ell}(\nabla_i h_{j\ell}+\nabla_j h_{i\ell}-\nabla_\ell h_{ij})\partial_k u \\ &= \frac{1}{2}g^{k\ell}(g^{ij}\nabla_i h_{j\ell}+g^{ji}\nabla_j h_{i\ell}-\nabla_\ell \operatorname{tr}_g h)\partial_k u \\ &= g^{k\ell}(g^{ji}\nabla_j h_{i\ell}-\frac{1}{2}\nabla_\ell \operatorname{tr}_g h)\partial_k u \\ &= g^{k\ell}(\operatorname{div}_g(h)_\ell-\frac{1}{2}\nabla_\ell \operatorname{tr}_g h)\partial_k u \\ &= \langle \operatorname{div}_g(h)^\sharp-\frac{1}{2}\nabla \operatorname{tr}_g h,\nabla u\rangle_g. \end{aligned}$$

This gives (2.2).

Now we compute the formula with respect to a conformal variation. Let $\psi\in C^\infty(M)$. Consider the conformal variation $h=\psi g$. Then we have $-\langle \psi g,\nabla^2 u\rangle_g=-\psi\Delta_g u$. Moreover,

$$\frac{1}{2}\nabla \operatorname{tr}_g \psi g=\frac{n}{2}\nabla\psi \quad \text{ and } \quad -\operatorname{div}_g(\psi g)^\sharp=-\nabla\psi,$$

which implies equation Eq. (2.3). For a traceless variation h, we just observe the Leibniz rule

$$\operatorname{div}_g(h(\nabla u,\cdot))=\langle \operatorname{div}_g h^\sharp,\nabla u\rangle_g+\langle h,\nabla^2 u\rangle_g.$$

Substituting this into equation (2.2) gives (2.4).

We now prove (2.5). By substituting $h=wF$ into (2.2) we get

$$\begin{aligned} \dot\Delta_g[wF](u) &= -\langle h,\nabla^2 u\rangle_g+\langle\frac{1}{2}\nabla(\operatorname{tr}_g h)-\operatorname{div}_g h^\sharp,\nabla u\rangle_g \\ &= -\langle F,\nabla^2 u\rangle_g w+\langle\frac{1}{2}\nabla(\operatorname{tr}_g Fw)-\operatorname{div}_g(Fw)^\sharp,\nabla u\rangle_g. \end{aligned} \tag{4.2}$$

We have

$$\begin{aligned} \langle\nabla \operatorname{tr}_g Fw,\nabla u\rangle_g &= \langle\nabla(\operatorname{tr}_g F)w+\operatorname{tr}_g F\cdot\nabla w,\nabla u\rangle_g \\ &= \langle\nabla \operatorname{tr}_g F,\nabla u\rangle_g w+\langle \operatorname{tr}_g F\nabla u,\nabla w\rangle_g \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \langle\operatorname{div}_g(wF)^\sharp,\nabla u\rangle_g &= \langle F(\nabla w,\cdot)^\sharp+(\operatorname{div}_g F)w^\sharp,\nabla u\rangle_g \\ &= \langle F(\nabla w,\cdot)^\sharp,\nabla u\rangle_g+\langle\operatorname{div}_g F^\sharp,\nabla u\rangle_g w. \end{aligned} \tag{4.4}$$

Moreover, we have

$$\langle F(\nabla w,\cdot)^\sharp,\nabla u\rangle_g=F(\nabla w,\nabla u)=F(\nabla u,\nabla w)=\langle F(\nabla u,\cdot)^\sharp,\nabla w\rangle_g. \tag{4.5}$$

Substituting (4.3), (4.4) and (4.5) in (4.2) concludes the proof. □

4.2. **Proof of Lemma 2.13.** Let $A\coloneqq JX_0(x_0)$. Then every eigenvalue of $A+\mu$ has negative real part. Thus, the matrix

$$P=\int_0^{+\infty} e^{(A^T+\mu\mathrm{Id})t}e^{(A+\mu\mathrm{Id})t}dt$$

is well defined, symmetric and positive definite. In particular, it satisfies the Lyapunov equation

$$(A^T+\mu\mathrm{Id})P+P(A+\mu\mathrm{Id})=-\mathrm{Id},$$

which can be written as

$$A^TP+PA=-\mathrm{Id}-2\mu P \tag{4.6}$$

Refer to the proof of [11, Theorem 4.6] for the details. Let $\epsilon > 0$ be small enough such that

$$2\|P\|\epsilon < \frac{1}{2}, \tag{4.7}$$

where $\|P\|$ is the operator norm of the matrix P associated to the standard Euclidean norm. Let $x \in \mathbb{R}^n$ and B be a matrix such that $\|B - A\| < \epsilon$. Then,

$$\begin{aligned} x^T(B^TP + PB)x &\leq x^T((B-A)^TP + P(B-A))x + x^T(A^TP + PA)x \\ &\leq 2\epsilon\|P\|\|x\|^2 + x^T(-\mathrm{Id} - 2\mu P)x \\ &\leq -\frac{\|x\|^2}{2} - 2\mu x^TPx \\ &\leq -2\mu x^TPx, \end{aligned} \tag{4.8}$$

where we have used (4.7) in the third inequality.

We now construct a positively invariant neighborhood for X_0. Consider the function

$$V_0(x) = (x - x_0)^TP(x - x_0).$$

Let $r > 0$ be a scalar. Since P is positive definite, the ellipsoid $U = V_0^{-1}([0, r))$ centered in x_0 has compact closure and is convex. Moreover, we can pick r small enough such that

$$\|JX_0(x) - A\| \leq \frac{\epsilon}{4}$$

for every $x \in \overline{U}$ and $X_0(x) \neq 0$ for every $x \in \overline{U} \setminus \{x_0\}$. Now we want to prove

$$DV_0(x)[X_0(x)] \leq -2\mu V_0(x) < 0 \tag{4.9}$$

for every $x \in \overline{U} \setminus \{x_0\}$. This will immediately imply that U is a Φ_t^0–positively invariant neighborhood. Proving the inequality in the middle is the only nontrivial step. We have

$$\begin{aligned} X_0(x) &= X_0(x) - X_0(x_0) \\ &= \int_0^1 \frac{d}{ds} X_0(x_0 + s(x - x_0))ds \\ &= \int_0^1 JX_0(x_0 + s(x - x_0))ds(x - x_0) = B_x(x - x_0), \end{aligned} \tag{4.10}$$

where $B_x = \int_0^1 DX_0(x_0 + s(x - x_0))ds$ is considered as an endomorphism of $\mathbb{R}^n$. This gives

$$\begin{aligned} DV_0(x)[X_0(x)] &= (B_x(x - x_0))^TP(x - x_0) + (x - x_0)^TPB_x(x - x_0) \\ &= (x - x_0)^T(B_x^TP + PB_x)(x - x_0) \end{aligned} \tag{4.11}$$

By convexity of U, we have

$$\|B_x - A\| \leq \int_0^1 \|JX_0(x_0 + s(x - x_0)) - A\|ds \leq \frac{\epsilon}{4} < \epsilon \tag{4.12}$$

Therefore, we can substitute (4.8) into (4.11) and obtain that for every $x \in \overline{U} \setminus \{x_0\}$

$$DV_0(x)[X_0(x)] \leq -2\mu(x - x_0)^TP(x - x_0) = -2\mu V_0(x) < 0. \tag{4.13}$$

This means exactly that U is Φ_t^0–positively invariant. Indeed, assume by contradiction that there exists $x \in U$ and $t \geq 0$ such that $V_0(\Phi_0^t(x)) > r$. Then there exists $t_0 \geq 0$ s.t. $V_0(\Phi_0^{t_0}(x))$ is increasing in t, which contradicts (4.13). Finally, since $\overline{U}$ is compact and positively invariant, [11, Theorem 3.3] implies that $\Phi_t^0(x)$ is defined for every $t \geq 0$ and $x \in U$.

Now we show that this holds also for every vector field sufficiently C^1–close to X_0. Let δ be as follows. For every vector field X_1 such that $\|X_0 - X_1\|_{C^1} < \delta$,

$$\|JX_1(x) - A\| \leq \frac{\epsilon}{2} \tag{4.14}$$

for every $x \in \overline{U}$. Moreover, by (4.13), we can assume that

$$DV_0(x)[X_1(x)] < 0$$

for every $x \in \partial U$ and X_1 as above. Such a δ exists by compactness of $\overline{U}$. Finally, we pick δ small enough such that every such X_1 has a critical point x_1 in $\overline{U}$. Since X_0 is nondegenerate in 0 and X_1 is C_1–close to X_0, we can assume that x_1 is the only critical point of X_1 in $\overline{U}$. As before, this implies that U is a positively invariant neighborhood for every such X_1 and that the flow of such vector fields starting at a point in U is defined for every positive t. Fix one such X_1 and let x_1 be its equilibrium in U. We now prove the first part of (2.16). Consider the quadratic form

$$V_1(x) = (x - x_1)^T P(x - x_1).$$

We want to prove that

$$\frac{d}{dt}V_1(\Phi_t^1(x)) = DV_1(\Phi_t^1(x))[X_1(\Phi_t^1(x))] \leq -2\mu V_1(\Phi_t^1(x)) < 0 \tag{4.15}$$

for every $x \in U \setminus \{x_1\}$. This is obtained exactly as we did to prove (4.13), but this time we need to use that $\Phi_t^1(x)$ belongs to U to have the estimate (4.12). Applying Grönwall lemma to (4.15) gives

$$V_1(\Phi_t^1(x)) \leq V_1(x)e^{-2\mu t}.$$

By definition of V_1, we have

$$\lambda_{\min}(P)\|\Phi_t^1(x) - x_1\|^2 \leq V_1(\Phi_t^1(x)) \leq V_1(x)e^{-2\mu t} \leq \lambda_{\max}(P)\|x_1 - x\|^2 e^{-2\mu t}.$$

Thus, we have obtained

$$\|\Phi_t^1(x) - x_1\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}e^{-\mu t}\|x - x_1\| \leq c_0 e^{-\mu t}, \tag{4.16}$$

where $c_0 = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\operatorname{diam}(U)$ depends only on X_0 through P and U.

It remains only to prove (2.16) for the flow of X_1. We prove it by induction. To simplify notation, we will write $X = X_1$ and denote its flow by simply Φ_t. Differentiating the equation characterizing the flow of X, we obtain

$$D\dot{\Phi}_t(x) = D(X(\Phi_t(x))) = JX(\Phi_t(x))D\Phi_t(x). \tag{4.17}$$

Observe that (4.17) is a linear system with associated matrix $JX(\Phi_t(x))$ and unknown $D\Phi_t(x)$. Since (4.14) holds by construction for (4.17), the Lyapunov estimates we gave for $m = 0$ still hold. Explicitly, let $v \in \mathbb{R}^n$ be a vector of norm 1 and define $Y_{v,t}(x) = D\Phi_t(x)v$. Then we have

$$\dot{Y}_{v,t}(x) = JX(\Phi_t(x))Y_{v,t}(x). \tag{4.18}$$

Proceeding as in the first part of the proof with the Lyapunov function

$$V_{v,t}(x) = Y_{v,t}(x)^T P Y_{v,t}(x)$$

and taking the supremum over v gives

$$\|D\Phi_t(x)\| \leq \|D\Phi_0(x)\|c_1 e^{-\mu t} = c_1 e^{-\mu t}, \tag{4.19}$$

where the second inequality is given by $\Phi_0(x) = x$. This implies the estimate in (2.16).

We now consider higher order derivatives. Let $m \geq 2$ and assume that the last part of the statement has been proven for every smaller integer. For simplicity, we will denote by c_m every constant which can be made dependent only on m, δ and X_0. We have

$$D^m \dot{\Phi}_t(x) = D^m(X(\Phi_t(x))) = JX(\Phi_t(x))D^m\Phi_t(x) + F_{m,t}(x), \tag{4.20}$$

where $F_{m,t}(x)$ is a finite sum of terms in the form

$$D^k X(\Phi_t(x))[D^{j_1}\Phi_t(x), \ldots, D^{j_k}\Phi_t(x)]$$

with $2 \leq k \leq m$, $j_i \geq 1$ and $j_1 + \ldots + j_k = m$. The coefficients and number of these terms depend only on m. This means that

$$\|F_{m,t}\| \leq c_m \sum_{k=2}^{m} \|D^k X\| \sum_{j_1+\ldots+j_k=m} \prod_{i=1}^{k} \|D^{j_i}\Phi_t\|. \tag{4.21}$$

By induction, we have

$$\|D^{j_i}\Phi_t\| \leq c_{j_i} e^{-\mu t}(1 + \|X\|_{j_i}) \leq c_m e^{-\mu t}(1 + \|X\|_{j_i}). \tag{4.22}$$

Applying (4.22) to (4.21) gives

$$\begin{aligned}
\|F_{m,t}\| &\leq c_m \sum_{k=2}^{m} \|X\|_k \sum_{j_1+\ldots+j_k=m} \prod_{i=1}^{k} e^{-\mu t}(1 + \|X\|_{j_i}) \\
&\leq c_m e^{-2\mu t} \sum_{k=2}^{m} \|X\|_k \sum_{j_1+\ldots+j_k=m} \prod_{i=1}^{k} (1 + \|X\|_{j_i}) \\
&\leq c_m e^{-2\mu t} \sum_{k=2}^{m} \sum_{j_1+\ldots+j_k=m} \left(\|X\|_k + \sum_{\ell=1}^{k} \sum_{f_1 \neq \ldots \neq f_\ell} \|X\|_k \|X\|_{j_{f_1}} \cdots \|X\|_{j_{f_\ell}} \right), \qquad (4.23)
\end{aligned}$$

where all the combinatorial coefficients have been incorporated in the constant c_m. For $1 \leq j \leq m$ and $m \geq 2$, Proposition 1.6 gives

$$\|X\|_j \leq c_m \|X\|_1^{\frac{m-j}{m-1}} \cdot \|X\|_m^{\frac{j-1}{m-1}}.$$

This gives

$$\begin{aligned}
\|X\|_k \|X\|_{j_{f_1}} \cdots \|X\|_{j_{f_\ell}} &\leq c_m \|X\|_1^{\frac{(\ell+1)m-(k+\sum j_f)}{m-1}} \cdot \|X\|_m^{\frac{k-1-\ell+\sum j_f}{m-1}} \\
&\leq c_m \|X\|_m^{\frac{k-1-\ell+\sum j_f}{m-1}} \qquad (4.24)
\end{aligned}$$

Since all the j indices appearing in (4.23) are positive, we have

$$\sum_{i=1}^{\ell} j_{f_i} \leq \sum_{i=1}^{k} j_i - (k-\ell) = m - (k-\ell).$$

In particular, $0 \leq \frac{k-1-\ell+\sum j_f}{m-1} \leq 1$. Then,

$$\|X\|_m^{\frac{k-1-\ell+\sum j_f}{m-1}} \leq 1 + \|X\|_m. \tag{4.25}$$

Substituting (4.24) and (4.25) in (4.23) gives

$$\begin{aligned}
\|F_{m,t}\| &\leq c_m e^{-2\mu t} \sum_{k=2}^{m} \sum_{j_1+\ldots+j_k=m} \left(\|X\|_k + \sum_{\ell=1}^{k} \sum_{f_1 \neq \ldots \neq f_\ell} (1 + \|X\|_m) \right) \\
&\leq c_m e^{-2\mu t}(1 + \|X\|_m) \qquad (4.26)
\end{aligned}$$

where we estimated $\|X\|_k \leq \|X\|_m$ and incorporated all the combinatorial constants in c_m.

It remains only to estimate the first summand in the right hand side of (4.20). We have

$$\|JX(\Phi_t)D^m\Phi_t\| \leq \|JX(\Phi_t)\| \cdot \|D^m\Phi_t\| \leq c_m\|D^m\Phi_t\|.$$

Therefore, it is enough to make estimates on $D^m\Phi_t$. To do so, we'll write it explicitly as a solution of (4.20). Define the evolution operator $S(t,s,x)$ as the solution of

$$\dot{S}(t,s,x) = JX(\Phi_t(x))S(t,s,x), \quad S(s,s,x) = \mathrm{Id}_n \forall s \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n. \tag{4.27}$$

By the standard theory of ODEs, the solution of (4.27) exists and is unique given initial conditions. Moreover, (4.14) holds for (4.27). Thus, we can obtain the following Lyapunov estimates for $S(t,s,x)$ in the same way as we did for (4.19)

$$\|S(t,s,x)\| \leq c_m e^{-\mu(t-s)}\|S(s,s,x)\| = c_m e^{-\mu(t-s)}, \tag{4.28}$$

where the first equality is an immediate consequence of (4.27). An easy computation shows that the function

$$f_t(x) = \int_0^t S(t,s,x)F_{m,s}(x)ds \tag{4.29}$$

is a solution to (4.20). In particular, since $m \geq 2$ and $\Phi_0(x) = x$ for every x, we have $f_0(x) = D^m\Phi_0(x) = 0$. By uniqueness of the solution, it follows that $f_t(x) = D^m\Phi_t(x)$ for every positive t. We can now make estimates on $D^m\Phi_t(x)$ using this explicit expression. Applying (4.26) and (4.28) gives

$$\begin{aligned}
\|D^m\Phi_t\| &\leq \int_0^t \|S(t,s,x)\| \cdot \|F_{m,s}\|ds \\
&\leq c_m(1+\|X\|_m)\int_0^t e^{-\mu(t-s)}e^{-2\mu s}ds \\
&= c_m(1+\|X\|_m)e^{-\mu t}\int_0^t e^{-\mu s}ds \\
&\leq c_m(1+\|X\|_m)e^{-\mu t}.
\end{aligned}$$

This concludes the proof. □

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