A “Gauss–Bonnet Formula” for Contact Sub-Riemannian Manifolds

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Abstract

We study 3-dimensional manifolds endowed with oriented contact sub-Riemannian structures. The Euler characteristic class of the contact structure is presented as the rotation class of a volume preserving vector field constructed in terms of fundamental differential invariants of the sub-Riemannian metric.

1. Let $M$ be a smooth 3-dimensional manifold. A contact sub-Riemannian structure is a pair $\Delta, \langle \cdot | \cdot \rangle$, where $\Delta = \{ \Delta_q \}_{q \in M}$, $\Delta_q \subset T_q M$, is a contact structure on $M$ and $\langle \cdot | \cdot \rangle = \{ \langle \cdot | \cdot \rangle_q \}_{q \in M}$ is a smooth with respect to $q$ family of Euclidean inner products $(v_1, v_2) \mapsto \langle v_1 | v_2 \rangle_q$, $v_1, v_2 \in \Delta_q$.

A Lipschitzian curve $\xi : [0, 1] \to M$ is called admissible for $\Delta$ if $\frac{d \xi(t)}{dt} \in \Delta_{\xi(t)}$ for almost all $t \in [0, 1]$. The length of an admissible curve $\xi$ is the integral $\int_0^1 \| \frac{d \xi}{dt} \| dt$, where $\| v \| = \sqrt{\langle v | v \rangle_q}$ $\forall v \in T_q M$. The infimum of the lengths of admissible curves connecting two points is the Carnot–Caratheodory distance between these points.

An important class of the sub-Riemannian structures is provided by magnetic fields on Riemannian surfaces. In this case $M$ is the total space of a principal $U(1)$-bundle over a Riemannian surface $N$ and $\Delta$ is a connection on the principal bundle. In other words, $\Delta$ is a transversal to fibers $U(1)$-invariant rank 2 distribution on $M$. The distribution is contact if and only if the curvature of the connection doesn’t vanish. The inner product of a pair of vectors in $\Delta_q$ equals the scalar product of their projections in $T_q N$.

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The length minimization problem for admissible curves is equivalent to the least action principle for the charged particles in the magnetic field (see [5] for details). The special case of the constant magnetic field (when the curvature of the connection is the area form multiplied by a constant) is locally equivalent to the classical Dido isoperimetric problem on the Riemannian surface.

The local structure of the Carnot–Caratheodory metric was studied in detail in the papers [1, 2, 3, 4]; it was shown that this structure is controlled by some fundamental differential invariants. In this note the same local invariants serve to express a global one: the Euler characteristic class of $\Delta$. We assume that $\Delta$ is an oriented contact structure. Then there exists a unique contact form $\omega$ on $M$ such that $\Delta$ is the annihilator of $\omega$, $\Delta_q = \omega_q^\perp$  \forall q \in M and the form $d\omega|_{\Delta_q}$ coincides with the area form on the oriented Euclidean plane $\Delta_q$. Let $X_0$ be the characteristic (or Reeb) vector field of $\omega$; it is defined by the relations $X_0|d\omega = 0$, $X_0|\omega = 1$.

Let $\delta(q_0, q_1)$ be the Carnot–Caratheodory distance between $q_0$ and $q_1$ and $C_{q_0} = \{ q_1 \in M : \text{the function } q \mapsto \delta(q_0, q) \text{ is not } C^1 \text{ at } q_1 \}$.

One can show that the line $\mathbb{R}X_0(q)$ is the tangent cone$^1$ to the $C_q$ at $q$, $\mathbb{R}X_0(q) = T_qC_q$.

The curve $t \mapsto e^{tX_0}(q)$ is transversal to the distribution $\Delta$. Let $N_q \subset M$ be (a germ of) a 2-dimentional submanifold such that $\Delta_q = T_qN$; then a neighborhood of $q$ is sliced by the submanifolds $e^{tX_0}(N_q)$. One of the basic invariants of the sub-Riemannian structure arises from the asymptotics of the distance between $q$ and $e^{tX_0}(N_q) \cap C_q$ as $t \rightarrow 0$:

$$\delta(q, e^{tX_0}(N_q) \cap C_q) = (2 - \kappa(q)t)|\pi t|^{1/2} + O(t^2),$$

where $\kappa$ is a smooth function on $M$.

Let $X_1(q), X_2(q)$ be an orthonormal frame in $\Delta_q$. The explicit expression of $\kappa$ in terms of the moving frame $X_0, X_1, X_2$ is as follows:

$$\kappa = X_1c_{12}^2 - X_2c_{12}^1 - (c_{12}^1)^2 - (c_{12}^2)^2 + \frac{1}{2}(c_{02}^1 - c_{01}^2),$$

where $[X_i, X_j] = \sum_k c_{ij}^k X_k$.

$^1$There are many definitions of the tangent cone to the closed set; all of them provide one and the same cone in this particular case.
In the special case of the constant magnetic field on the Riemannian surface (see Sec. 1) \( X_0 \) is a generator of the structural group of the principle bundle, \( \kappa \) is constant on the fibers of the bundle and is actually the pullback of the Gaussian curvature of the Riemannian surface.

3. Now we take the dual object to the sub-Riemannian structure, the Hamiltonian \( h \) on the cotangent bundle \( T^*M \):

\[
h(\lambda) = \frac{1}{2} \left( \max \{ \langle \lambda, v \rangle : v \in \Delta_q, \ |v| = 1 \} \right)^2, \quad \lambda \in T^*_q M, \ q \in M.
\]

This Hamiltonian serves to describe sub-Riemannian geodesics, i.e. admissible curves whose small pieces are length minimizers. It follows from the Pontryagin Maximum Principle that geodesics are exactly projections to \( M \) of the trajectories of the Hamiltonian system in \( T^*M \) associated with \( h \).

Let \( u_i(\lambda) = \langle \lambda, X_i(q) \rangle, \ i = 1, 2 \); then

\[
h(\lambda) = \frac{1}{2} \left( u_1(\lambda)^2 + u_2(\lambda)^2 \right), \ \lambda \in T^*_q M.
\]

The Hamiltonian keeps all the information on the sub-Riemannian structure: both \( \Delta \) and the inner product are easily recovered from \( h \).

We denote:

\[
u_0(\lambda) = \langle \lambda, X_0(q) \rangle, \ \lambda \in T^*_q M, \ q \in M, \ \text{and} \ \Delta^* = u^{-1}(0).
\]

Then \( \Delta^* \) is a rank 2 linear subbundle of \( T^*M \) with the fibers \( \Delta^*_q = \Delta^* \cap T^*_q M \). Obviously, \( (\lambda, \xi) \mapsto (\lambda, \xi), \ \lambda \in \Delta^*_q, \ \xi \in \Delta_q, \) is a nondegenerate pairing. Moreover, quadratic form \( 2h|_{\Delta^*_q} \) defines the Euclidean structure on \( \Delta^*_q \) dual to the given Euclidean structure on \( \Delta_q \).

We’ll deal with homogeneous polynomials on the plane \( \Delta^*_q \). The group \( \text{SO}(\Delta^*_q) \) acts on the polynomials by the changing of variables. Irreducible components of this (real) action are 2-dimensional spaces of polynomials having the following expression in polar coordinates \((r, \theta)\):

\[
\text{span}\{r^n \cos(k\theta), \ r^n \sin(k\theta)\}, \quad 1)
\]

where \( n - k \) is a nonnegative even number. Note that \( r^2 = 2h|_{\Delta^*_q} \). So any homogeneous degree \( n \) polynomial \( \phi \) has a unique presentation as a sum of isotopic components: \( \phi = \sum_{i=0}^{[n/2]} \phi^{n-2i} \), where \( \phi^k \) belongs to the space (1). This presentation is actually equivalent to the Fourier expansion of the restriction of \( \phi \) to the unit circle.

4. Recall that \( h, u_0 \) are functions on the cotangent bundle \( T^*M \), where restrictions of \( h \) to the fibers \( T^*_q M \) are quadratic forms and restrictions of \( u_0 \) to the fibers are linear forms. Hence the Poisson bracket \( \{h, u_0\} \) is quadratic and the double Poisson bracket \( \{h, \{h, u_0\}\} \) is cubic on the fibers. Let \( \phi_q = \{h, \{h, u_0\}\}|_{\Delta^*_q} \); then \( \phi_q = \phi^1_q + \phi^3_q \), where \( \phi^1_q \) is the product of \( h|_{\Delta^*_q} \) and a
linear form on $\Delta^*_q$, according to (1). In other words, $\phi^1_q(\lambda) = h(\lambda)\langle \lambda, f(q) \rangle$ for some $f(q) \in \Delta_q$.

We thus obtain an intrinsically defined vector field $f$ with values in $\Delta$ in addition to the transversal to $\Delta$ field $X_0$.

**Theorem 1** The 2-form on $M$

$$\left( \frac{\kappa}{2\pi}X_0 - \frac{1}{\pi}f \right) \wedge \omega$$

is closed and represents the Euler characteristic class of the oriented linear bundle $\Delta$.

**Remark 1.** The statement of the theorem can be also formulated as follows: The flow generated by the field

$$\left( \frac{\kappa}{2\pi}X_0 - \frac{1}{\pi}f \right)$$

preserves the volume form $\omega \wedge d\omega$ and the rotation class (see [6]) of the field (3) is equal to the Euler class of $\Delta$.

**Remark 2.** In the case of the principal bundle $M \xrightarrow{U(1)} N$ and the sub-Riemannian structure defined by a constant magnetic field (see sec.1) we have $\{h, u_0\} = 0$ and hence $f = 0$. Then (2) takes the form: $rac{\kappa}{2\pi}X_0|\omega \wedge d\omega = \frac{\kappa}{2\pi} d\omega$. Moreover, $d\omega$ and $\kappa$ are the pullbacks of the area form and of the Gaussian curvature on $N$ so that the form (2) turns into the pullback of the Gauss–Bonnet form on $N$.

The proof of Theorem 1 consists of a calculation with moving frames in $T^*M$. The idea is to construct an appropriate linear connection on the bundle $\Delta^* \subset T^*M$ via the Hamiltonian vector fields associated with $h$ and $u_0$ and a vertical vector field generating rotations of the fibers $\Delta^*_q$. The form (2) is the curvature form of the correspondent linear connection.

**References**


